NEW DERIVATION OF CONSERVED QUANTITIES FOR HIGHER ORDER DIFFERENTIAL SYSTEM

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Abstract. Through the application a suitable version of Noether's theorem to the composite variational principle, a new method was investigated for the derivation of conserved quantities for second order differential system. In this article, the variables provided for higher order differential system are arranged to convert the system into the second order one. And then the method is applied effectively to construct conserved quantities of the higher order differential system.

Introduction. Noether's theorem (Noether [7]) has been extensively initiated for the derivation of conserved quantities based on the symmetries in the Lagrangian or the Hamiltonian structures. However, without using the structures (which may fail to exist), Caviglia ([2], [3]) determined the new operative procedure for the quantities via the application of a suitable version of Noether's theorem to the composite variational principle. The procedure was analyzed by Mimura and Nôno [6] with various viewpoints for the derivation of the quantities of a given second order differential system. Following Sarlet and Cantrijn [8], Mimura, Ikeda and Fujiwara [5] introduced some geometric notions associating with the equation field of the differential system to construct the quantities. The local version of the result in [6] was reformulated in [5] with the geometric notions.

In this paper, we give a further derivation of conserved quantities of higher order differential system by virtue of the method in [6], while some results in [6] were translated by Crâşmăreanu [4] for higher order one with the extended notion of adjoint for a linear operator. Arranging the original variables in higher order differential system, the system is converted into a second order one to apply the method (Remark 2 of Theorem 1 in [6]). The result in the consideration is carried into the case of higher order differential system, and one arrives at the theorem 1. Some postulations are imposed on unknown functions which give rise to the conserved quantities. Then the theorem 1 deduces to the theorem 2. In the context of the deduction, it can be observed that the theorem 2 is equivalent to the result of Crâşmăreanu [4]. Moreover the theorem 2 yields the theorem 3 by imposing an arbitrary degree of homogeneity on the higher order differential system. As illustrations for two types of third order differential equations, conserved quantities are constructed through which the solutions of the equations are determined completely.

1 Conserved quantity for higher order differential system. We set our starting point for a given k-th order differential system

$$F^A(t,\, x,\, \dot{x}, \cdots, x^{(k)}) = 0 \qquad (A = 1, \cdots, n;\ k \geq 2), \tag{1}$$

where $x = (x^\sigma(t)),\ \dot{x} = (\dot{x}^\sigma(t)) = (dx^\sigma/dt),\ x^{(k)} = (d^k x^\sigma/dt^k),\ (\sigma = 1, \cdots, m;\ m \geq n)$. A conserved quantity of the system (1) is a quantity $\Omega(t,\, x,\, \dot{x}, \cdots, x^{(k-1)})$ satisfying $d\Omega/dt = 0$

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on solutions to (1), where d/dt denotes the total differentiation with respect to t:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \cdots + x^{(k+1)}\frac{\partial}{\partial x^{(k)}},$$

where $\dot{x}\partial/\partial x = \sum_{\sigma=1}^{m} \dot{x}^\sigma \partial/\partial x^\sigma$, $\ddot{x}\partial/\partial \dot{x} = \sum_{\sigma=1}^{m} \ddot{x}^\sigma \partial/\partial \dot{x}^\sigma$ and so on. Particularly for a second order differential system, new derivation of conserved quantity has been given in ([6], Theorem 1 and its Remark 2). So by putting (in what follows the bracket [ ] denotes Gauss's symbol)

$$\frac{d^{2p}x^\sigma}{dt^{2p}} = y_p^\sigma \qquad (\sigma = 1, \cdots, m;\ p = 1, \cdots, [\tfrac{k}{2}]), \tag{2}$$

the k-th order system (1) is converted into a second order one:

$$\begin{cases} F^A(t,\, y_0,\, y_1, \cdots, y_{[\frac{k}{2}]},\, \dot{y}_0,\, \dot{y}_1, \cdots, \dot{y}_{[\frac{k-1}{2}]}) = 0 \\ \ddot{y}^\sigma_{p-1} - y^\sigma_p = 0 \end{cases} \tag{3}$$

$$(A = 1, \cdots n;\ \sigma = 1, \cdots, m,\ m \geq n;\ p = 1, \cdots, [\tfrac{k}{2}]),$$

where $y_0 = (y_0^\sigma(t)) = (x^\sigma(t))$, $y_p = (y_p^\sigma(t))$ $(p = 1, \cdots, [\frac{k}{2}])$, $\dot{y}_0 = (\dot{y}_0^\sigma(t)) = (\dot{x}^\sigma(t))$, $\dot{y}_r = (\dot{y}_r^\sigma(t))$ $(r = 1, \cdots, [\frac{k-1}{2}])$. Then by regarding the variables $y = (y_0,\, y_1, \cdots, y_{[\frac{k}{2}]})$ as $q = (q^\kappa)$, $(\kappa = 1, \cdots, ([\frac{k}{2}] + 1)n)$, the method in ([6], Remark 2 of Theorem 1) is applied to (3) to obtain the following theorem.

Theorem 1 *Let* $\mu = (\mu_A^0,\, \mu_\sigma^1, \cdots, \mu_\sigma^{[\frac{k}{2}]})$ *and* $\xi = (\xi_0^\sigma,\, \xi_1^\sigma, \cdots, \xi^\sigma_{[\frac{k}{2}]})$ *be functions of* $t,\, y_0,\, y_1, \cdots, y_{[\frac{k-1}{2}]},\, \dot{y}_0,\, \dot{y}_1, \cdots, \dot{y}_{[\frac{k}{2}]-1}$ *satisfying the following system of equations on solutions to* (3):

$$\ddot{\mu}^1_\sigma + \sum_{A=1}^{n} \mu_A^0 \frac{\partial F^A}{\partial y_0^\sigma} - \frac{d}{dt}\left(\sum_{A=1}^{n} \mu_A^0 \frac{\partial F^A}{\partial \dot{y}_0^\sigma}\right) = 0 \qquad (\sigma = 1, \cdots, m), \tag{4}$$

$$\mu_\sigma^{[\frac{k}{2}]} = \sum_{A=1}^{n} \mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]}} - \frac{d}{dt}\left(\sum_{A=1}^{n} \mu_A^0 \frac{\partial F^A}{\partial \dot{y}^\sigma_{[\frac{k}{2}]}}\right) \qquad (\sigma = 1, \cdots, m), \tag{5}$$

$$\sum_{A=1}^{n}\sum_{\sigma=1}^{m} \mu_A^0 \left(\sum_{\ell=0}^{[\frac{k}{2}]} \frac{\partial F^A}{\partial y_\ell^\sigma}\xi_\ell^\sigma + \sum_{\ell=0}^{[\frac{k-1}{2}]} \frac{\partial F^A}{\partial \dot{y}_\ell^\sigma}\dot{\xi}_\ell^\sigma\right) + \sum_{\sigma=1}^{m}\sum_{\ell=1}^{[\frac{k}{2}]} \mu_\sigma^\ell(\ddot{\xi}^\sigma_{\ell-1} - \xi^\sigma_\ell) = \frac{dK}{dt}, \tag{6}$$

where K *is a function of* $t,\, y_0,\, y_1, \cdots, y_{[\frac{k-1}{2}]},\, \dot{y}_0,\, \dot{y}_1, \cdots, \dot{y}_{[\frac{k}{2}]-1}$; *and if* $k \geq 4$, *moreover*

$$\mu_\sigma^\ell = \ddot{\mu}_\sigma^{\ell+1} + \sum_{A=1}^{n} \mu_A^0 \frac{\partial F^A}{\partial y_\ell^\sigma} - \frac{d}{dt}\left(\sum_{A=1}^{n} \mu_A^0 \frac{\partial F^A}{\partial \dot{y}_\ell^\sigma}\right) \tag{7}$$

$$(\sigma = 1, \cdots, m;\ \ell = 1, \cdots, [\tfrac{k}{2}] - 1).$$

Then, by using of the above μ and ξ, the following quantity Ω satisfing $\dot{\Omega}=0$ on solutions to (3) *is constructed*:

$$\Omega=\sum_{\sigma=1}^{m}\sum_{\ell=1}^{[\frac{k}{2}]}(\mu_\sigma^\ell\dot{\xi}_{\ell-1}^\sigma-\dot{\mu}_\sigma^\ell\xi_{\ell-1}^\sigma)+\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\sum_{\ell=0}^{[\frac{k-1}{2}]}\mu_A^0\frac{\partial F^A}{\partial\dot{y}_\ell^\sigma}\xi_\ell^\sigma-K, \tag{8}$$

which gives rise to a conserved quantity of (1) *by denoting $\mu_\sigma^\ell,\ \xi_\ell^\sigma,\ F^A$ and K as functions of the original variables $t,\ x,\ \dot{x},\cdots,x^{(k-1)}$.*

Remark 1 Here note that $[\frac{k-1}{2}]$ is equal to $[\frac{k}{2}]-1$ (k: even), or to $[\frac{k}{2}]$ (k: odd). Accordingly, if k is even, F^A in (3) does not have the variables $\dot{y}^\sigma_{[\frac{k}{2}]}$, i.e., $\partial F^A/\partial\dot{y}^\sigma_{[\frac{k}{2}]}=0$. So that the term $\sum_{A=1}^n\mu_A^0\partial F^A/\partial\dot{y}^\sigma_{[\frac{k}{2}]}$ may be deleted in the appearance of (5).

When $k\geq 4$, at first, (5) is substituted for $\ddot{\mu}_\sigma^{[\frac{k}{2}]}$ in (7) with $\ell=[\frac{k}{2}]-1$ to see

$$\begin{aligned}
\mu_\sigma^{[\frac{k}{2}]-1} &= \ddot{\mu}_\sigma^{[\frac{k}{2}]}+\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-1}}-\frac{d}{dt}\left(\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]-1}}\right)\\
&=\frac{d^2}{dt^2}\left(\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]}}-\frac{d}{dt}\left(\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]}}\right)\right)\\
&\qquad+\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-1}}-\frac{d}{dt}\left(\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]-1}}\right)\\
&=\sum_{A=1}^n\sum_{i=0}^1\frac{d^{2(1-i)}}{dt^{2(1-i)}}\left(\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}}-\frac{d}{dt}\left(\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]-i}}\right)\right),
\end{aligned}$$

where d^0F/dt^0 denotes that $d^0F/dt^0=F$ for an arbitrary function F. And then the appearance of $\mu_\sigma^{[\frac{k}{2}]-1}$ is substituted for $\ddot{\mu}_\sigma^{[\frac{k}{2}]-1}$ in (7) with $\ell=[\frac{k}{2}]-2$ to see

$$\begin{aligned}
\mu_\sigma^{[\frac{k}{2}]-2} &= \ddot{\mu}_\sigma^{[\frac{k}{2}]-1}+\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-2}}-\frac{d}{dt}\left(\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]-2}}\right)\\
&=\sum_{A=1}^n\sum_{i=0}^1\frac{d^{2(1-i)+2}}{dt^{2(1-i)+2}}\left(\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}}-\frac{d}{dt}\left(\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]-i}}\right)\right)\\
&\qquad+\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-2}}-\frac{d}{dt}\left(\sum_{A=1}^n\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]-2}}\right)\\
&=\sum_{A=1}^n\sum_{i=0}^2\frac{d^{2(2-i)}}{dt^{2(2-i)}}\left(\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}}-\frac{d}{dt}\left(\mu_A^0\frac{\partial F^A}{\partial\dot{y}^\sigma_{[\frac{k}{2}]-i}}\right)\right),
\end{aligned}$$

and so on. Finally, it follows that

$$\mu_\sigma^\ell = \sum_{A=1}^{n} \sum_{i=0}^{[\frac{k}{2}]-\ell} \left(\frac{d^{2([\frac{k}{2}]-\ell-i)}}{dt^{2([\frac{k}{2}]-\ell-i)}} \left(\mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}} \right) - \frac{d^{2([\frac{k}{2}]-\ell-i)+1}}{dt^{2([\frac{k}{2}]-\ell-i)+1}} \left(\mu_A^0 \frac{\partial F^A}{\partial \dot{y}^\sigma_{[\frac{k}{2}]-i}} \right) \right) \tag{9}$$

$$(\sigma = 1, \cdots, m;\ \ell = 1, \cdots, [\tfrac{k}{2}];\ k \geq 4).$$

For $k \geq 4$ and $\ell = 1$, (9) is written as

$$\mu_\sigma^1 = \sum_{A=1}^{n} \sum_{i=0}^{[\frac{k}{2}]-1} \left(\frac{d^{2([\frac{k}{2}]-i-1)}}{dt^{2([\frac{k}{2}]-i-1)}} \left(\mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}} \right) - \frac{d^{2([\frac{k}{2}]-i)-1}}{dt^{2([\frac{k}{2}]-i)-1}} \left(\mu_A^0 \frac{\partial F^A}{\partial \dot{y}^\sigma_{[\frac{k}{2}]-i}} \right) \right). \tag{10}$$

When $k = 2$ or $k = 3$, (10) is coincide with (5). Therefore, (10) is valid for $k \geq 2$ (while it is derived from (5) and (7) for $k \geq 4$). The total differentiation of (10) with respect to t is substituted for (4) to have

$$\sum_{A=1}^{n} \sum_{i=0}^{[\frac{k}{2}]} \left(\frac{d^{2([\frac{k}{2}]-i)}}{dt^{2([\frac{k}{2}]-i)}} \left(\mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}} \right) - \frac{d^{2([\frac{k}{2}]-i)+1}}{dt^{2([\frac{k}{2}]-i)+1}} \left(\mu_A^0 \frac{\partial F^A}{\partial \dot{y}^\sigma_{[\frac{k}{2}]-i}} \right) \right) = 0 \tag{11}$$

$$(\sigma = 1, \cdots, m;\ k \geq 2).$$

Remark 2 By putting $y_0^\sigma \equiv x^\sigma$ and $y_1^\sigma = \ddot{x}^\sigma$, a second order differential system (1) with $k = 2$:

$$F^A(t,\, x,\, \dot{x},\, \ddot{x}) = 0$$

is converted into

$$\begin{cases} F^A(t,\, y_0,\, y_1,\, \dot{y}_0) = 0 \\ \ddot{y}_0^\sigma - y_1^\sigma = 0 \end{cases} \qquad (A = 1, \cdots, n;\ \sigma = 1, \cdots, m). \tag{3$'$}$$

In this case, (11) reduces to

$$\sum_{A=1}^{n} \left(\mu_A^0 \frac{\partial F^A}{\partial y_0^\sigma} - \frac{d}{dt} \left(\mu_A^0 \frac{\partial F^A}{\partial \dot{y}_0^\sigma} \right) + \frac{d^2}{dt^2} \left(\mu_A^0 \frac{\partial F^A}{\partial y_1^\sigma} \right) \right) = 0. \tag{11$'$}$$

And (10) (or (5) with $k = 2$) is substituted for (6) with $k = 2$ to see

$$\sum_{A=1}^{n} \sum_{\sigma=1}^{m} \mu_A^0 \left(\frac{\partial F^A}{\partial y_0^\sigma} \xi_0^\sigma + \frac{\partial F^A}{\partial \dot{y}_0^\sigma} \dot{\xi}_0^\sigma + \frac{\partial F^A}{\partial y_1^\sigma} \ddot{\xi}_0^\sigma \right) = \frac{dK}{dt}. \tag{12}$$

Therefore, the functions $\mu_A^0(t,\, y_0,\, \dot{y}_0)$ and $\xi_0^\sigma(t,\, y_0,\, \dot{y}_0)$ satisfying (11)$'$ and (12) on solutions to (3)$'$ yields the following quantity Ω satisfing $\dot{\Omega} = 0$ on solutions to (3)$'$:

$$\Omega = \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \left(\mu_A^0 \frac{\partial F^A}{\partial y_1^\sigma} \dot{\xi}_0^\sigma + \left(\mu_A^0 \frac{\partial F^A}{\partial \dot{y}_0^\sigma} - \frac{d}{dt} \left(\mu_A^0 \frac{\partial F^A}{\partial y_1^\sigma} \right) \right) \xi_0^\sigma \right) - K(t,\, y_0,\, \dot{y}_0). \tag{8$'$}$$

By denoting $y_0^\sigma, \dot{y}_0^\sigma$ and y_1^σ as the original variables x^σ, $\dot{x}^\sigma$ and $\ddot{x}^\sigma$ respectively, (8)$'$ turns into the conserved quantities obtained in ([6], Remark 2 of Theorem 1):

$$\Omega = \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \left(\mu_A^0 \frac{\partial F^A}{\partial \ddot{x}^\sigma} \dot{\xi}_0^\sigma + \left(\mu_A^0 \frac{\partial F^A}{\partial \dot{x}^\sigma} - \frac{d}{dt} \left(\mu_A^0 \frac{\partial F^A}{\partial \ddot{x}^\sigma} \right) \right) \xi_0^\sigma \right) - K(t,\, x,\, \dot{x}).$$

For an arbitrary given ξ_0^σ in (6), particularly put $\xi_\ell^\sigma = d^{2\ell}\xi_0^\sigma/dt^{2\ell}$ i.e., $\xi_\ell^\sigma = \ddot{\xi}_{\ell-1}^\sigma$ ($\sigma = 1,\cdots,m;\ \ell = 1,\cdots,[\frac{k}{2}]$). Then (6) reduces to

$$\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\mu_A^0\left(\sum_{\ell=0}^{[\frac{k}{2}]}\frac{\partial F^A}{\partial y_\ell^\sigma}\frac{d^{2\ell}\xi_0^\sigma}{dt^{2\ell}}+\sum_{\ell=0}^{[\frac{k-1}{2}]}\frac{\partial F^A}{\partial \dot{y}_\ell^\sigma}\frac{d^{2\ell+1}\xi_0^\sigma}{dt^{2\ell+1}}\right)=\frac{dK}{dt}, \tag{13}$$

which is just the equation (1.10b) in (Crâşmăreanu [4], Theorem). Therefore the result of Crâşmăreanu is concluded completely in the theorem 1:

Theorem 2 *Let μ_A^0 and ξ_0^σ be functions of t, y_0, $y_1,\cdots,y_{[\frac{k-1}{2}]}$, $\dot{y}_0$, $\dot{y}_1,\cdots,\dot{y}_{[\frac{k}{2}]-1}$ satisfying the equations* (11) *and* (13) *on solutions to* (3)*. Then the following quantity Ω satisfing $\dot{\Omega}=0$ on solutions to* (3) *is constructed*:

(14)

$$\begin{aligned}\Omega = &\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\sum_{\ell=1}^{[\frac{k}{2}]}\sum_{i=0}^{[\frac{k}{2}]-\ell}\left(\frac{d^{2([\frac{k}{2}]-\ell-i)}}{dt^{2([\frac{k}{2}]-\ell-i)}}\left(\mu_A^0\frac{\partial F^A}{\partial y_{[\frac{k}{2}]-i}^\sigma}\right)-\frac{d^{2([\frac{k}{2}]-\ell-i)+1}}{dt^{2([\frac{k}{2}]-\ell-i)+1}}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}_{[\frac{k}{2}]-i}^\sigma}\right)\right)\frac{d^{2\ell-1}\xi_0^\sigma}{dt^{2\ell-1}}\\
&-\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\sum_{\ell=1}^{[\frac{k}{2}]}\sum_{i=0}^{[\frac{k}{2}]-\ell}\left(\frac{d^{2([\frac{k}{2}]-\ell-i)+1}}{dt^{2([\frac{k}{2}]-\ell-i)+1}}\left(\mu_A^0\frac{\partial F^A}{\partial y_{[\frac{k}{2}]-i}^\sigma}\right)-\frac{d^{2([\frac{k}{2}]-\ell-i+1)}}{dt^{2([\frac{k}{2}]-\ell-i+1)}}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}_{[\frac{k}{2}]-i}^\sigma}\right)\right)\frac{d^{2(\ell-1)}\xi_0^\sigma}{dt^{2(\ell-1)}}\\
&+\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\sum_{\ell=0}^{[\frac{k-1}{2}]}\mu_A^0\frac{\partial F^A}{\partial \dot{y}_\ell^\sigma}\frac{d^{2\ell}\xi_0^\sigma}{dt^{2\ell}}-K \qquad (k\geq 2),\end{aligned}$$

which gives rise to a conserved quantity of (1) *by denoting μ_A^0, ξ_0^σ, F^A and K as functions of the original variables $t,\ x,\ \dot{x},\cdots,x^{(k-1)}$.*

Remark 3 If k is even, the term $\partial F^A/\partial\dot{y}_{[\frac{k}{2}]}^\sigma$ in (11), (13) and (14) may be deleted (see Remark 1).

Remark 4 Here note that (13) is derived by putting $\xi_\ell^\sigma = d^{2\ell}\xi_0^\sigma/dt^{2\ell}$ ($\sigma = 1,\cdots,n;$ $\ell = 1,\cdots,[\frac{k}{2}]$). But whenever $k=2$, it can be directely obtained by substituting (10) for (6) without the postulation $\xi_1^\sigma = \ddot{\xi}_0^\sigma$.

Here we impose on F^A an arbitrary degree s of homogeneity with respect to y_ℓ^σ ($\sigma = 1,\cdots,n;\ \ell=0,\cdots,[\frac{k}{2}]$) and $\dot{y}_\ell^\sigma$ ($\sigma=1,\cdots,n;\ \ell=0,\cdots,[\frac{k-1}{2}]$) to have the identity:

$$\sum_{\sigma=1}^{m}\left(\sum_{\ell=0}^{[\frac{k}{2}]}\frac{\partial F^A}{\partial y_\ell^\sigma}y_\ell^\sigma+\sum_{\ell=0}^{[\frac{k-1}{2}]}\frac{\partial F^A}{\partial \dot{y}_\ell^\sigma}\dot{y}_\ell^\sigma\right)=sF^A, \tag{15}$$

which vanishes on solutions to (3). In viewing of $y_p^\sigma = \ddot{y}_{p-1}^\sigma$ in (3), since

$$y_p^\sigma = \frac{d^4\ddot{y}_{p-2}^\sigma}{dt^4}=\cdots=\frac{d^{2p}\ddot{y}_0^\sigma}{dt^{2p}},$$

(15) guarantees that $\xi_0^\sigma = y_0^\sigma$ satisfies (13) with $K=0$ on solutions to (3). Therefore the theorem 2 reduces to

Theorem 3 *Let F^A in* (3) *be homogeneous function of degree s with respect to y_0, y_1, $\cdots$, $y_{[\frac{k-1}{2}]}$, $\dot{y}_0$, $\dot{y}_1$, $\cdots$, $\dot{y}_{[\frac{k}{2}]-1}$. Then the function μ_A^0 of t, y_0, $y_1 \cdots$, $y_{[\frac{k-1}{2}]}$, $\dot{y}_0$, $\dot{y}_1, \cdots$, $\dot{y}_{[\frac{k}{2}]-1}$ satisfying the equation* (11) *on solutions to* (3) *yields the following quantity Ω satisfing $\dot{\Omega} = 0$ on solutions to* (3):

(16)

$$\Omega = \sum_{A=1}^{n}\sum_{\sigma=1}^{m}\sum_{\ell=1}^{[\frac{k}{2}]}\sum_{i=0}^{[\frac{k}{2}]-\ell}\left(\frac{d^{2([\frac{k}{2}]-\ell-i)}}{dt^{2([\frac{k}{2}]-\ell-i)}}\left(\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}}\right) - \frac{d^{2([\frac{k}{2}]-\ell-i)+1}}{dt^{2([\frac{k}{2}]-\ell-i)+1}}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}^\sigma_{[\frac{k}{2}]-i}}\right)\right)\frac{d^{2\ell-1}y_0^\sigma}{dt^{2\ell-1}}$$

$$-\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\sum_{\ell=1}^{[\frac{k}{2}]}\sum_{i=0}^{[\frac{k}{2}]-\ell}\left(\frac{d^{2([\frac{k}{2}]-\ell-i)+1}}{dt^{2([\frac{k}{2}]-\ell-i)+1}}\left(\mu_A^0\frac{\partial F^A}{\partial y^\sigma_{[\frac{k}{2}]-i}}\right) - \frac{d^{2([\frac{k}{2}]-\ell-i+1)}}{dt^{2([\frac{k}{2}]-\ell-i+1)}}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}^\sigma_{[\frac{k}{2}]-i}}\right)\right)\frac{d^{2(\ell-1)}y_0^\sigma}{dt^{2(\ell-1)}}$$

$$+\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\sum_{\ell=0}^{[\frac{k-1}{2}]}\mu_A^0\frac{\partial F^A}{\partial \dot{y}^\sigma_\ell}\frac{d^{2\ell}y_0^\sigma}{dt^{2\ell}} \qquad (k \geq 2),$$

which gives rise to a conserved quantity of (1) *by denoting μ_A^0 and F^A as functions of the original variables $t,\ x,\ \dot{x}, \cdots, x^{(k-1)}$.*

2 A reduction to a third order differential system. Particularly consider third order differential system

$$(1)'' \qquad F^A(t,\, x,\, \dot{x},\, \ddot{x},\, \dddot{x}) = 0 \qquad (A = 1, \cdots, n).$$

By putting $x^\sigma = y_0^\sigma$ and $\ddot{x}^\sigma = y_1^\sigma$ $(\sigma = 1, \cdots, m)$, the system $(1)''$ can be converted into a second order one:

$$(3)'' \qquad \begin{cases} F^A(t,\, y_0,\, y_1,\, \dot{y}_0,\, \dot{y}_1) = 0 \\ \ddot{y}_0^\sigma - y_1^\sigma = 0 \end{cases} \qquad (A = 1, \cdots, n;\ \sigma = 1, \cdots, m).$$

Then the theorem 3 reduces to

Corollary. *Let F^A in $(3)''$ be homogeneous function of degree s with respect to $y_0,\, y_1,\, \dot{y}_0,\, \dot{y}_1$. Then the function μ_A^0 of $t,\, y_0,\, y_1,\, \dot{y}_0$ satisfying the equation*

$$(11)'' \qquad \sum_{A=1}^{n}\left(\frac{d^3}{dt^3}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}_1^\sigma}\right) - \frac{d^2}{dt^2}\left(\mu_A^0\frac{\partial F^A}{\partial y_1^\sigma}\right) + \frac{d}{dt}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}_0^\sigma}\right) - \mu_A^0\frac{\partial F^A}{\partial y_0^\sigma}\right) = 0$$

on solutions to $(3)''$ yields the following quantity Ω satisfying $\dot{\Omega} = 0$ on solutions to $(3)''$ is constructed:

$(16)'$

$$\Omega = \sum_{A=1}^{n}\sum_{\sigma=1}^{m}\left(\mu_A^0\frac{\partial F^A}{\partial y_1^\sigma} - \frac{d}{dt}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}_1^\sigma}\right)\right)\dot{y}_0^\sigma - \sum_{A=1}^{n}\sum_{\sigma=1}^{m}\left(\frac{d}{dt}\left(\mu_A^0\frac{\partial F^A}{\partial y_1^\sigma}\right) - \frac{d^2}{dt^2}\left(\mu_A^0\frac{\partial F^A}{\partial \dot{y}_1^\sigma}\right)\right)y_0^\sigma$$
$$+\sum_{A=1}^{n}\sum_{\sigma=1}^{m}\mu_A^0\left(\frac{\partial F^A}{\partial \dot{y}_0^\sigma} + \frac{\partial F^A}{\partial \dot{y}_1^\sigma}\ddot{y}_0^\sigma\right),$$

which gives rise to a conserved quantity of $(1)''$ by denoting μ_A^0 and F^A as functions of the original variables $t,\, x,\, \dot{x},\, \ddot{x}$.

Example 1. First consider the following linear differential equation

$$\dddot{x} - \frac{a}{t^2}\dot{x} = 0 \qquad (a\text{: const.},\ a > -\tfrac{1}{4}). \tag{17}$$

By putting $x = y_0$ and $\ddot{x} = y_1$, (17) is converted into a second order system

$$\begin{cases} \dot{y}_1 - \dfrac{a}{t^2}\dot{y}_0 = 0 \\ \ddot{y}_0 - y_1 = 0, \end{cases} \tag{18}$$

in which $\dot{y}_1 - (a/t^2)\dot{y}_0$ is homogeneous function of degree one with respect to $\dot{y}_0$ and $\dot{y}_1$. The equation $(11)''$ reduces to

$$\dddot{\mu}^0 - \frac{d}{dt}\left(\frac{a}{t^2}\mu^0\right) = 0, \tag{19}$$

which is integrated as

$$\ddot{\mu}^0 - \frac{a}{t^2}\mu^0 = C \qquad (C\text{: const.}). \tag{20}$$

Here put $\mu^0 = t^m$ $(m$: const.). Then the homogeneous equation of (20) is written as

$$\ddot{\mu}^0 - \frac{a}{t^2}\mu^0 = (m^2 - m - a)t^{m-2} = 0, \tag{21}$$

whose solution is

$$\mu^0 = C_1 t^{m_1} + C_2 t^{m_2} \qquad (C_1,\ C_2\text{: const.}),$$

where m_1 and m_2 are the constants:

$$m_1 = \frac{1 + \sqrt{1 + 4a}}{2}, \quad m_2 = \frac{1 - \sqrt{1 + 4a}}{2}.$$

Accordingly, since $\mu^0 = t^2$ is a solution of (19) (also (20)), the solution of (19) is determined as

$$\mu^0 = C_1 t^{m_1} + C_2 t^{m_2} + C_3 t^2 \qquad (C_1,\ C_2,\ C_3\text{: const.}),$$

which is substituted for $(16)'$ to obtain the conserved quantity:

(22)

$$\Omega = C_1(t^{m_1}y_1 - m_1 t^{m_1-1}\dot{y}_0) + C_2(t^{m_2}y_1 - m_2 t^{m_2-1}\dot{y}_0) + C_3(t^2 y_1 - 2t\dot{y}_0 + (2-a)y_0).$$

Since C_1, C_2 and C_3 are arbitrary constants, (22) includes the following three conserved quantities:

$$\Omega_1 = t^{m_1}\ddot{x} - m_1 t^{m_1-1}\dot{x},$$
$$\Omega_2 = t^{m_2}\ddot{x} - m_2 t^{m_2-1}\dot{x},$$
$$\Omega_3 = t^2\ddot{x} - 2t\dot{x} + (2-a)x,$$

which are independent if

$$\begin{vmatrix} t^{m_1} & -m_1 & 0 \\ t^{m_2} & -m_2 & 0 \\ t^2 & -2t & 2-a \end{vmatrix} = (2-a)(m_1 t^{m_2} - m_2 t^{m_1}) \neq 0.$$

When $a \neq 2$, by eliminating $\ddot{x}$ and $\dot{x}$ in Ω_1, Ω_2 and Ω_3, the solution of (17) is determined immediately as

$$x = A_1 t^{2-m_1} + A_2 t^{2-m_2} + A_3,$$

where $A_1,\ A_2$ and A_3 are the constants:

$$A_1 = \frac{2-m_2}{(2-a)(m_2-m_1)}\Omega_1, \quad A_2 = \frac{2-m_1}{(2-a)(m_2-m_1)}\Omega_2, \quad A_3 = \frac{1}{2-a}\Omega_3.$$

When $a = 2$, it follows that $\Omega_1 = \Omega_3$, which and Ω_2 lead respectively to

$$\Omega_1 = \Omega_3 = t^2\ddot{x} - 2t\dot{x},$$

$$\Omega_2 = \frac{\ddot{x}}{t} + \frac{\dot{x}}{t^2}.$$

In Ω_1 and Ω_2, $\ddot{x}$ is eliminated to have

$$\dot{x} = \frac{a_2}{3}t^2 - \frac{a_1}{3}\frac{1}{t},$$

which is integrated as

$$x = B_1 \log|t| + B_2 t^3 + B_3,$$

where $B_1 = -\Omega_1/3,\ B_2 = \Omega_2/9$ and B_3 are arbitrary constants.

Example 2. Next consider the following linear differential equation

$$\dddot{x} + f(t)\ddot{x} = 0. \tag{23}$$

By putting $x = y_0$ and $\ddot{x} = y_1$, (23) is converted into a second order system

$$\begin{cases} \dot{y}_1 + f(t)y_1 = 0 \\ \ddot{y}_0 - y_1 = 0. \end{cases} \tag{24}$$

Then $(11)''$ reduces to

$$\frac{d^3\mu^0}{dt^3} - \frac{d^2}{dt^2}f(t)\mu^0 = 0, \quad \text{i.e.,} \quad \dot{\mu}^0 - f(t)\mu^0 = C_1 t + C_2 \qquad (C_1,\, C_2\text{: const.}),$$

whose solution

$$\mu^0 = e^{\int f(t)dt}\left(C_1 \int te^{-\int f(t)dt}dt + C_2 \int e^{\int f(t)dt}dt + C_3\right) \qquad (C_1,\, C_2,\, C_3\text{: const.})$$

is substituted for $(16)'$ to construct the conserved quantity

$$\begin{aligned} \Omega \ &= C_1\left(-ty_0 + y_0 + y_1 e^{\int f(t)dt}\int te^{-\int f(t)dt}dt\right) \\ &\quad + C_2\left(-\dot{y}_0 + y_1 e^{\int f(t)dt}\int e^{-\int f(t)dt}dt\right) + C_3 y_1 e^{\int f(t)dt}. \end{aligned} \tag{25}$$

Since C_1, C_2 and C_3 are arbitrary constants, (25) includes the following conserved quantities:

$$\Omega_1 = -t\dot{x} + x + \ddot{x}e^{\int f(t)dt}\int te^{-\int f(t)dt}dt,$$

$$\Omega_2 = -\dot{x} + \ddot{x}e^{\int f(t)dt}\int e^{-\int f(t)dt}dt,$$

$$\Omega_3 = \ddot{x}e^{\int f(t)dt},$$

in which $\ddot{x}$ and $\dot{x}$ is eliminated to determine the solution of (23):

$$x = \Omega_1 - \Omega_2 t + \Omega_3\left(t\int e^{-\int f(t)dt}dt - \int te^{-\int f(t)dt}dt\right),$$

where $\Omega_1,\ \Omega_2$ and Ω_3 are arbitrary constants.

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