FUZZY BCK-FILTERS INDUCED BY FUZZY SETS

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ABSTRACT. We give the definition of fuzzy BCK-filter induced by a fuzzy set in a bounded BCK-algebra. We verify that the family of fuzzy BCK-filters is a completely distributive lattice. Using the BCK-filter $\langle U(\mu;\alpha)\rangle$ for a given fuzzy set μ, we construct the fuzzy BCK-filter induced by μ.

1. INTRODUCTION

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers. L. A. Zadeh [10] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group theory, functional analysis, probability theory, topology, and so on. In 1991, O. G. Xi [9] applied this concept to BCK-algebra, and he introduced the notion of fuzzy subalgebra (ideal) of the BCK-algebra. J. Meng [7] introduced the notion of BCK-filter in BCK-algebra and gave a description of the BCK-filter generated by a set. The present authors have studied several notions and properties about fuzzy BCK-filters together with S. S. Ahn, S. M. Hong, H. S. Kim, J. Meng, X. L. Xin and F. L. Zhang in [1]∼ [6].

In this paper, we give the definition of fuzzy BCK-filter induced by a fuzzy set in bounded BCK-algebras. We verify that the family of fuzzy BCK-filters is a completely distributive lattice. Using the BCK-filter $\langle U(\mu;\alpha)\rangle$ for a given fuzzy set μ, we construct the fuzzy BCK-filter induced by μ.

2. PRELIMINARIES

We recall some definitions and preliminary results.

A nonempty set X with a constant 0 and a binary operation denoted by juxtaposition is called a *BCK-algebra* if for all $x, y, z \in X$ the following conditions hold:

(I) $((xy)(xz))(zy) = 0$,
(II) $(x(xy))y = 0$,
(III) $xx = 0$,
(IV) $0x = 0$,
(V) $xy = 0$ and $yx = 0$ imply $x = y$.

A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $xy = 0$. This ordering is called *BCK-ordering*. The following statements are true in any BCK-algebra: for all x, y, z,

(1) $x0 = x$.
(2) $(xy)z = (xz)y$.
(3) $xy \leq x$.
(4) $(xy)z \leq (xz)(yz)$.

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(5) $x \le y$ implies $xz \le yz$ and $zy \le zx$.

A BCK-algebra X satisfying the identity $x(xy) = y(yx)$ is said to be *commutative*. If there is a special element e of a BCK-algebra X satisfying $x \le e$ for all $x \in X$, then e is called *unit* of X. A BCK-algebra with unit is said to be *bounded*. In a bounded BCK-algebra X, we denote ex by x^* for every $x \in X$.

In a bounded BCK-algebra, we have

(6) $e^* = 0$ and $0^* = e$.

(7) $y \le x$ implies $x^* \le y^*$.

(8) $x^*y^* \le yx$.

In what follows, let X denote a bounded BCK-algebra unless otherwise specified. A non-empty subset F of X is called a *BCK-filter* of X ([7, Definition 3]) if

(i) $e \in F$,

(ii) $(x^*y^*)^* \in F$ and $y \in F$ imply $x \in F$ for all $x, y \in X$.

3. Fuzzy BCK-filters induced by fuzzy sets

A *fuzzy set* in X is a function $\mu : X \to [0,1]$. Let μ be a fuzzy set in X. For $\alpha \in [0,1]$, the set $U(\mu;\alpha) = \{x \in X \mid \mu(x) \ge \alpha\}$ is called a *level subset* of μ.

Definition 3.1. ([3, Definition 3.2]) Let μ be a fuzzy set in X. Then μ is called a *fuzzy BCK-filter* of X if, for all $x, y \in X$,

(i) $\mu(e) \ge \mu(x)$,

(ii) $\mu(x) \ge \min\{\mu((x^*y^*)^*), \mu(y)\}$.

For a family $\{\mu_\alpha \mid \alpha \in \Lambda\}$ of fuzzy sets in X, define the join $\bigvee_{\alpha\in\Lambda} \mu_\alpha$ and the meet $\bigwedge_{\alpha\in\Lambda} \mu_\alpha$ as follows:

$$(\bigvee_{\alpha\in\Lambda} \mu_\alpha)(x) = \sup_{\alpha\in\Lambda} \mu_\alpha(x) \text{ and } (\bigwedge_{\alpha\in\Lambda} \mu_\alpha)(x) = \inf_{\alpha\in\Lambda} \mu_\alpha(x)$$

for all $x \in X$, where Λ is any index set.

Theorem 3.2. *The family of fuzzy BCK-filters in X is a completely distributive lattice with respect to the meet "$\wedge$" and the join "$\vee$".*

Proof. Let $\{\mu_\alpha \mid \alpha \in \Lambda\}$ be a family of fuzzy BCK-filters of X. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\bigvee_{\alpha\in\Lambda} \mu_\alpha$ and $\bigwedge_{\alpha\in\Lambda} \mu_\alpha$ are fuzzy BCK-filters of X. For any $x \in X$, we have

$$(\bigvee_{\alpha\in\Lambda} \mu_\alpha)(e) = \sup_{\alpha\in\Lambda} \mu_\alpha(e) \ge \sup_{\alpha\in\Lambda} \mu_\alpha(x) = (\bigvee_{\alpha\in\Lambda} \mu_\alpha)(x)$$

and

$$(\bigwedge_{\alpha\in\Lambda} \mu_\alpha)(e) = \inf_{\alpha\in\Lambda} \mu_\alpha(e) \ge \inf_{\alpha\in\Lambda} \mu_\alpha(x) = (\bigwedge_{\alpha\in\Lambda} \mu_\alpha)(x).$$

Let $x, y \in X$. Then

$$\begin{aligned}(\bigvee_{\alpha\in\Lambda} \mu_\alpha)(x) &= \sup_{\alpha\in\Lambda} \mu_\alpha(x) \ge \sup_{\alpha\in\Lambda}\{\min\{\mu_\alpha((x^*y^*)^*), \mu_\alpha(y)\}\} \\ &\ge \min\{\sup_{\alpha\in\Lambda} \mu_\alpha((x^*y^*)^*), \sup_{\alpha\in\Lambda} \mu_\alpha(y)\} = \min\{(\bigvee_{\alpha\in\Lambda} \mu_\alpha)((x^*y^*)^*), (\bigvee_{\alpha\in\Lambda} \mu_\alpha)(y)\}\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{\alpha\in\Lambda} \mu_\alpha)(x) &= \inf_{\alpha\in\Lambda} \mu_\alpha(x) \ge \inf_{\alpha\in\Lambda}\{\min\{\mu_\alpha((x^*y^*)^*), \mu_\alpha(y)\}\} \\ &= \min\{\inf_{\alpha\in\Lambda} \mu_\alpha((x^*y^*)^*), \inf_{\alpha\in\Lambda} \mu_\alpha(y)\} = \min\{(\bigwedge_{\alpha\in\Lambda} \mu_\alpha)((x^*y^*)^*), (\bigwedge_{\alpha\in\Lambda} \mu_\alpha)(y)\}.\end{aligned}$$

Hence $\bigvee_{\alpha\in\Lambda} \mu_\alpha$ and $\bigwedge_{\alpha\in\Lambda} \mu_\alpha$ are fuzzy BCK-filters of X. □

Lemma 3.3. ([3, Theorem 3.8]) *Let μ be a fuzzy set in X. Then the following are equivalent:*

(i) *μ is a fuzzy BCK-filter of X;*

(ii) *Each non-empty $U(\mu;\alpha)$ is a BCK-filter of X.*

Lemma 3.4. *For a fuzzy set μ in X, we have*

$$\mu(x) = \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} \text{ for all } x \in X.$$

Proof. Let $q := \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}$ and let $\varepsilon > 0$ be given. Then $q-\varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu;\alpha)$, and so $q-\varepsilon < \mu(x)$. Since ε is arbitrary, it follows that $q \le \mu(x)$. Now let $\mu(x) = \beta$. Then $x \in U(\mu;\beta)$ and hence $\beta \in \{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}$. Therefore

$$\mu(x) = \beta \le \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} = q.$$

Consequently $\mu(x) = q$, as desired. □ □

Let Γ be a non-empty subset of $[0,1]$.

Theorem 3.5. *Let $\{F_\alpha \mid \alpha \in \Gamma\}$ be a collection of BCK-filters of X such that*

(i) $X = \bigcup_{\alpha\in\Gamma} F_\alpha$,

(ii) *$\alpha > \beta$ if and only if $F_\alpha \subseteq F_\beta$ for all $\alpha,\beta \in \Gamma$.*

Define a fuzzy set ν in X by $\nu(x) := \sup\{\alpha \in \Gamma \mid x \in F_\alpha\}$ for all $x \in X$. Then ν is a fuzzy BCK-filter of X.

Proof. Using Lemma 3.3, it is sufficient to show that $U(\nu;\alpha)$ $(\neq \emptyset)$ is a BCK-filter of X for every $\alpha \in [0,1]$. We consider the following two cases:

$$\text{(i) } \alpha = \sup\{\beta \in \Gamma \mid \beta < \alpha\} \text{ and (ii) } \alpha \neq \sup\{\beta \in \Gamma \mid \beta < \alpha\}.$$

Case (i) implies that

$$x \in U(\nu;\alpha) \Leftrightarrow x \in F_\beta \text{ for all } \beta < \alpha \Leftrightarrow x \in \bigcap_{\beta<\alpha} F_\beta,$$

whence $U(\nu;\alpha) = \bigcap_{\beta<\alpha} F_\beta$, which is a BCK-filter of X. For the case (ii), there exists $\varepsilon > 0$ such that $(\alpha-\varepsilon,\alpha)\bigcap\Gamma = \emptyset$. We claim that $U(\nu;\alpha) = \bigcup_{\beta\ge\alpha} F_\beta$. If $x \in \bigcup_{\beta\ge\alpha} F_\beta$, then $x \in F_\beta$ for some $\beta \ge \alpha$. It follows that $\nu(x) \ge \beta \ge \alpha$ so that $x \in U(\nu;\alpha)$. Conversely if $x \notin \bigcup_{\beta\ge\alpha} F_\beta$, then $x \notin F_\beta$ for all $\beta \ge \alpha$, which implies that $x \notin F_\beta$ for all $\beta > \alpha-\varepsilon$, that is, if $x \in F_\beta$ then $\beta \le \alpha-\varepsilon$. Thus $\nu(x) \le \alpha-\varepsilon$ and so $x \notin U(\nu;\alpha)$. Consequently $U(\nu;\alpha) = \bigcup_{\beta\ge\alpha} F_\beta$ which is a BCK-filter of X. This completes the proof. □ □

Let A be a subset of X. The least BCK-filter containing A is called the *BCK-filter generated* by A, written $\langle A\rangle$. Note from [7, Theorem 15] that if A is a non-empty subset of X then

$$\langle A\rangle = \{x \in X \mid (\cdots((x^*a_1^*)a_2^*)\cdots)a_n^* = 0 \text{ for some } a_1, a_2, \cdots, a_n \in A\}.$$

Lemma 3.6. ([7, Theorem 14]) *Let A and B be subsets of X. Then*

(i) *$\langle\{e\}\rangle = \{e\}$ and $\langle\emptyset\rangle = \{e\}$,*

(ii) *$\langle X\rangle = X$ and $\langle\{0\}\rangle = X$,*

(iii) *$A \subseteq B$ implies $\langle A\rangle \subseteq \langle B\rangle$,*

(iv) *$x \le y$ implies $\langle\{y\}\rangle \subseteq \langle\{x\}\rangle$,*

(v) *if A is a BCK-filter then $\langle A\rangle = A$.*

Definition 3.7. Let μ be a fuzzy set in X. The least fuzzy BCK-filter of X cantaining μ is called a fuzzy BCK-filter of X *induced* by μ, denoted by $\langle\mu\rangle$.

Theorem 3.8. *Let μ be a fuzzy set in X. Then the fuzzy set μ^* in X defined by*

$$\mu^*(x) = \sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}$$

for all $x \in X$ is the fuzzy BCK-filter $\langle\mu\rangle$ induced by μ.

Proof. We first show that μ^* is a fuzzy BCK-filter of X. For any $\gamma \in \mathrm{Im}(\mu^*)$, let $\gamma_n = \gamma - \frac{1}{n}$ for any $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers, and let $x \in U(\mu^*;\gamma)$. Then $\mu^*(x) \geq \gamma$, which implies that

$$\sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\} \geq \gamma > \gamma - \frac{1}{n} = \gamma_n$$

for any $n \in \mathbf{N}$. Hence there exists $\beta \in \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}$ such that $\beta > \gamma_n$. Thus $U(\mu;\beta) \subseteq U(\mu;\gamma_n)$ and so $x \in \langle U(\mu;\beta)\rangle \subseteq \langle U(\mu;\gamma_n)\rangle$ for all $n \in \mathbf{N}$. Consequently $x \in \bigcap_{n\in\mathbf{N}} \langle U(\mu;\gamma_n)\rangle$. On the other hand, if $x \in \bigcap_{n\in\mathbf{N}} \langle U(\mu;\gamma_n)\rangle$, then $\gamma_n \in \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}$ for any $n \in \mathbf{N}$. Therefore

$$\gamma - \frac{1}{n} = \gamma_n \leq \sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\} = \mu^*(x)$$

for all $n \in \mathbf{N}$. Since n is an arbitrary positive integer, it follows that $\gamma \leq \mu^*(x)$ so that $x \in U(\mu^*;\gamma)$. Hence $U(\mu^*;\gamma) = \bigcap_{n\in\mathbf{N}} \langle U(\mu;\gamma_n)\rangle$, which is a BCK-filter of X. Using Lemma 3.3, we know that μ^* is a fuzzy BCK-filter of X. We now prove that μ^* contains μ. For any $x \in X$, let $\beta \in \{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}$. Then $x \in U(\mu;\beta)$ and so $x \in \langle U(\mu;\beta)\rangle$. Thus we get $\beta \in \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}$, which implies that

$$\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} \subseteq \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}.$$

Using Lemma 3.4, we conclude that

$$\mu(x) = \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} \leq \sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\} = \mu^*(x).$$

Hence $\mu \subseteq \mu^*$. Finally let ν be a fuzzy BCK-filter of X containing μ. Let $x \in X$. If $\mu^*(x) = 0$, then obviously $\mu^*(x) \leq \nu(x)$. Assume that $\mu^*(x) = \gamma \neq 0$. Then $x \in U(\mu^*;\gamma) = \bigcap_{n\in\mathbf{N}} \langle U(\mu;\gamma_n)\rangle$, i.e., $x \in U(\mu;\gamma_n)$ for all $n \in \mathbf{N}$. It follows that $\nu(x) \geq \mu(x) \geq \gamma_n = \gamma - \frac{1}{n}$ for all $n \in \mathbf{N}$ so that $\nu(x) \geq \gamma = \mu^*(x)$ since n is arbitrary. This shows that $\mu^* \subseteq \nu$, ending the proof. □

Combining Lemma 3.6(v) and Theorems 3.5 and 3.8, we have the following corollary.

Corollary 3.9. *If μ is a fuzzy BCK-filter of X, then*

$$\langle\mu\rangle(x) = \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}$$

for all $x \in X$.

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