AN IDENTIFICATION METHOD OF DIPOLAR SOURCES IN HOMOGENEOUS SPACE FOR TIME-HARMONIC MAXWELL'S EQUATIONS

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Abstract. This paper discusses an inverse source problem for time-harmonic Maxwell's equations. We consider the propagation of electromagnetic waves in a homogeneous space. Here, the source term is expressed by electric current dipoles. For the above problem, we propose an identification method of dipoles from the data of electric and magnetic fields. This method is based on boundary integrals using vector-valued weighting functions. By the proper choice of weighting functions, we can identify locations and moments of dipoles without any iterative procedures. The error estimates for identified results are also obtained. The effectiveness of our method is shown by numerical examples.

1 Introduction. Many researchers study various inverse problems for Maxwell's equations, e.g., identification of current source distribution, determination of conductivity and permeability, and so on [8]. Especially, one of the important problems is to identify the electrical activity in the human brain from observation data of electric and magnetic fields called electroencephalogram and magnetoencephalogram. This problem is often expressed as the identification of dipolar sources for a quasi-static approximation of Maxwell's equations. Numerical methods for the above case have been proposed in many papers [2, 6, 9].

In recent years, inverse source problems for time-harmonic Maxwell's equations and Helmholtz equation are considered. For such problems, noniterative methods are developed in [1, 3, 5]. These methods are based on boundary integrals using weighting functions. In [3], He and Romanov show the identification of single dipole. Here, weighting functions satisfy Helmholtz equation. They also consider the case where a dipole and a quadrupole are located at the same position. Ammari, Bao, and Fleming [1] discuss the inverse analysis and the identification of single dipole. Here, weighting functions are vector-valued functions, and each component satisfies Laplace equation. In [5], we propose an identification method of point sources using weighting functions which satisfy Helmholtz equation.

This paper discusses an identification problem of the current source in a homogeneous space from observations of electric and magnetic fields. We consider the case where the propagation of electromagnetic wave is governed by time-harmonic Maxwell's equations. The current source is modeled by a sum of electric current dipoles. For this model, the unknown parameters are locations and moments of dipoles. Here, number of dipoles is also unknown. Under the above conditions, we propose a numerical method for identifying locations and moments of dipoles without any a priori information about unknown parameters and number of dipoles. Our method is based on boundary integrals using vector-valued weighting functions. Each component of weighting functions satisfies Helmholtz equation. By the proper choice of weighting functions, we can identify locations and moments of

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dipoles without any iterative procedures. The error estimates for identified results are also obtained.

The contents of this paper are as follows. Section 2 shows the identification problem of electric current dipoles in a homogeneous space for time-harmonic Maxwell's equations. In Section 3, we propose a direct identification method based on boundary integrals using vector-valued weighting functions. Section 4 shows an algorithm for our problem, and our algorithm is applied to several numerical examples in Section 5.

2 Mathematical formulation. We consider the propagation of electromagnetic waves in a homogeneous space. The wave propagation is governed by Maxwell's equations. For the time-harmonic case, Maxwell's equations are expressed by

$$\operatorname{rot} \boldsymbol{E}(\boldsymbol{x}) = iw\mu \boldsymbol{H}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3, \tag{1}$$

$$\operatorname{rot} \boldsymbol{H}(\boldsymbol{x}) = \sigma \boldsymbol{E}(\boldsymbol{x}) - iw\epsilon \boldsymbol{E}(\boldsymbol{x}) + \boldsymbol{J}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3, \tag{2}$$

where $\boldsymbol{E}(\boldsymbol{x})$ is the electric field, $\boldsymbol{H}(\boldsymbol{x})$ is the magnetic field, $\boldsymbol{J}(\boldsymbol{x})$ is the current source, w is the frequency, and σ, ϵ, and μ are the electric conductivity, the electric permittivity, and the magnetic permeability, respectively (see [1]). Here, $w(\neq 0)$ is a real constant, and σ, ϵ, and μ are positive constants. The electric field $\boldsymbol{E}(\boldsymbol{x})$ and magnetic field $\boldsymbol{H}(\boldsymbol{x})$ are assumed to satisfy Silver-Müller radiation condition [1, 7]:

$$\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left\{ w\mu \boldsymbol{H}(\boldsymbol{x}) \times \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - k\boldsymbol{E}(\boldsymbol{x}) \right\} = 0, \quad k^2 = w^2\epsilon\mu + iw\mu\sigma, \quad \operatorname{Im} k > 0. \tag{3}$$

Let $\Omega \subset \mathbb{R}^3$ be a simply connected domain with smooth boundary Γ. The current source $\boldsymbol{J}(\boldsymbol{x})$ is assumed to be a sum of dipoles such that

$$\boldsymbol{J}(\boldsymbol{x}) = \sum_{j=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{p}_j)\boldsymbol{q}_j, \quad \boldsymbol{p}_j \in \Omega, \quad \boldsymbol{q}_j \in \mathbb{C}^3, \quad |\boldsymbol{q}_j| \neq 0, \tag{4}$$

where N is the number of dipoles, and $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$ are the location and moment of the j-th dipole, respectively. We assume that a priori information about locations, moments, and number of dipoles is not given. Our purpose is to identify unknown parameters $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$ ($j = 1, 2, \ldots, N$) from the data of $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$ on Γ, where N is also unknown.

3 Identification of dipolar sources using weighted integral. From eqs.(1) and (2), we have

$$\operatorname{rot}\operatorname{rot} \boldsymbol{E}(\boldsymbol{x}) - k^2\boldsymbol{E}(\boldsymbol{x}) = iw\mu \boldsymbol{J}(\boldsymbol{x}). \tag{5}$$

Multiply both sides by a vector-valued weighting function $\boldsymbol{u}(\boldsymbol{x})$ and integrate over Ω, then

$$\int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \{\operatorname{rot}\operatorname{rot} \boldsymbol{E}(\boldsymbol{x}) - k^2\boldsymbol{E}(\boldsymbol{x})\} dV = iw\mu \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{J}(\boldsymbol{x}) dV, \tag{6}$$

where $\boldsymbol{u}(\boldsymbol{x})$ satisfies

$$\operatorname{rot}\operatorname{rot} \boldsymbol{u}(\boldsymbol{x}) - k^2\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}. \tag{7}$$

The left-hand side of eq.(6) becomes

$$\begin{aligned}
&\int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \{\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{E}(\boldsymbol{x}) - k^2\boldsymbol{E}(\boldsymbol{x})\}dV \\
&\quad = \int_\Omega [\boldsymbol{u}(\boldsymbol{x}) \cdot \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{E}(\boldsymbol{x}) \cdot \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u}(\boldsymbol{x})]\,dV \\
&\quad = -\int_\Gamma \{iw\mu\boldsymbol{H}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}) \cdot \mathrm{rot}\,\boldsymbol{u}(\boldsymbol{x})\}\,dS,
\end{aligned}$$

where $\boldsymbol{n}(\boldsymbol{x})$ denotes the outward unit normal vector to Γ. From eq.(4), the right-hand side of eq.(6) is

$$iw\mu \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{J}(\boldsymbol{x})dV = iw\mu \sum_{j=1}^{N} \boldsymbol{q}_j \cdot \boldsymbol{u}(\boldsymbol{p}_j).$$

Then, we have

$$\begin{aligned}
I(\boldsymbol{u}) &\equiv \int_\Gamma \left\{ -\boldsymbol{H}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) + \frac{i}{w\mu}\boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}) \cdot \mathrm{rot}\,\boldsymbol{u}(\boldsymbol{x}) \right\} dS \\
&= \sum_{j=1}^{N} \boldsymbol{q}_j \cdot \boldsymbol{u}(\boldsymbol{p}_j).
\end{aligned} \tag{8}$$

In the following, we construct an identification method for unknown dipolar sources using the boundary integral $I(\boldsymbol{u})$. Note that $I(\boldsymbol{u})$ is calculated from the data of $\boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x})$ on Γ.

3.1 Identification of location. As the weighting functions satisfying eq.(7), we use

$$\begin{aligned}
&\boldsymbol{u}_{1\xi}(\boldsymbol{x}) = v_{1\xi}(\boldsymbol{x})\boldsymbol{e}_0 - v_{2\xi}(\boldsymbol{x})\boldsymbol{e}_\xi, \quad \boldsymbol{u}_{2\xi}(\boldsymbol{x}) = v_{2\xi}(\boldsymbol{x})\boldsymbol{e}_0 + v_{1\xi}(\boldsymbol{x})\boldsymbol{e}_\xi, \\
&\boldsymbol{u}_{3\xi}(\boldsymbol{x}) = v_{3\xi}(\boldsymbol{x})\boldsymbol{e}_0 - v_{4\xi}(\boldsymbol{x})\boldsymbol{e}_\xi, \quad \boldsymbol{u}_{4\xi}(\boldsymbol{x}) = v_{4\xi}(\boldsymbol{x})\boldsymbol{e}_0 + v_{3\xi}(\boldsymbol{x})\boldsymbol{e}_\xi,
\end{aligned} \qquad \xi = 1, 2, \tag{9}$$

where real vectors $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$ satisfy

$$|\boldsymbol{e}_0| = |\boldsymbol{e}_1| = |\boldsymbol{e}_2| = 1, \quad \boldsymbol{e}_0 \cdot \boldsymbol{e}_1 = 0, \quad \boldsymbol{e}_2 = \boldsymbol{e}_0 \times \boldsymbol{e}_1.$$

Here, $v_{1\xi}(\boldsymbol{x})$, $v_{2\xi}(\boldsymbol{x})$, $v_{3\xi}(\boldsymbol{x})$, and $v_{4\xi}(\boldsymbol{x})$ are scalar functions with parameter $\lambda_\xi (> 0)$ such that

$$\begin{aligned}
v_{1\xi}(\boldsymbol{x}) &= e^{\lambda_\xi(\boldsymbol{x}\cdot\boldsymbol{e}_0)} \cos \lambda_\xi(\boldsymbol{x} \cdot \boldsymbol{e}_\xi) e^{ik(\boldsymbol{x}\cdot\boldsymbol{e}_\eta)}, \\
v_{2\xi}(\boldsymbol{x}) &= e^{\lambda_\xi(\boldsymbol{x}\cdot\boldsymbol{e}_0)} \sin \lambda_\xi(\boldsymbol{x} \cdot \boldsymbol{e}_\xi) e^{ik(\boldsymbol{x}\cdot\boldsymbol{e}_\eta)}, \\
v_{3\xi}(\boldsymbol{x}) &= (\boldsymbol{x} \cdot \boldsymbol{e}_0)v_{1\xi}(\boldsymbol{x}) - (\boldsymbol{x} \cdot \boldsymbol{e}_\xi)v_{2\xi}(\boldsymbol{x}), \\
v_{4\xi}(\boldsymbol{x}) &= (\boldsymbol{x} \cdot \boldsymbol{e}_0)v_{2\xi}(\boldsymbol{x}) + (\boldsymbol{x} \cdot \boldsymbol{e}_\xi)v_{1\xi}(\boldsymbol{x}),
\end{aligned} \qquad \eta = 3 - \xi. \tag{10}$$

We assume that $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$ satisfy the following conditions for some l:

(i) $\boldsymbol{p}_l \cdot \boldsymbol{e}_0 - \max_{j \neq l} \boldsymbol{p}_j \cdot \boldsymbol{e}_0 > 0.$

(ii) $\sqrt{|\boldsymbol{q}_l \cdot \boldsymbol{e}_0|^2 + |\boldsymbol{q}_l \cdot \boldsymbol{e}_1|^2} \neq 0, \quad \sqrt{|\boldsymbol{q}_l \cdot \boldsymbol{e}_0|^2 + |\boldsymbol{q}_l \cdot \boldsymbol{e}_2|^2} \neq 0.$

Under the above conditions, we consider the identification of location $\boldsymbol{p}_l$ and moment $\boldsymbol{q}_l$.

From eqs.(9) and (10), the boundary integrals become

$$
(11)\quad
\begin{aligned}
I(\boldsymbol{u}_{1\xi}) &= \sum_{j=1}^{N} e^{\lambda_\xi p_{j0}} (q_{l0}\cos\lambda_\xi p_{j\xi} - q_{l\xi}\sin\lambda_\xi p_{j\xi}) e^{ikp_{j\eta}} \equiv \sum_{j=1}^{N} e^{\lambda_\xi p_{j0}} \Psi_{j\xi},\\
I(\boldsymbol{u}_{2\xi}) &= \sum_{j=1}^{N} e^{\lambda_\xi p_{j0}} (q_{l0}\sin\lambda_\xi p_{j\xi} + q_{l\xi}\cos\lambda_\xi p_{j\xi}) e^{ikp_{j\eta}} \equiv \sum_{j=1}^{N} e^{\lambda_\xi p_{j0}} \Phi_{j\xi},\\
I(\boldsymbol{u}_{3\xi}) &= \sum_{j=1}^{N} e^{\lambda_\xi p_{j0}} (p_{j0}\Psi_{j\xi} - p_{j\xi}\Phi_{j\xi}),\\
I(\boldsymbol{u}_{4\xi}) &= \sum_{j=1}^{N} e^{\lambda_\xi p_{j0}} (p_{j0}\Phi_{j\xi} + p_{j\xi}\Psi_{j\xi}),
\end{aligned}
$$

where

$$
p_{jm} = \boldsymbol{p}_j \cdot \boldsymbol{e}_m, \quad q_{jm} = \boldsymbol{q}_j \cdot \boldsymbol{e}_m, \qquad m = 0, 1, 2.
$$

Substituting $e^{\lambda_\xi p_{l0}}\Psi_{l\xi} = I(\boldsymbol{u}_{1\xi}) - \sum_{j\neq l} e^{\lambda_\xi p_{j0}}\Psi_{j\xi}$ and $e^{\lambda_\xi p_{l0}}\Phi_{l\xi} = I(\boldsymbol{u}_{2\xi}) - \sum_{j\neq l} e^{\lambda_\xi p_{j0}}\Phi_{j\xi}$ into the right-hand sides of $I(\boldsymbol{u}_{3\xi})$ and $I(\boldsymbol{u}_{4\xi})$, the following equations are obtained:

$$
\begin{aligned}
I(\boldsymbol{u}_{3\xi}) &= p_{l0} I(\boldsymbol{u}_{1\xi}) - p_{l\xi} I(\boldsymbol{u}_{2\xi}) - \mathcal{R}_{A\xi},\\
I(\boldsymbol{u}_{4\xi}) &= p_{l0} I(\boldsymbol{u}_{2\xi}) + p_{l\xi} I(\boldsymbol{u}_{1\xi}) - \mathcal{R}_{B\xi},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R}_{A\xi} &= \sum_{j\neq l} e^{\lambda_\xi p_{j0}} \{(p_{l0} - p_{j0})\Psi_{j\xi} - (p_{l\xi} - p_{j\xi})\Phi_{j\xi}\},\\
\mathcal{R}_{B\xi} &= \sum_{j\neq l} e^{\lambda_\xi p_{j0}} \{(p_{l0} - p_{j0})\Phi_{j\xi} + (p_{l\xi} - p_{j\xi})\Psi_{j\xi}\}.
\end{aligned}
$$

Then, we have

$$
(12)\quad
\begin{pmatrix} e^{-\lambda_\xi p_{l0}} I(\boldsymbol{u}_{1\xi}) & -e^{-\lambda_\xi p_{l0}} I(\boldsymbol{u}_{2\xi}) \\ e^{-\lambda_\xi p_{l0}} \overline{I}(\boldsymbol{u}_{2\xi}) & e^{-\lambda_\xi p_{l0}} \overline{I}(\boldsymbol{u}_{1\xi}) \end{pmatrix}
\begin{pmatrix} p_{l0} \\ p_{l\xi} \end{pmatrix}
=
\begin{pmatrix} e^{-\lambda_\xi p_{l0}} I(\boldsymbol{u}_{3\xi}) \\ e^{-\lambda_\xi p_{l0}} \overline{I}(\boldsymbol{u}_{4\xi}) \end{pmatrix}
+
\begin{pmatrix} e^{-\lambda_\xi p_{l0}} \mathcal{R}_{A\xi} \\ e^{-\lambda_\xi p_{l0}} \overline{\mathcal{R}}_{B\xi} \end{pmatrix}.
$$

The notations $\overline{I}(\boldsymbol{u}_{1\xi})$, $\overline{I}(\boldsymbol{u}_{2\xi})$, $\overline{I}(\boldsymbol{u}_{4\xi})$, and $\overline{\mathcal{R}}_{B\xi}$ denote the complex conjugates of $I(\boldsymbol{u}_{1\xi})$, $I(\boldsymbol{u}_{2\xi})$, $I(\boldsymbol{u}_{4\xi})$, and $\mathcal{R}_{B\xi}$, respectively. Under the conditions (i) and (ii), $e^{-\lambda_\xi p_{l0}}\mathcal{R}_{A\xi}$ and $e^{-\lambda_\xi p_{l0}}\overline{\mathcal{R}}_{B\xi}$ converge to 0 as $\lambda_\xi \to \infty$, and the determinant of the coefficient matrix in eq.(12) becomes

$$
e^{-2\lambda_\xi p_{l0}} \left(|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2\right) \to \left(|q_{l0}|^2 + |q_{l\xi}|^2\right) |e^{ikp_{l\eta}}|^2 \neq 0, \quad \text{as } \lambda_\xi \to \infty.
$$

From eq.(12), p_{l0} and $p_{l\xi}$ are expressed by

$$
(13)\quad
\begin{aligned}
p_{l0} &= \frac{e^{-2\lambda_\xi p_{l0}}\{\overline{I}(\boldsymbol{u}_{1\xi}) I(\boldsymbol{u}_{3\xi}) + I(\boldsymbol{u}_{2\xi}) \overline{I}(\boldsymbol{u}_{4\xi})\}}{e^{-2\lambda_\xi p_{l0}}\{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2\}} + \frac{e^{-2\lambda_\xi p_{l0}}\{\overline{I}(\boldsymbol{u}_{1\xi}) \mathcal{R}_{A\xi} + I(\boldsymbol{u}_{2\xi}) \overline{\mathcal{R}}_{B\xi}\}}{e^{-2\lambda_\xi p_{l0}}\{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2\}},\\
p_{l\xi} &= \frac{e^{-2\lambda_\xi p_{l0}}\{I(\boldsymbol{u}_{1\xi}) \overline{I}(\boldsymbol{u}_{4\xi}) - \overline{I}(\boldsymbol{u}_{2\xi}) I(\boldsymbol{u}_{3\xi})\}}{e^{-2\lambda_\xi p_{l0}}\{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2\}} + \frac{e^{-2\lambda_\xi p_{l0}}\{I(\boldsymbol{u}_{1\xi}) \overline{\mathcal{R}}_{B\xi} - \overline{I}(\boldsymbol{u}_{2\xi}) \mathcal{R}_{A\xi}\}}{e^{-2\lambda_\xi p_{l0}}\{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2\}}.
\end{aligned}
$$

Let

$$(14)\quad \widetilde{p}_{l0}^{(\xi)} = \frac{\mathrm{Re}\{\overline{I}(\boldsymbol{u}_{1\xi})I(\boldsymbol{u}_{3\xi}) + I(\boldsymbol{u}_{2\xi})\overline{I}(\boldsymbol{u}_{4\xi})\}}{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2}, \quad \widetilde{p}_{l\xi} = \frac{\mathrm{Re}\{I(\boldsymbol{u}_{1\xi})\overline{I}(\boldsymbol{u}_{4\xi}) - \overline{I}(\boldsymbol{u}_{2\xi})I(\boldsymbol{u}_{3\xi})\}}{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2}.$$

From eqs.(13) and (14), the following inequality holds:

$$(15)\quad \begin{aligned} &\sqrt{(p_{l0} - \widetilde{p}_{l0}^{(\xi)})^2 + (p_{l\xi} - \widetilde{p}_{l\xi})^2} \\ &= \frac{e^{-2\lambda_\xi p_{l0}} \sqrt{\left[\mathrm{Re}\left\{\overline{I}(\boldsymbol{u}_{1\xi})\mathcal{R}_{A\xi} + I(\boldsymbol{u}_{2\xi})\overline{\mathcal{R}}_{B\xi}\right\}\right]^2 + \left[\mathrm{Re}\left\{I(\boldsymbol{u}_{1\xi})\overline{\mathcal{R}}_{B\xi} - \overline{I}(\boldsymbol{u}_{2\xi})\mathcal{R}_{A\xi}\right\}\right]^2}}{e^{-2\lambda_\xi p_{l0}}\{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2\}} \\ &\le \frac{e^{-\lambda_\xi p_{l0}} \sqrt{|\mathcal{R}_{A\xi}|^2 + |\mathcal{R}_{B\xi}|^2}}{e^{-\lambda_\xi p_{l0}} \sqrt{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2}} \\ &\le \frac{\sum_{j\ne l} e^{-\lambda_\xi(p_{l0} - p_{j0})} \sqrt{(p_{l0} - p_{j0})^2 + (p_{l\xi} - p_{j\xi})^2} \sqrt{|q_{j0}|^2 + |q_{j\xi}|^2} |e^{ikp_{j\eta}}|}{e^{-\lambda_\xi p_{l0}} \sqrt{|I(\boldsymbol{u}_{1\xi})|^2 + |I(\boldsymbol{u}_{2\xi})|^2}} \equiv \varepsilon_{pl}^{(\xi)}. \end{aligned}$$

Under the conditions (i) and (ii), $\varepsilon_{pl}^{(\xi)}$ exponentially converges to 0 as $\lambda_\xi \to \infty$. For sufficiently large λ_1 and λ_2, we obtain $\widetilde{p}_{l0}^{(1)}$ and $\widetilde{p}_{l1}$ as the approximations of p_{l0} and p_{l1} for $\xi = 1$, and also obtain $\widetilde{p}_{l0}^{(2)}$ and $\widetilde{p}_{l2}$ for $\xi = 2$.

Finally, we obtain the identified result of $\boldsymbol{p}_l$ such that

$$\widetilde{\boldsymbol{p}}_l = \frac{\widetilde{p}_{l0}^{(1)} + \widetilde{p}_{l0}^{(2)}}{2}\boldsymbol{e}_0 + \widetilde{p}_{l1}\boldsymbol{e}_1 + \widetilde{p}_{l2}\boldsymbol{e}_2 \equiv \widetilde{p}_{l0}\boldsymbol{e}_0 + \widetilde{p}_{l1}\boldsymbol{e}_1 + \widetilde{p}_{l2}\boldsymbol{e}_2.$$

For the identified result $\widetilde{\boldsymbol{p}}_l$, the error is bounded by

$$(16)\quad \begin{aligned} |\boldsymbol{p}_l - \widetilde{\boldsymbol{p}}_l| &= \sqrt{\left\{p_{l0} - \frac{1}{2}(\widetilde{p}_{l0}^{(1)} + \widetilde{p}_{l0}^{(2)})\right\}^2 + (p_{l1} - \widetilde{p}_{l1})^2 + (p_{l2} - \widetilde{p}_{l2})^2} \\ &= \sqrt{\left\{(p_{l0} - \widetilde{p}_{l0}^{(1)}) + \frac{1}{2}(\widetilde{p}_{l0}^{(1)} - \widetilde{p}_{l0}^{(2)})\right\}^2 + (p_{l1} - \widetilde{p}_{l1})^2 + (p_{l2} - \widetilde{p}_{l2})^2} \\ &\le \sqrt{(p_{l0} - \widetilde{p}_{l0}^{(1)})^2 + (p_{l1} - \widetilde{p}_{l1})^2 + (p_{l2} - \widetilde{p}_{l2})^2} + \frac{1}{2}|\widetilde{p}_{l0}^{(1)} - \widetilde{p}_{l0}^{(2)}| \\ &\le \sqrt{\{\varepsilon_{pl}^{(1)}\}^2 + \{\varepsilon_{pl}^{(2)}\}^2} + \frac{1}{2}|\widetilde{p}_{l0}^{(1)} - \widetilde{p}_{l0}^{(2)}| \equiv \varepsilon_{pl}. \end{aligned}$$

Obviously, ε_{pl} exponentially converges to 0 as $\lambda_1,\ \lambda_2 \to \infty$ since

$$|\widetilde{p}_{l0}^{(1)} - \widetilde{p}_{l0}^{(2)}| \le |p_{l0} - \widetilde{p}_{l0}^{(1)}| + |p_{l0} - \widetilde{p}_{l0}^{(2)}| \le \varepsilon_{pl}^{(1)} + \varepsilon_{pl}^{(2)} \to 0, \quad \text{as } \lambda_1,\ \lambda_2 \to \infty.$$

3.2 Identification of moment. From eq.(11), the following equations hold:

$$\begin{aligned} &I(\boldsymbol{u}_{1\xi})\cos\lambda_\xi p_{l\xi} + I(\boldsymbol{u}_{2\xi})\sin\lambda_\xi p_{l\xi} \\ &\quad = q_{l0}e^{\lambda_\xi p_{l0}}e^{ikp_{l\eta}} + \sum_{j\ne l} e^{\lambda_\xi p_{j0}}\{q_{j0}\cos\lambda_\xi(p_{l\xi} - p_{j\xi}) + q_{j\xi}\sin\lambda_\xi(p_{l\xi} - p_{j\xi})\}e^{ikp_{j\eta}}, \\ &I(\boldsymbol{u}_{2\xi})\cos\lambda_\xi p_{l\xi} - I(\boldsymbol{u}_{1\xi})\sin\lambda_\xi p_{l\xi} \\ &\quad = q_{l\xi}e^{\lambda_\xi p_{l0}}e^{ikp_{l\eta}} + \sum_{j\ne l} e^{\lambda_\xi p_{j0}}\{q_{j\xi}\cos\lambda_\xi(p_{l\xi} - p_{j\xi}) - q_{j0}\sin\lambda_\xi(p_{l\xi} - p_{j\xi})\}e^{ikp_{j\eta}}. \end{aligned}$$

Solving for q_{l0} and $q_{l\xi}$, we have

$$
\begin{aligned}
q_{l0} &= e^{-\lambda_\xi p_{l0}}\{I(\boldsymbol{u}_{1\xi})\cos\lambda_\xi p_{l\xi} + I(\boldsymbol{u}_{2\xi})\sin\lambda_\xi p_{l\xi}\}e^{-ikp_{l\eta}} \\
&\quad - \sum_{j\neq l} e^{-\lambda_\xi(p_{l0}-p_{j0})}\{q_{j0}\cos\lambda_\xi(p_{l\xi}-p_{j\xi}) + q_{j\xi}\sin\lambda_\xi(p_{l\xi}-p_{j\xi})\}e^{-ik(p_{l\eta}-p_{j\eta})},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
q_{l\xi} &= e^{-\lambda_\xi p_{l0}}\{I(\boldsymbol{u}_{2\xi})\cos\lambda_\xi p_{l\xi} - I(\boldsymbol{u}_{1\xi})\sin\lambda_\xi p_{l\xi}\}e^{-ikp_{l\eta}} \\
&\quad - \sum_{j\neq l} e^{-\lambda_\xi(p_{l0}-p_{j0})}\{q_{j\xi}\cos\lambda_\xi(p_{l\xi}-p_{j\xi}) - q_{j0}\sin\lambda_\xi(p_{l\xi}-p_{j\xi})\}e^{-ik(p_{l\eta}-p_{j\eta})}.
\end{aligned}
\tag{18}
$$

Let

$$
\begin{aligned}
\widehat{q}_{l0}^{(\xi)} &= e^{-\lambda_\xi p_{l0}}\{I(\boldsymbol{u}_{1\xi})\cos\lambda_\xi p_{l\xi} + I(\boldsymbol{u}_{2\xi})\sin\lambda_\xi p_{l\xi}\}e^{-ikp_{l\eta}}, \\
\widehat{q}_{l\xi} &= e^{-\lambda_\xi p_{l0}}\{I(\boldsymbol{u}_{2\xi})\cos\lambda_\xi p_{l\xi} - I(\boldsymbol{u}_{1\xi})\sin\lambda_\xi p_{l\xi}\}e^{-ikp_{l\eta}}.
\end{aligned}
\tag{19}
$$

From eqs.(17), (18), and (19), $\sqrt{|q_{l0}-\widehat{q}_{l0}^{(\xi)}|^2 + |q_{l\xi}-\widehat{q}_{l\xi}|^2}$ is bounded by

$$
\sqrt{|q_{l0}-\widehat{q}_{l0}^{(\xi)}|^2 + |q_{l\xi}-\widehat{q}_{l\xi}|^2} \leq \sum_{j\neq l} e^{-\lambda_\xi(p_{l0}-p_{j0})}\sqrt{|q_{j0}|^2+|q_{j\xi}|^2}|e^{-ik(p_{l\eta}-p_{j\eta})}|.
\tag{20}
$$

Under the conditions (i) and (ii), the right-hand side of eq.(20) converges to 0 as $\lambda_\xi \to \infty$. However, the right-hand sides of eq.(19) contain unknown parameters p_{l0}, p_{l1}, and p_{l2}. Using the identified results instead of these unknown parameters, the approximations of q_{l0} and $q_{l\xi}$ are obtained by

$$
\begin{aligned}
\widetilde{q}_{l0}^{(\xi)} &= e^{-\lambda_\xi \widetilde{p}_{l0}}\{I(\boldsymbol{u}_{1\xi})\cos\lambda_\xi \widetilde{p}_{l\xi} + I(\boldsymbol{u}_{2\xi})\sin\lambda_\xi \widetilde{p}_{l\xi}\}e^{-ik\widetilde{p}_{l\eta}}, \\
\widetilde{q}_{l\xi} &= e^{-\lambda_\xi \widetilde{p}_{l0}}\{I(\boldsymbol{u}_{2\xi})\cos\lambda_\xi \widetilde{p}_{l\xi} - I(\boldsymbol{u}_{1\xi})\sin\lambda_\xi \widetilde{p}_{l\xi}\}e^{-ik\widetilde{p}_{l\eta}}.
\end{aligned}
\tag{21}
$$

Now, we consider the estimation of $\sqrt{|q_{l0}-\widetilde{q}_{l0}^{(\xi)}|^2 + |q_{l\xi}-\widetilde{q}_{l\xi}|^2}$. From eqs.(17), (18), and (21), the following inequality holds:

$$
\begin{aligned}
&\sqrt{|q_{l0}-\widetilde{q}_{l0}^{(\xi)}|^2 + |q_{l\xi}-\widetilde{q}_{l\xi}|^2} \\
&\quad \leq \sqrt{|q_{l0}-\widehat{q}_{l0}^{(\xi)}|^2 + |q_{l\xi}-\widehat{q}_{l\xi}|^2} + \sqrt{|\widehat{q}_{l0}^{(\xi)}-\widetilde{q}_{l0}^{(\xi)}|^2 + |\widehat{q}_{l\xi}-\widetilde{q}_{l\xi}|^2}.
\end{aligned}
\tag{22}
$$

Since ε_{pl} exponentially converges to 0 as λ_1, $\lambda_2 \to \infty$, we approximate $\widehat{q}_{l0}^{(\xi)} - \widetilde{q}_{l0}^{(\xi)}$ and $\widehat{q}_{l\xi} - \widetilde{q}_{l\xi}$ using Taylor's theorem:

$$
\begin{aligned}
\widehat{q}_{l0}^{(\xi)} - \widetilde{q}_{l0}^{(\xi)} &\simeq -\lambda_\xi \widehat{q}_{l0}^{(\xi)}(p_{l0}-\widetilde{p}_{l0}) + \lambda_\xi \widehat{q}_{l\xi}(p_{l\xi}-\widetilde{p}_{l\xi}) - ik\widehat{q}_{l0}^{(\xi)}(p_{l\eta}-\widetilde{p}_{l\eta}), \\
\widehat{q}_{l\xi} - \widetilde{q}_{l\xi} &\simeq -\lambda_\xi \widehat{q}_{l\xi}(p_{l0}-\widetilde{p}_{l0}) - \lambda_\xi \widehat{q}_{l0}^{(\xi)}(p_{l\xi}-\widetilde{p}_{l\xi}) - ik\widehat{q}_{l\xi}(p_{l\eta}-\widetilde{p}_{l\eta}).
\end{aligned}
$$

Then, the second term of the right-hand side of eq.(22) becomes

$$
\begin{aligned}
&\sqrt{|\widehat{q}_{l0}^{(\xi)}-\widetilde{q}_{l0}^{(\xi)}|^2 + |\widehat{q}_{l\xi}-\widetilde{q}_{l\xi}|^2} \\
&\quad \lesssim \sqrt{\lambda_\xi^2+|k|^2}\sqrt{|\widehat{q}_{l0}^{(\xi)}|^2+|\widehat{q}_{l\xi}|^2}\sqrt{(p_{l0}-\widetilde{p}_{l0}^{(\xi)})^2+(p_{l\xi}-\widetilde{p}_{l\xi})^2+(p_{l\eta}-\widetilde{p}_{l\eta})^2} \\
&\quad \leq \sqrt{\lambda_\xi^2+|k|^2}\sqrt{|\widehat{q}_{l0}^{(\xi)}|^2+|\widehat{q}_{l\xi}|^2}\,\varepsilon_{pl}.
\end{aligned}
\tag{23}
$$

From eqs.(20), (22), and (23), we have the following estimate:

$$\sqrt{|q_{l0}-\widetilde{q}_{l0}^{(\xi)}|^2+|q_{l\xi}-\widetilde{q}_{l\xi}|^2} \lesssim \sum_{j\neq l} e^{-\lambda_\xi(p_{l0}-p_{j0})}\sqrt{|q_{j0}|^2+|q_{j\xi}|^2}|e^{-ik(p_{l\eta}-p_{j\eta})}| + \sqrt{\lambda_\xi^2+|k|^2}\sqrt{|\widehat{q}_{l0}^{(\xi)}|^2+|\widehat{q}_{l\xi}|^2}\varepsilon_{pl} \equiv \varepsilon_{ql}^{(\xi)}. \tag{24}$$

Finally, we obtain the identified result of $\boldsymbol{q}_l$ such that

$$\widetilde{\boldsymbol{q}}_l = \frac{\widetilde{q}_{l0}^{(1)}+\widetilde{q}_{l0}^{(2)}}{2}\boldsymbol{e}_0 + \widetilde{q}_{l1}\boldsymbol{e}_1 + \widetilde{q}_{l2}\boldsymbol{e}_2 \equiv \widetilde{q}_{l0}\boldsymbol{e}_0 + \widetilde{q}_{l1}\boldsymbol{e}_1 + \widetilde{q}_{l2}\boldsymbol{e}_2.$$

For the identified result $\widetilde{\boldsymbol{q}}_l$, the error is estimated by

$$|\boldsymbol{q}_l - \widetilde{\boldsymbol{q}}_l| \lesssim \sqrt{\{\varepsilon_{ql}^{(1)}\}^2+\{\varepsilon_{ql}^{(2)}\}^2} + \frac{1}{2}|\widetilde{q}_{l0}^{(1)} - \widetilde{q}_{l0}^{(2)}| \equiv \varepsilon_{ql}. \tag{25}$$

In the same way as shown in Section 3.1, ε_{ql} converges to 0 as λ_1, $\lambda_2 \to \infty$.

3.3 Deflation. As shown in Sections 3.1 and 3.2, we can identify dipoles satisfying condition (i). If there exist dipoles not to satisfy condition (i) for any $\boldsymbol{e}_0$, it is necessary to remove the information about already identified dipoles from the data of $\boldsymbol{U}(\boldsymbol{x}) \equiv \boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x}) \equiv \boldsymbol{H}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x})$ on Γ. In this section, we show the *deflation* process [6] for our problem.

From eqs.(1)-(4), $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$ are expressed by [1]

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}) &= iw\mu \sum_{j=1}^{N}\left[\boldsymbol{q}_j g(\boldsymbol{x};\boldsymbol{p}_j) + \frac{1}{k^2}\mathrm{grad}\,\{\boldsymbol{q}_j\cdot \mathrm{grad}\, g(\boldsymbol{x};\boldsymbol{p}_j)\}\right], \\ \boldsymbol{H}(\boldsymbol{x}) &= -\sum_{j=1}^{N}\boldsymbol{q}_j \times \mathrm{grad}\, g(\boldsymbol{x};\boldsymbol{p}_j), \end{aligned} \tag{26}$$

where

$$g(\boldsymbol{x};\boldsymbol{x}') = \frac{e^{ik|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}.$$

The deflation is to remove the information about already identified dipoles from $\boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x})$ such that:

$$\begin{aligned} \boldsymbol{U}_d(\boldsymbol{x}) &\equiv \boldsymbol{U}(\boldsymbol{x}) - \sum_l \widetilde{\boldsymbol{E}}_l(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}), \\ \boldsymbol{V}_d(\boldsymbol{x}) &\equiv \boldsymbol{V}(\boldsymbol{x}) - \sum_l \widetilde{\boldsymbol{H}}_l(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}), \end{aligned} \qquad \boldsymbol{x}\in\Gamma, \tag{27}$$

where $\sum_l$ is a sum of already identified dipoles, and $\widetilde{\boldsymbol{E}}_l(\boldsymbol{x})$ and $\widetilde{\boldsymbol{H}}_l(\boldsymbol{x})$ are

$$\begin{aligned} \widetilde{\boldsymbol{E}}_l(\boldsymbol{x}) &= iw\mu\left[\widetilde{\boldsymbol{q}}_l g(\boldsymbol{x};\widetilde{\boldsymbol{p}}_l) + \frac{1}{k^2}\mathrm{grad}\,\{\widetilde{\boldsymbol{q}}_l\cdot\mathrm{grad}\, g(\boldsymbol{x};\widetilde{\boldsymbol{p}}_l)\}\right], \\ \widetilde{\boldsymbol{H}}_l(\boldsymbol{x}) &= -\widetilde{\boldsymbol{q}}_l \times \mathrm{grad}\, g(\boldsymbol{x};\widetilde{\boldsymbol{p}}_l). \end{aligned}$$

For the identification of remaining dipoles, we use $\boldsymbol{U}_d(\boldsymbol{x})$ and $\boldsymbol{V}_d(\boldsymbol{x})$ instead of $\boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x})$.

4 Algorithm. In this section, we describe our algorithm for the identification of unknown parameters $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$ $(j = 1, 2, \dots, N)$ from observations of $\boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x})$ on Γ, where N is also unknown.

As shown in Section 3, it is necessary to choose $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$ under the conditions (i) and (ii). We can choose $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ so that the identified result satisfies the condition (ii). However, it is not easy to choose $\boldsymbol{e}_0$. We show a criterion for the proper choice of $\boldsymbol{e}_0$. If $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$ is properly chosen for the l-th dipole, we expect that the same dipole is also identified using the following vectors for weighting functions:

$$\boldsymbol{e}_0^{(1)} = \boldsymbol{e}_0 \cos \triangle\psi - \boldsymbol{e}_1 \sin \triangle\psi, \quad \boldsymbol{e}_1^{(1)} = \boldsymbol{e}_0 \sin \triangle\psi + \boldsymbol{e}_1 \cos \triangle\psi, \qquad \boldsymbol{e}_2^{(1)} = \boldsymbol{e}_2,$$
$$\boldsymbol{e}_0^{(2)} = \boldsymbol{e}_0 \cos \triangle\psi + \boldsymbol{e}_1 \sin \triangle\psi, \quad \boldsymbol{e}_1^{(2)} = -\boldsymbol{e}_0 \sin \triangle\psi + \boldsymbol{e}_1 \cos \triangle\psi, \quad \boldsymbol{e}_2^{(2)} = \boldsymbol{e}_2,$$

where the constant $\triangle\psi > 0$ is sufficiently small. The vectors $\boldsymbol{e}_0^{(m)}$ and $\boldsymbol{e}_1^{(m)}$ $(m = 1, 2)$ are obtained by a rotation of $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ through angles $\pm\triangle\psi$. Similarly, the following vectors are obtained by a rotation of $\boldsymbol{e}_0$ and $\boldsymbol{e}_2$:

$$\boldsymbol{e}_0^{(3)} = \boldsymbol{e}_0 \cos \triangle\psi - \boldsymbol{e}_2 \sin \triangle\psi, \quad \boldsymbol{e}_1^{(3)} = \boldsymbol{e}_1, \quad \boldsymbol{e}_2^{(3)} = \boldsymbol{e}_0 \sin \triangle\psi + \boldsymbol{e}_2 \cos \triangle\psi,$$
$$\boldsymbol{e}_0^{(4)} = \boldsymbol{e}_0 \cos \triangle\psi + \boldsymbol{e}_2 \sin \triangle\psi, \quad \boldsymbol{e}_1^{(4)} = \boldsymbol{e}_1, \quad \boldsymbol{e}_2^{(4)} = -\boldsymbol{e}_0 \sin \triangle\psi + \boldsymbol{e}_2 \cos \triangle\psi.$$

Let $\widetilde{\boldsymbol{p}}_l^{(m)}$ be identified results of $\boldsymbol{p}_l$ using $\boldsymbol{e}_0^{(m)}$, $\boldsymbol{e}_1^{(m)}$, and $\boldsymbol{e}_2^{(m)}$ $(m = 1, 2, 3, 4)$, and $\widetilde{\boldsymbol{p}}_l^{(0)}$ be a candidate of identified results of $\boldsymbol{p}_l$ using $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$. Here, we compute

$$h_l = \max_{m \in \{1,2,3,4\}} |\widetilde{\boldsymbol{p}}_l^{(0)} - \widetilde{\boldsymbol{p}}_l^{(m)}| / \triangle\psi.$$

If h_l is sufficiently small, we adopt $\widetilde{\boldsymbol{p}}_l^{(0)}$ as the identified results $\widetilde{\boldsymbol{p}}_l$, and compute $\widetilde{\boldsymbol{q}}_l$.

In the following, we show our algorithm.

Algorithm

Step 1 Input observations of $\boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x})$ on Γ. Set λ_1, λ_2, $\triangle\psi$, α, ρ, C_E, C_H, and $\boldsymbol{z}_j$ $(j = 1, \dots, K)$, where $\boldsymbol{z}_j$ is the j-th candidate of $\boldsymbol{e}_0$.

Step 2 Set $\boldsymbol{U}_d(\boldsymbol{x}) = \boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{V}_d(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{x})$, and initialize $j = 1$, $n^* = 0$, and $\widetilde{N} = 0$.

Step 3 Using the data of $\boldsymbol{U}_d(\boldsymbol{x})$ and $\boldsymbol{V}_d(\boldsymbol{x})$, calculate $\widetilde{\boldsymbol{p}}_{\widetilde{N}+1}$, $\widetilde{\boldsymbol{q}}_{\widetilde{N}+1}$, and $h_{\widetilde{N}+1}$ for $\boldsymbol{e}_0 = \boldsymbol{z}_j$. If $h_{\widetilde{N}+1} \geq \alpha$, then go to *Step* 7.

Step 4 If $\min_{1 \leq l \leq n^*} |\widetilde{\boldsymbol{p}}_l - \widetilde{\boldsymbol{p}}_{\widetilde{N}+1}| < \rho$ for $n^* \neq 0$, then go to *Step* 7.

Step 5 If $\widetilde{N} = n^*$ or $\min_{n^* < l \leq \widetilde{N}} |\widetilde{\boldsymbol{p}}_l - \widetilde{\boldsymbol{p}}_{\widetilde{N}+1}| \geq \rho$ for $\widetilde{N} > n^*$, then $\widetilde{N} \leftarrow \widetilde{N} + 1$ and go to *Step* 7.

Step 6 If there exists $m \in \{n^* + 1, \dots, \widetilde{N}\}$ such that $|\widetilde{\boldsymbol{p}}_m - \widetilde{\boldsymbol{p}}_{\widetilde{N}+1}| = \min_{n^* < l \leq \widetilde{N}} |\widetilde{\boldsymbol{p}}_l - \widetilde{\boldsymbol{p}}_{\widetilde{N}+1}|$ and $h_m > h_{\widetilde{N}+1}$, then $\widetilde{\boldsymbol{p}}_m \leftarrow \widetilde{\boldsymbol{p}}_{\widetilde{N}+1}$, $\widetilde{\boldsymbol{q}}_m \leftarrow \widetilde{\boldsymbol{q}}_{\widetilde{N}+1}$, and $h_m \leftarrow h_{\widetilde{N}+1}$.

Step 7 If $j < K$, then $j \leftarrow j + 1$ and go to *Step* 3.

Step 8 If $\widetilde{N} = n^*$, then go to *Step* 10. Else, update $n^* \leftarrow \widetilde{N}$, and compute $\boldsymbol{U}_d(\boldsymbol{x})$ and $\boldsymbol{V}_d(\boldsymbol{x})$ from eq.(27).

Step 9 If $\|\boldsymbol{U}_d\|/\|\boldsymbol{U}\| \geq C_E$ and $\|\boldsymbol{V}_d\|/\|\boldsymbol{V}\| \geq C_H$, then $j = 1$, and go to *Step* 3. Here, the norm $\|\cdot\|$ denotes the L^2-norm on Γ.

Step 10 Compute ε_{pl} and ε_{ql} for each l using identified results $\widetilde{\boldsymbol{p}}_j$ and $\widetilde{\boldsymbol{q}}_j$ $(j = 1, \ldots, \widetilde{N})$ instead of true dipoles, and stop.

5 Numerical Examples. Let $\Omega = \{\boldsymbol{x}; |\boldsymbol{x}| < R\}$. Our method needs to compute $I(\boldsymbol{u})$ from the observation data of $\boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x})$ on the boundary of Ω. Using spherical coordinates, we express the integrand in eq.(8) by $F(r, \theta, \phi)$. Then, the boundary integral $I(\boldsymbol{u})$ becomes

$$I(\boldsymbol{u}) = R^2 \int_0^{\pi} \left[\int_0^{2\pi} F(R, \theta, \phi) d\phi \right] \sin\theta d\theta.$$

We compute $I(\boldsymbol{u})$ by combining the trapezoidal rule and Lobatto quadrature [4] such that

$$I(\boldsymbol{u}) \simeq R^2 \sum_{n=0}^{L} s_n \frac{2\pi}{M_n} \sum_{m=1}^{M_n} F(R, \mathrm{Cos}^{-1}\zeta_n, \frac{2\pi m}{M_n}),$$

where s_n and ζ_n $(n = 0, \ldots, L)$ are weights and abscissas of Lobatto quadrature, and M_n is the number of abscissas for trapezoidal rule on the n-th latitude line. If M_n is independent of n, then observation points are accumulated at the neighbourhood of $\theta = 0, \pi$. So, we consider the following form for M_n:

$$M_n = \begin{cases} 1, & n = 0, L, \\ \lceil (s_n / \max_j s_j)^{\gamma} M \rceil, & \text{otherwise}, \end{cases}$$

where $\lceil \cdot \rceil$ denotes the ceiling function, and γ is a positive constant.

Let $R = 0.1$, $\sigma = 0.2$, $\epsilon = 8.854 \times 10^{-12}$, $\mu = 4\pi \times 10^{-7}$, and $w = 100$. For the arrangement of observation points, we set $L = 26$ and $M = 52$. The power γ is experimentally determined as $\gamma = 0.7$. Then, the number of observation points is 974. The observations of $\boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x})$ are analytically generated using eq.(26). In the algorithm, we set

$$\lambda_1 = \lambda_2 = 120, \quad \triangle\psi = \pi/52, \quad \alpha = 0.02, \quad \rho = 0.02, \quad C_E = C_H = 0.1.$$

The candidates of $\boldsymbol{e}_0$ are given by $(1, \mathrm{Cos}^{-1}\zeta_n, 2\pi m/M_n)$ $(n = 0, \ldots, L,\ m = 1, \ldots, M_n)$ in spherical coordinates. Therefore, the number of the candidates of $\boldsymbol{e}_0$ is also 974.

First, our algorithm is applied to the case of 2 dipoles such as

Case 1

$$\begin{aligned} \boldsymbol{p}_1 &= (0.04,\ \pi/4,\ \pi/3), \quad & \boldsymbol{q}_1 &= (0.1,\ 3\pi/4,\ \pi/3), \\ \boldsymbol{p}_2 &= (0.02,\ 3\pi/4,\ 4\pi/3), \quad & \boldsymbol{q}_2 &= (0.2,\ \pi/2,\ 5\pi/6). \end{aligned}$$

Here, we express $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$ $(j = 1, 2)$ using spherical coordinates. Table 1 shows the identified results for *Case* 1. As shown in Table 1, 2 dipoles are well identified, and $|\boldsymbol{p}_j - \widetilde{\boldsymbol{p}}_j|$ and $|\boldsymbol{q}_j - \widetilde{\boldsymbol{q}}_j|$ are bounded by ε_{pj} and ε_{qj}, respectively.

Next, we generate 25 examples by uniform random number under the following condition:

Case 2

$$\left. \begin{array}{ll} |\boldsymbol{p}_j| \leq 0.07\,, & |\boldsymbol{p}_j - \boldsymbol{p}_{j'}| \geq 0.03 \ \ (j \neq j'), \\ 0.1 \leq |\boldsymbol{q}_j| \leq 0.3\,, & \boldsymbol{q}_j \in \mathbb{C}^3, \end{array} \right\} \quad j,\, j' = 1, \ldots, N.$$

Table 1: Identified results for *Case* 1

j	$\lvert \boldsymbol{p}_j - \widetilde{\boldsymbol{p}}_j \rvert$	ε_{pj}	$\lvert \boldsymbol{q}_j - \widetilde{\boldsymbol{q}}_j \rvert$	ε_{qj}
1	1.64×10^{-4}	1.79×10^{-4}	6.06×10^{-4}	2.96×10^{-3}
2	3.85×10^{-5}	4.57×10^{-5}	2.92×10^{-4}	1.52×10^{-3}

Table 2: The number of identified dipoles for *Case* 2

	$\widetilde{N} = N - 1$	$\widetilde{N} = N$	$\widetilde{N} = N+1$
$N = 3$	0	25	0
$N = 4$	4	20	1

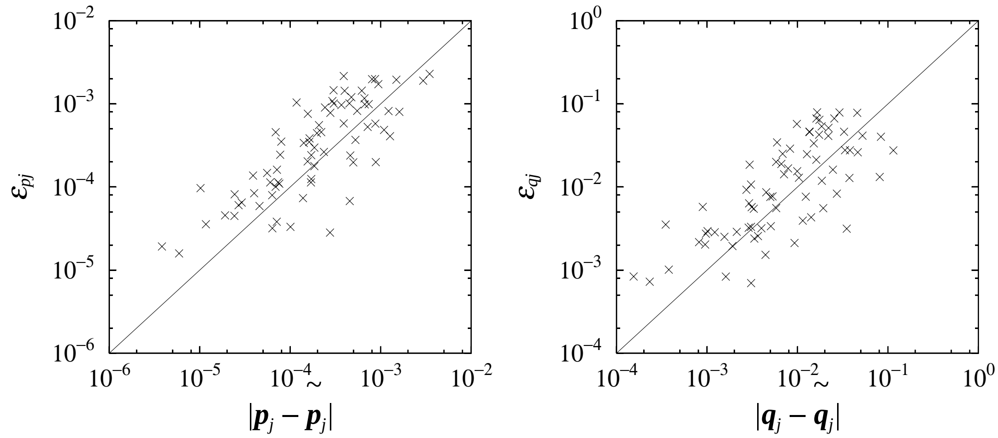

Figure 1: The error estimation of $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$ for *Case* 2 ($N = 3$)

We consider two cases of $N = 3$ and $N = 4$. Table 2 shows the number of identified dipoles. As shown in Table 2, our method identifies all dipoles for 25 examples in the case of $N = 3$. In the case of $N = 4$, the number of identified dipoles is not always equal to the true number of dipoles. For 4 examples, 3 dipoles can be identified, but 1 dipole cannot be identified. For 1 example, 4 dipoles are reasonably obtained, and a ghost dipole is additionally obtained. Figure 1 shows the error estimation of $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$ for $N = 3$. As shown in Figure 1, our method gives practical error estimates for many identified results. However, estimated error bounds ε_{pj} and ε_{qj} are unsuccessful for a few dipoles. These failures may be caused by the error of numerical integrations of $I(\boldsymbol{u})$.

6 Conclusions. This paper proposes a direct identification method of electric current dipoles in a homogeneous space for time-harmonic Maxwell's equations. Our method is based on boundary integrals using vector-valued weighting functions, and the electric and magnetic fields are observed on the smooth boundary of the bounded convex domain which includes unknown dipoles. By the proper choice of weighting functions, locations and moments of dipoles can be identified without any iterative procedures. In the numerical examples, we consider the case where observations are obtained on a spherical surface. This reason is that the human brain is often modeled as a sphere in the field of bioengineering. For various examples, identified results and their error estimates are reasonably obtained without using a priori information about locations, moments, and number of dipoles. We also obtain good results for the number of identified dipoles since location and moment of each dipole are well identified. Therefore, our method is reliable and effective for the identification of electric current dipoles in a homogeneous space for time-harmonic Maxwell's equations.

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