SIMPLE INDUCTIVE LIMIT C^*-ALGEBRAS WITH STABLE RANK FINITE

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Abstract. We construct simple inductive limits of homogeneous C^*-algebras without slow dimension growth and with stable rank finite. We also consider similarly such inductive limits with real rank finite.

Introduction The theory of (topological) stable rank for C^*-algebras was initiated by Rieffel [Rf1]. Successively, some results on the stable rank were obtained by Nistor [Ns] and Rieffel [Rf2] (among other related papers). One of the most interesting problems about the stable rank was whether there exists a (stably finite) simple C^*-algebra with stable rank more than one (Blackadar [Bl2, Questions 4.2.4 to 4.2.6]). For this problem, Villadsen [Vl] is the first to construct simple inductive limits of homogeneous C^*-algebras without slow dimension growth and with stable rank more than one. Our first attempt is to understand his construction and proofs since the proofs are much complicated and technical. With some effort since some time before, we have obtained a simple, comprehensive proof for the problem without using the argument of Villadsen from differential geometry involving the Euler class of vector bundles. Namely, we construct simple inductive limits of homogeneous C^*-algebras without slow dimension growth and bounded dimension and with stable rank finite and real rank finite. Although our construction is standard (in a sense) and similar with that of Villadsen, this paper could be a remedy for [Vl]. See also [RS, Example 3.1.7] for the Goodearl's construction of inductive limits, and [Ln, Definition 2.7]. For the proof, we use a formula of the stable rank (and connected stable rank) for matrix algebras over C^*-algebras ([Rf1], [Rf2]) and that of the real rank (Beggs-Evans [BE]).

For the convenience to the readers, we recall that a unital C^*-algebra $\mathfrak{A}$ has stable rank $n = \mathrm{sr}(\mathfrak{A})$ if n is the smallest such that $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, where $L_n(\mathfrak{A})$ is the set of all elements $(a_j)_{j=1}^n$ of $\mathfrak{A}^n$ such that $\sum_{j=1}^n a_j^* a_j$ are invertible in $\mathfrak{A}$. Also, $\mathfrak{A}$ has connected stable rank $n = \mathrm{csr}(\mathfrak{A})$ if n is the smallest such that $L_m(\mathfrak{A})$ is connected for any $m \geq n$ (see [Rf1]). Furthermore, $\mathfrak{A}$ has real rank $n = \mathrm{RR}(\mathfrak{A})$ if n is the smallest such that any self-adjoint element of $\mathfrak{A}^{n+1}$ is approximated by self-adjoint elements $(b_j)_{j=0}^n$ of $\mathfrak{A}^{n+1}$ such that $\sum_{j=0}^n b_j^2$ are invertible in $\mathfrak{A}$ (see Brown-Pedersen [BP]). See also [Bl1] and [RLL].

1 The main results Let $C(\mathbb{T}^{2n})$ be the C^*-algebra of continuous functions on the $2n$-torus $\mathbb{T}^{2n}$. Let $M_{2n}(C(\mathbb{T}^{2n^2}))$ be the $2n \times 2n$ matrix algebra over $C(\mathbb{T}^{(2n)^2})$. Let $\{x_j^1\}$ be a dense sequence of $\mathbb{T}^{2n}$. Then we define a homomorphism φ_1 from $C(\mathbb{T}^{2n})$ to $M_{2n}(C(\mathbb{T}^{2n^2}))$ by

$$\varphi_1(f) = (f \circ \pi_1)p_1 \oplus f(x_1^1)p_2 \oplus \cdots \oplus f(x_{2n-1}^1)p_{2n}$$

for $f \in C(\mathbb{T}^{2n})$, where π_1 is the canonical projection from $\mathbb{T}^{(2n)^2}$ to $\mathbb{T}^{2n}$, and p_j $(1 \leq j \leq 2n)$ are mutually orthogonal rank 1 projections in $M_{2n}(\mathbb{C})$, and $\oplus$ means the diagonal sum. Note

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that by [Rf1, Proposition 1.7 and Theorem 6.1],

$$\begin{aligned}\mathrm{sr}(C(\mathbb{T}^{2n})) &= [\dim \mathbb{T}^{2n}/2] + 1 = n+1, \\ \mathrm{sr}(M_{2n}(C(\mathbb{T}^{(2n)^2}))) &= \{(\mathrm{sr}(C(\mathbb{T}^{(2n)^2})) - 1)/2n\} + 1 \\ &= \{[(2n)^2/2]/2n\} + 1 = n+1,\end{aligned}$$

where $[x]$ means the least integer $\leq x$, and $\{y\}$ means the least integer $\geq y$ (we use this notation in what follows). We continue the process above inductively as follows. Define a homomorphism φ_2 from $M_{2n}(C(\mathbb{T}^{(2n)^2}))$ to $M_{(2n)^2}(C(\mathbb{T}^{(2n)^3}))$ by

$$\varphi_2(g) = (g \circ \pi_2) \oplus g(x^2_{2n}) \oplus \cdots \oplus g(x^2_{(2n)^2-1})$$

for $g \in M_{2n}(C(\mathbb{T}^{(2n)^2}))$, and $\pi_1(x^2_j) = x^1_j$ for $1 \leq j \leq 2n-1$, where π_2 is the canonical projection from $\mathbb{T}^{(2n)^3}$ to $\mathbb{T}^{(2n)^2}$, and $\{x^2_j\}$ is a dense sequence of $\mathbb{T}^{(2n)^2}$ chosen as desired. Note that by [Rf1, Theorem 6.1],

$$\mathrm{sr}(M_{(2n)^2}(C(\mathbb{T}^{(2n)^3}))) = \{[(2n)^3/2]/(2n)^2\} + 1 = n+1.$$

In the general step, a homomorphism φ_k from $M_{(2n)^{k-1}}(C(\mathbb{T}^{(2n)^k}))$ to $M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))$ is defined by

$$\varphi_k(h) = (h \circ \pi_k) \oplus h(x^k_{(2n)^{k-1}}) \oplus \cdots \oplus h(x^k_{(2n)^k-1})$$

for $h \in M_{(2n)^{k-1}}(C(\mathbb{T}^{(2n)^k}))$, and $\pi_{k-1}(x^k_j) = x^{k-1}_j$ for $1 \leq j \leq (2n)^{k-1} - 1$, where π_k is the canonical projection from $\mathbb{T}^{(2n)^{k+1}}$ to $\mathbb{T}^{(2n)^k}$, and $\{x^k_j\}$ is a dense sequence of $\mathbb{T}^{(2n)^k}$ chosen as desired. Note that by [Rf1, Theorem 6.1],

$$\mathrm{sr}(M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))) = \{[(2n)^{k+1}/2]/(2n)^k\} + 1 = n+1.$$

For $n \in \boldsymbol{N}$ we define the C^*-algebra $\mathfrak{A}_n = \varinjlim(M_{(2n)^{k-1}}(C(\mathbb{T}^{(2n)^k})), \varphi_k)$ to be the inductive limit of the homogeneous C^*-algebras $M_{(2n)^{k-1}}(C(\mathbb{T}^{(2n)^k}))$ for $k \geq 1$ with φ_k the connecting homomorphisms.

Theorem 1.1 *Let $\mathfrak{A}_n$ be the inductive limit C^*-algebra defined above. Then $\mathfrak{A}_n$ is a simple inductive limit without slow dimension growth and bounded dimension, and*

$$\mathrm{sr}(\mathfrak{A}_n) = n+1, \quad \textit{and} \quad \mathrm{csr}(\mathfrak{A}_n) \leq n+1.$$

Proof. We use a criterion for inductive limits of homogeneous C^*-algebras $\mathfrak{H}_j$ on spaces X_j with φ_j connecting homomorphisms to be simple, that is, for any nonzero $f \in \mathfrak{H}_j$ in $\varinjlim \mathfrak{H}_j$, there exists $k \geq j$ such that $\varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_j(f)(x) \neq 0$ for any $x \in X_{k+1}$. See [RS, Proposition 3.1.2] for this. Note the following identification:

$$M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}})) \cong C(\mathbb{T}^{(2n)^{k+1}}, M_{(2n)^k}(\mathbb{C})),$$

where the right hand side means the C^*-algebra of continuous $M_{(2n)^k}(\mathbb{C})$-valued functions on $\mathbb{T}^{(2n)^{k+1}}$ (cf. [Mp, Theorem 6.4.17]). By the construction of $\mathfrak{A}_n$ using the density of the points $\{x^k_j\}$ in $\mathbb{T}^{(2n)^k}$, it is easy to see that $\mathfrak{A}_n$ is simple. For the building blocks $M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))$ of $\mathfrak{A}_n$, the ratios of dimensions of the base spaces and matrix sizes are $(2n)^{k+1}/(2n)^k = 2n$ nonzero constant. Thus, $\mathfrak{A}_n$ is an inductive limit without slow dimension growth and bounded dimension (see [RS, Definition 3.1.1] for their definitions). Since

$\mathrm{sr}(M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))) = n+1$ for $k \geq 1$, we have $\mathrm{sr}(\mathfrak{A}_n) \leq n+1$ by [Rf1, Theorem 5.1]. It follows from $\mathrm{sr}(M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))) = n+1$ that there exists $(f_j) \in M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))^n$ such that the distance between (f_j) and $L_n(M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))$ is nonzero $K > 0$, that is,

$$d((f_j), L_n(M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}})))) = K.$$

Then we have the norm $\|f_j - g_j\| \geq K$ for any $(g_j) \in M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}}))^n$. Note that it is easy to see that the homomorphisms φ_k are injective. This implies that $\|\varphi_{k+1}(f_j) - \varphi_{k+1}(g_j)\| \geq K$. For any $(h_j) \in M_{(2n)^{k+1}}(C(\mathbb{T}^{(2n)^{k+2}}))^n$, we have

$$\|\varphi_{k+1}(f_j) - h_j\| \geq \|\varphi_{k+1}(f_j) - \varphi_{k+1}(h_j|_{\mathbb{T}^{(2n)^{k+1}}})\| \geq K,$$

where $h_j|_{\mathbb{T}^{(2n)^{k+1}}}(z) = h_j(z, 1)$ for $z \in \mathbb{T}^{(2n)^{k+1}}$ and $1 = (1)_{l=1}^n \in \mathbb{T}^{2n}$. To check this, we can use the identification of $M_{(2n)^{k+1}}(C(\mathbb{T}^{(2n)^{k+2}}))$ as above. Therefore, we obtain $d((\varphi_{k+1}(f_j)), L_n(M_{(2n)^{k+1}}(C(\mathbb{T}^{(2n)^{k+2}})))) \geq K$. This implies that $\mathrm{sr}(\mathfrak{A}_n) \geq n+1$. In fact, if $\mathrm{sr}(\mathfrak{A}_n) \leq n$, then the element (f_j) is approximated by $(l_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^n l_j^* l_j$ is invertible in $\mathfrak{A}_n$, and (l_j) can be replaced with elements of $L_n(M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}})))$, which implies that there exists no such constant $K > 0$. This is the contradiction.

For the estimate of the connected stable rank, we use [Rf2, Theorem 4.7] and [Ns, Corollary 2.5] to obtain

$$\begin{aligned}\mathrm{csr}(M_{(2n)^{k-1}}(C(\mathbb{T}^{(2n)^k}))) &\leq \{(\mathrm{csr}(C(\mathbb{T}^{(2n)^k})) - 1)/(2n)^{k-1}\} + 1 \\ &\leq \{[((2n)^k + 1)/2]/(2n)^{k-1}\} + 1 = n+1.\end{aligned}$$

Therefore, we have $\mathrm{csr}(\mathfrak{A}_n) \leq n+1$ (cf. [Ns, The formula (1.6)]). □

Corollary 1.1 *There exists an inductive limit of discrete groups such that a simple inductive limit of their group C^*-algebras has stable rank $n+1$.*

Proof. Note that $C(\mathbb{T}^{2n}) \cong C^*(\mathbb{Z}^{2n})$ the group C^*-algebra of $\mathbb{Z}^{2n}$, and

$$\begin{aligned}M_{(2n)^k}(C(\mathbb{T}^{(2n)^{k+1}})) &\cong C(\mathbb{T}^{(2n)^{k+1}}) \otimes M_{(2n)^k}(\mathbb{C}) \\ &\cong C^*(\mathbb{Z}^{(2n)^{k+1}}) \otimes C^*(\mathbb{Z}_{(2n)^k} \rtimes_\tau \mathbb{Z}_{(2n)^k}),\end{aligned}$$

where $\mathbb{Z}_{(2n)^k} \rtimes_\tau \mathbb{Z}_{(2n)^k}$ is the semi-direct product by cyclic groups $\mathbb{Z}_{(2n)^k}$ with τ the action by the translation, and

$$C^*(\mathbb{Z}^{(2n)^{k+1}}) \otimes C^*(\mathbb{Z}_{(2n)^k} \rtimes_\tau \mathbb{Z}_{(2n)^k}) \cong C^*(\mathbb{Z}^{(2n)^{k+1}} \times (\mathbb{Z}_{(2n)^k} \rtimes_\tau \mathbb{Z}_{(2n)^k})).$$

Thus, the required inductive limit of discrete groups is given by the direct product groups $\mathbb{Z}^{(2n)^{k+1}} \times (\mathbb{Z}_{(2n)^k} \rtimes_\tau \mathbb{Z}_{(2n)^k})$ for $k \geq 0$, where $\mathbb{Z}_0 \rtimes_\tau \mathbb{Z}_0 = 0$. □

Remark. It is evident that the connecting homomorphisms (φ_k) of the group C^*-algebras $C^*(\mathbb{Z}^{(2n)^{k+1}} \times (\mathbb{Z}_{(2n)^k} \rtimes_\tau \mathbb{Z}_{(2n)^k}))$ are not coming from the (natural) homomorphisms of the groups $\mathbb{Z}^{(2n)^{k+1}} \times (\mathbb{Z}_{(2n)^k} \rtimes_\tau \mathbb{Z}_{(2n)^k})$. Therefore, the interpretation above in the corollary seems to be not useful. But it suggests that inductive limits of group C^*-algebras might be of interest.

Similarly as Theorem 1.1, we obtain

Theorem 1.2 *There exists a simple inductive limit C^*-algebra $\mathfrak{B}_n$ without slow dimension growth and bounded dimension such that*

$$\mathrm{RR}(\mathfrak{B}_n) = n + 1.$$

Proof. The construction of $\mathfrak{B}_n$ is almost the same as that of $\mathfrak{A}_n$ of Theorem 1.1. We take the inductive limit of $\mathfrak{B}_n$ as follows:

$$C(\mathbb{T}^n) \xrightarrow{\psi_1} M_n(C(\mathbb{T}^{2n^2})) \xrightarrow{\psi_2} M_{n^2}(C(\mathbb{T}^{2n^3})) \xrightarrow{\psi_3} \cdots$$

where the connecting homomorphisms ψ_k are defined by the same as φ_k of $\mathfrak{A}_n$. Note that $\mathrm{RR}(C(\mathbb{T}^n)) = n$ by [BP, Proposition 1.1], and

$$\mathrm{RR}(M_{n^k}(C(\mathbb{T}^{2n^{k+1}}))) = \{2n^{k+1}/(2n^k - 1)\} = \{n + (n/(2n^k - 1))\} = n + 1$$

by [BE, Corollary 3.2]. Hence, $\mathrm{RR}(\mathfrak{B}_n) \le n + 1$. The rest of the proof is similar with that of Theorem 1.1 just by considering self-adjoint elements $(b_j)_{j=0}^n$ of $(\mathfrak{B}_n)^{n+1}$ with $\sum_{j=0}^n b_j^2$ invertible in $\mathfrak{B}_n$. □

Remark. The equality in the statement is better than Villadsen's estimate of the real rank in [Vl].

Similarly as Corollary 1.1, we obtain

Corollary 1.2 *There exists an inductive limit of discrete groups such that a simple inductive limit of their group C^*-algebras has real rank $n + 1$.*

Remark. We in fact have

$$\mathrm{RR}(\mathfrak{A}_n) = n + 1, \quad \mathrm{sr}(\mathfrak{B}_n) = n + 1, \quad \text{and} \quad \mathrm{csr}(\mathfrak{B}_n) \le n + 1$$

by using the methods of Theorems 1.2 and 1.1 respectively.

Furthermore,

Theorem 1.3 *There exists a simple inductive limit C^*-algebra $\mathfrak{C}_\infty$ without slow dimension growth and bounded dimension such that*

$$\mathrm{sr}(\mathfrak{C}_\infty) = \infty, \quad \text{and} \quad \mathrm{RR}(\mathfrak{C}_\infty) = \infty.$$

Proof. We take an inductive limit for $\mathfrak{C}_\infty$ as follows:

$$\cdots \xrightarrow{\chi_{k-1}} M_{n^k}(C(\mathbb{T}^{2kn^{k+1}})) \xrightarrow{\chi_k} M_{n^{k+1}}(C(\mathbb{T}^{2(k+1)n^{k+2}})) \xrightarrow{\chi_{k+1}} \cdots$$

where the connecting homomorphisms χ_k $(k \ge 1)$ are defined similarly as φ_k and ψ_k in Theorems 1.1 and 1.2 respectively. Note that

$$\mathrm{sr}(M_{n^k}(C(\mathbb{T}^{2kn^{k+1}}))) = \{[2kn^{k+1}/2]/n^k\} + 1 = kn + 1,$$
$$\mathrm{RR}(M_{n^k}(C(\mathbb{T}^{2kn^{k+1}}))) = \{2kn^{k+1}/(2n^k - 1)\} = \{kn + (kn/(2n^k - 1))\} = kn.$$

Thus, the claim follows from showing $\mathrm{sr}(\mathfrak{C}_\infty) \ge kn + 1$ and $\mathrm{RR}(\mathfrak{C}_\infty) \ge kn$ as given in the proof of Theorem 1.1. □

References

[BE] E.J. Beggs and D.E. Evans, *The real rank of algebras of matrix valued functions*, Internat. J. Math. **2** (1991), 131–138.

[Bl1] B. Blackadar, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

[Bl2] ——— , *Comparison theory for simple C^*-algebras*, Operator Algebras and Applications, L.M.S. Lecture Notes Series, **135** (1989), 21–54.

[BP] L.G. Brown and G.K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.

[Ln] H. Lin, *Tracially quasidiagonal extensions.* Canad. Math. Bull. **46** (2003), 388–399.

[Mp] G.J. Murphy, *C^*-algebras and Operator theory*, Academic Press, (1990).

[Ns] V. Nistor, *Stable range for tensor products of extensions of $\mathfrak{K}$ by $C(X)$*, J. Operator Theory, **16** (1986), 387–396.

[Rf1] M.A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), 301–333.

[Rf2] ——— , *The homotopy groups of the unitary groups of non-commutative tori*, J. Operator Theory **17** (1987), 237–254.

[RLL] M. Rørdam, F. Larsen and N.J. Laustsen, *An Introduction to K-Theory for C^*-Algebras*, LMSST **49**, Cambridge (2000).

[RS] M. Rørdam and E. Størmer, *Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras* EMS **126** Operator Algebras and Non-Commutative Geometry VII, Springer, London-New York-San Francisco (2002).

[Vl] J. Villadsen, *On the stable rank of simple C^*-algebras*, J. Amer. Math. Soc. **12** (1999), 1091–1102.

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