MINIMAL CHARACTERISTIC ALGEBRAS FOR K-NORMALITY

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Abstract. A property p of identities of a fixed type τ is said to be hereditary if for every set I of identities having the property p, every consequence of I (under the usual derivation rules for identities) also has the property. A characteristic algebra for such a hereditary property is an algebra $\mathcal{A}$ such that for any variety V of type τ, we have $\mathcal{A} \in V$ iff every identity satisfied by V has the property p. Płonka has produced minimal characteristic algebras for a number of hereditary properties, including regularity, normality,uniformity, biregularity, outermost, and external-compatibility. We study characteristic algebras for the hereditary property of k-normality, for $k \geq 1$, which extends the usual normality property. For type (2) and the usual depth valuation of terms, we produce minimal characteristic algebras for $k = 1, 2, 3$ and 4.

1 Introduction and Definitions Throughout this paper we consider a type $\tau = (n_i)_{i\in I}$ of algebras and identities, with f_i an n_i-ary operation symbol of the type for each $i \in I$. We denote by $Id\,\tau$ the set of all identities of type τ. For any set Σ of identities of type τ and any class K of algebras of type τ, we will denote by $Mod\,\Sigma$ and $Id\,K$ respectively the class of all algebras satisfying the identities in Σ and the set of identities satisfied by all algebras in K.

Let p be a structural property of identities. We will say that a variety V of type τ has property p when the set $Id\ V$ of its identities has property p. A property p is said to be hereditary (for type τ) if for every set I of identities of type τ all having the property p, any consequence of I, derived according to the usual five derivation rules for identities, also has property p. Equivalently, p is hereditary when the set $p(\tau)$ of all type τ identities having property p is an equational theory. Let $V_p = Mod\ p(\tau)$. When $p(\tau)$ is an equational theory we have $Id\ V_p = p(\tau)$, which means that V_p is the smallest non-trivial variety to have property p.

Let p be a hereditary property. A characteristic algebra for p is an algebra $\mathcal{A}$ such that for any variety V of type τ, every identity of V has property p iff $\mathcal{A}$ is in V. This is equivalent to the property that the set $Id\ \mathcal{A}$ of identities satisfied by $\mathcal{A}$ is exactly the set $p(\tau)$ of type τ identities having property p. Another way to express this is that $\mathcal{A}$ is a generating algebra for the variety V_p, since we have $Id\ \mathcal{A} = p(\tau) = Id\ V_p$.

It is well known that the properties of regularity and normality of identities are hereditary, and these properties have been much studied (see for instance [C], [G1], [G2], [M], [P1]).

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In [P2] and [P3] Płonka produced characteristic algebras for some properties of identities, including normality, regularity, biregularity, external-compatibility, uniformity and rectangularity, and for some combinations of these. In [P3] the focus was on finding minimal characteristic algebras for these properties, that is, characteristic algebras of the smallest cardinality possible.

In this paper we study characteristic algebras for the property of k-normality, for $k \geq 1$. This property is a generalization of the well-studied property of normality of identities. To describe the concepts of normality and k-normality, we need some notation. We let $X = \{x_1, x_2, x_3, \dots\}$ be a countably infinite alphabet of variables, and we denote by $W_\tau(X)$ the set of all terms of type τ over X. There is an obvious distinction between variable terms x_j for $j \geq 1$, and compound or composite terms having the form $f_i(t_1, \dots, t_{n_i})$ for some $i \in I$. An identity $s \approx t$ of terms from $W_\tau(X)$ is said to be non-normal if s and t are different variables or if only one of s and t is a variable and the other is a compound term, and is called a normal identity otherwise. A variety V of type τ is called a normal variety if the set IdV of all its identities contains only normal identities. For any variety V there is a least normal variety $N(V)$ containing V, called the normalization of V.

Since the definition of a normal identity distinguishes the simplest kind of terms, the variables, from all others, it is in fact based on a very simple measurement of the complexity of terms. Another common measurement of complexity of a term is the depth of a term. Regarding a term as a tree diagram, the depth is the length of the longest path from the root of the term to a variable; inductively, variable terms and 0-ary operation symbols have depth 0, and a compound term of the form $f_i(t_1, \dots, t_{n_i})$ has depth $1 + max\{depth(t_j) \mid 1 \leq j \leq n_i\}$. Several other complexity measurements, including the minimum depth, the number of occurrences of operation symbols, and the number of occurrences of variables, have been studied in [DW1] and [DW2], along with a general theory of such complexity measurements.

Let v be a valuation, or complexity measurement, which assigns to each term $t \in W_\tau(X)$ a non-negative integer value $v(t)$. Let $k \geq 0$ be a natural number. An identity $s \approx t$ is called k-normal, with respect to the valuation v, if either $s = t$ or both $v(s)$ and $v(t)$ are $\geq k$. A non-trivial variety V will be called k-normal if all its identities are k-normal, and non-k-normal otherwise. The theory of k-normal varieties, with respect to any valuation v, was developed by Denecke and Wismath in [DW2]. They showed that for any variety V and $k \geq 0$, the variety $N_k(V)$ determined by all the k-normal identities of V is the least k-normal variety to contain V. Moreover, for $k = 1$ and v the depth valuation, the concepts of k-normal and the k-normalization coincide with the usual concepts of normality and normalization described above.

Denecke and Wismath also identified in [DW2] two conditions on the valuation v which ensure that the k-normality property of identities is hereditary. In particular, they showed that the usual depth valuation of terms ensures that the set of all k-normal identities is an equational theory, for all $k \geq 1$, and thus k-normality is a hereditary property. In [DW3] a characterization was given of the non-trivial minimal variety V_p when p is the property of k-normality. This characterization uses the concept of a k-constant algebra. An algebra $\mathcal{A}$ of type τ will be called a k-constant algebra (with respect to a valuation v) if there is an element $a_k \in A$ such that for every n and for all n-ary terms t with $v(t) \geq k$ and for all inputs $a_1, \dots, a_n$, we have $t^A(a_1, \dots, a_n) = a_k$. Then the collection of all k-constant algebras of type τ is precisely the smallest non-trivial k-normal variety of type τ. The special element a_k will be called a k-absorbing element, or simply an absorbing element, for the algebra $\mathcal{A}$.

Example 1.1 Let $k \geq 1$, and let $Id_k(\tau)$ be the set of all k-normal identities of type τ. Let X be a countably infinite alphabet of variables. Let $\mathcal{F}_\tau(X)$ be the absolutely free algebra of type τ over the alphabet X, and let $\mathcal{A}$ be the algebra $\mathcal{A} = \mathcal{F}_\tau(X)/Id_k(\tau)$. Since $Id_k(\tau)$ is a fully invariant congruence, $\mathcal{A}$ is a relatively free algebra of type τ. Its universe can be regarded as the set

$$W_k = \{t \in W_\tau(X) \mid v(t) < k\} \cup \{w_k\},$$

where w_k is any one term representing the equivalence class of all terms with value $\geq k$. It is clear that this algebra is k-constant; when v is the depth valuation and there are terms with value of $k-1$, this algebra is k-constant but not $(k-1)$-constant.

2 Characteristic Algebras for k-normality In this section we begin our study of characteristic algebras for the property of k-normality. Here k is a natural number, and k-normality is measured with respect to the usual depth measurement for terms. We will denote by V_k the smallest k-normal variety (of fixed type τ). This variety has a countably infinite generator, namely the relatively free algebra $\mathcal{F}_{V_k}(X)$ constructed in Example 1.1. This algebra is then a characteristic algebra for the property of k-normality. A basic question is whether is it possible to find a smaller, finite, characteristic algebra. For the case $k = 1$, when k-normality is the usual normality property, an affirmative answer to this question is known. In fact there is a two-element characteristic algebra for normality, as shown by Płonka in [P3].

Example 2.1 Let $\mathcal{A}_1 = (\{a, 0\}, (f_i^{A_1})_{i \in I})$ be the two-element zero algebra, defined by setting $f_i^{A_1}(a_1, \ldots, a_{n_i}) = 0$ for any index $i \in I$ and any inputs $a_1, \ldots, a_{n_i} \in \{a, 0\}$. This algebra satisfies all normal identities, but every non-normal identity is not satisfied in it. It is thus defined by the set of all normal identities of type τ, making it a characteristic algebra for the normality property. It is obviously also a minimal characteristic algebra, since any one-element algebra satisfies all identities of type τ, both normal and non-normal.

In Section 3 we shall show that for type (2) there is a finite minimal characteristic algebra for k-normality for $k \geq 2$, and produce minimal characteristic algebras for $k = 2, 3, 4$. To do this, we first give some Lemmas describing properties that any characteristic algebra $\mathcal{A}_k$ for k-normality must have, for any type τ. However, we assume that our type τ has at least one operation symbol whose arity is at least one. Most of the properties to be shown stem from the basic fact that $\mathcal{A}_k$ must satisfy all k-normal identities of type τ, but must break all non-k-normal identities.

Lemma 2.2 *If $\mathcal{A}_k$ is a characteristic algebra for k-normality, for $k \geq 2$, then it contains a k-absorbing element a_k.*

Proof: As noted above, a characteristic algebra for k-normality is a generator for the minimal variety V_k, which consists of all k-constant algebras of type τ. This means that the characteristic algebra must contain a k-absorbing element. ■

Next we introduce some notation for special kinds of terms to be considered.

Definition 2.3 The minimum depth of a term t is defined inductively by letting variable terms and nullary operation symbols have minimum depth 0, and compound terms of the form $f_i(t_1, \ldots, t_{n_i})$ have minimum depth $1+min\{depth(t_j) \mid 1 \leq j \leq n_i\}$. Just as the usual (maximum) depth measures the length of the longest path in the tree diagram representing a term, the minimum depth measures the length of the shortest path. A term t will be called a full term if the minimum and maximum depth of t are equal.

Definition 2.4 The shape of a term t of type τ is the term resulting from t by replacing every variable in t by the variable x.

The property of "having the same shape" is an equivalence relation on the set of all terms of type τ.

Definition 2.5 Let $i \in I$, and let $n_i \geq 1$. The full-shape term FS_i^j at depth j in the symbol f_i and the single variable x is defined inductively by

$$\begin{aligned} FS_i^0(x) &= x, \\ FS_i^{j+1}(x) &= f_i(FS_i^j(x), \ldots, FS_i^j(x)). \end{aligned}$$

We shall denoted the term operation induced on $\mathcal{A}_k$ by such a term by $(FS_i^j)^{A_k}$.

Lemma 2.6 *Let $\mathcal{A}_k$ be a characteristic algebra for k-normality, for $k \geq 2$. Then $\mathcal{A}_k$ contains $k+1$ distinct elements $a_0, \ldots, a_k$, with the property that for some $i \in I$, $a_{j+1} = (FS_i^1)^{A_k}(a_j) = (FS_i^{j+1})^{A_k}(a_0)$, for $0 \leq j \leq k-1$.*

Proof: Let $i \in I$. Since the identity $FS_i^{k-1}(x) \approx FS_i^k(x)$ is not k-normal, there is an element a_0 in $\mathcal{A}_k$ which breaks this identity. That is, $(FS_i^{k-1})^{A_k}(a_0) \neq (FS_i^k)^{A_k}(a_0) = a_k$. For $1 \leq j \leq k-1$, let $a_j = (FS_i^j)^{A_k}(a_0)$. Then the elements $a_0, a_1, \ldots, a_{k-1}$ must all be distinct from each other and from a_k, since otherwise we would not have $(FS_i^{k-1})^{A_k}(a_0) \neq a_k$. ■

Lemma 2.7 *Let $\mathcal{A}_k$ be a characteristic algebra for k-normality, for $k \geq 2$. Then the element a_{k-1} has the property that for any arity $n \geq 2$, any n-ary term t and any inputs $b_1, \ldots, b_n$, we have $t^{A_k}(b_1, \ldots, b_n) = a_k$ if any one of the inputs b_j is equal to a_{k-1}.*

Proof: Suppose without loss of generality that $b_1 = a_{k-1}$. Then for any inputs $b_2, \ldots, b_n$ we have $t^{A_k}(b_1, \ldots, b_n) = t^{A_k}(a_{k-1}, b_2, \ldots, b_n) = t^{A_k}((FS_i^{k-1})^{A_k}(a_0), b_2, \ldots, b_n)$. Since the latter is the output of a term of depth k, and the algebra $\mathcal{A}_k$ is k-constant, this output must equal a_k. ■

We shall refer to an element such as a_{k-1} with the property from Lemma 2.7 as a pre-absorbing element of the algebra.

Next we will show that we can assign to each element $b \in A_k$ a non-negative integer called its level. This level for specific elements of the algebra is analogous to the valuation, that is the depth, of the terms. We will denote by L_j the set of elements of the algebra $\mathcal{A}_k$ which have level j, for $0 \leq j \leq k$. We start by setting $L_k = \{a_k\}$, making the absorbing element a_k the only element to have level k. Now for any element $b \neq a_k$, we consider the set

$$T_b = \{t \in W_\tau(X) \mid t^{A_k}(b_1, \ldots, b_n) = b \text{ for some inputs } b_1, \ldots, b_n\}.$$

Since any term at depth $\geq k$ always gives output a_k and $b \neq a_k$, the set T_b is a subset of the set of terms of type τ of depth less than k. If T_b is empty, we say that b has level 0, and put $b \in L_0$. Otherwise, we define the level of b to be the maximum of the depths of the terms in T_b. In this way we assign a level to each element of $\mathcal{A}_k$, and we define $L_j = \{b \in A_k \mid b$ has level $j\}$, for $0 \leq j \leq k$.

Lemma 2.8 *Let $\mathcal{A}_k$ be a characteristic algebra for k-normality, for $k \geq 2$. For $0 \leq j \leq k-1$, the special element a_j has level j.*

Proof: By definition a_j is an output of the depth j term $FS_i^j(x)$, so that the level of a_j is at least j. Now suppose that a_j had level greater than j. Then $a_{j+1} = f_i^{A_k}(a_j, \ldots, a_j)$ would be the output of a term at depth at least $j+2$, so a_{j+1} would have level at least $j+2$. Similarly, a_{j+2} would have level at least $j+3$, and so on. But then a_{k-1} would have level greater than $k-1$, which is impossible since $a_{k-1} \neq a_k$. Therefore a_j must have level j. ■

The next Lemma shows that performing any operation of the algebra on elements at given levels increases the level by at least one.

Lemma 2.9 *Let $i \in I$, and let $b_1, \ldots, b_{n_i}$ be elements of $\mathcal{A}_k$, with b_p at level j_p respectively. Then $f_i^{A_k}(b_1, \ldots, b_{n_i})$ has level at least one greater than the maximum of the levels of $b_1, \ldots, b_{n_i}$.*

Proof: Since each element b_p is at level j_p, there is a term t_p at depth j_p and some inputs for which t_p gives output b_p. Then $f_i^{A_k}(b_1, \ldots, b_{n_i})$ is the output of the term $f_i(t_1, \ldots, t_{n_i})$, which in turn has depth equal to $1 + max\{\text{depth } t_p \mid 1 \leq p \leq n_i\}$. Thus the element $f_i^{A_k}(b_1, \ldots, b_{n_i})$ has level at least this number. (Note that the level could be higher, since there could be other ways to produce the output $f_i^{A_k}(b_1, \ldots, b_{n_i})$.) ■

Lemma 2.10 *Let s and t be two full terms of type τ, both at depth $k-1$ and both using the same set of variables $x_1, \ldots, x_m$ for some m. Then any elements $c_1, \ldots, c_m$ from $\mathcal{A}_k$ which break the identity $s(x_1, \ldots, x_m) \approx t(x_1, \ldots, x_m)$ must be at level 0 in $\mathcal{A}_k$.*

Proof: Since $s \approx t$ is a non-k-normal identity of type τ, it cannot hold in $\mathcal{A}_k$, and there must be some elements $c_1, \ldots, c_m$ in $\mathcal{A}_k$ such that $s^{A_k}(c_1, \ldots, c_m) \neq t^{A_k}(c_1, \ldots, c_m)$. Suppose that for some $1 \leq p \leq m$ we have c_p at level $j > 0$. Then there is an n-ary term u, for some $n \geq 1$, of depth at least one and some inputs $b_1, \ldots, b_n$ such that $u^{A_k}(b_1, \ldots, b_n) = c_p$. Now we can replace x_p by the term u in the identity $s \approx t$, to create a new identity $s' \approx t'$, with both sides at depth k or higher. Now using the inputs $c_1, \ldots, c_m, b_1, \ldots, b_n$ in this identity makes both sides come out to a_k, contradicting the fact that the inputs $c_1, \ldots, c_m$ break the identity $s \approx t$. ■

Next we introduce some constructions which will tell us more about what elements are required in $\mathcal{A}_k$. In each case we consider an identity which must be broken by $\mathcal{A}_k$.

Construction 1:
Let τ be a type containing at least one operation symbol f_i of arity at least 2. We construct a series of terms of this type, using the two variables x and y. First, let $t_{0,1}(x,y) = x$ and $t_{0,2}(x,y) = y$. Inductively, let

$$t_{j+1,1}(x,y) = f_i(t_{j,2}(x,y), t_{j,1}(x,y), \ldots, t_{j,1}(x,y))$$

and

$$t_{j+1,2}(x,y) = f_i(t_{j,1}(x,y), t_{j,2}(x,y), t_{j,1}(x,y), \ldots, t_{j,1}(x,y)).$$

Now consider the identity $t_{k-1,1}(x,y) \approx t_{k-1,2}(x,y)$. Both of these terms are full terms at depth $k-1$, so by Lemma 2.10, there must exist two elements c_0 and d_0 at level 0 in $\mathcal{A}_k$ to break this identity. The construction shows that we must have $c_0 \neq d_0$ and $t_{p,1}^{A_k}(c_0, d_0) \neq t_{p,2}^{A_k}(c_0, d_0)$ for all levels $1 \leq p \leq k-1$. Moreover, for $0 \leq p \leq k-2$ each pair of elements $t_{p,1}^{A_k}(c_0, d_0)$ and $t_{p,2}^{A_k}(c_0, d_0)$ must be at level p: they are outputs of terms at depth p, and hence at level at least p, but they cannot have higher level or we would get $t_{k-1,1}^{A_k}(c_0, d_0) = a_k = t_{k-1,2}^{A_k}(c_0, d_0)$. Notice however that at level $k-1$, all we need is that $t_{k-1,1}^{A_k}(c_0, d_0)$ and $t_{k-1,2}^{A_k}(c_0, d_0)$ are both at level at least $k-1$, but not both at level k. This proves the following result.

Lemma 2.11 *Let τ be a type with at least one operation symbol of arity at least two. Let $\mathcal{A}_k$ be a characteristic algebra for k-normality, for $k \geq 2$. Then for each $0 \leq j \leq k-2$, there must be at least two elements in $\mathcal{A}_k$ at level j.*

Corollary 2.12 *Let τ be a type with at least one operation symbol of arity at least two, and let $k \geq 2$. If $\mathcal{A}_k$ is a characteristic algebra for k-normality, then $\mathcal{A}_k$ has cardinality at least $2k$.*

Construction 2:
Let τ be a type containing at least one operation symbol f_i of arity at least two. We consider the identity

$$f_i(FS_i^{k-2}(x), FS_i^{k-2}(y), \ldots, FS_i^{k-2}(y)) \approx f_i(FS_i^{k-2}(y), FS_i^{k-2}(x), \ldots, FS_i^{k-2}(x)).$$

This identity consists of two full terms at depth $k-1$, and by Lemma 2.10 must be broken by two elements d_0 and e_0 at level 0. Note that we must have $d_0 \neq e_0$, and also that $(FS_i^p)^{A_k}(d_0)$ and $(FS_i^p)^{A_k}(e_0)$ must be distinct elements at level p, for each $1 \leq p \leq k-2$. This gives another proof of Lemma 2.11.

3 Examples for Type (2) Now we turn to type (2), where we have one binary operation symbol in our type. Throughout this section, we shall follow the usual convention of indicating this binary operation by juxtaposition. We shall also refer to breaking an identity between two terms as 'separating' the two terms, in this and subsequent sections. More generally, separating a set of terms will mean finding inputs which break the identity $s \approx t$ for any two distinct terms s and t in the set.

Theorem 3.1 *For type* (2)*, there is a minimal characteristic algebra of size* 4 *for the property of* 2*-normality.*

Proof: Let $\mathcal{A}_2$ be the type two algebra with universe set $A_2 = \{a_0, b_0, a_1, a_2\}$, with one binary operation, to be denoted by juxtaposition, given by the table below.

A_2	a_0	b_0	a_1	a_2
a_0	a_1	a_1	a_2	a_2
b_0	a_2	a_1	a_2	a_2
a_1	a_2	a_2	a_2	a_2
a_2	a_2	a_2	a_2	a_2

We will show that this algebra is characteristic for 2-normality, for type (2). Then by Corollary 2.12, it must be a minimal characteristic algebra. The algebra has an absorbing element a_2, a pre-absorbing element a_1 at level 1, and two elements a_0 and b_0 at level 0. It is clear from the table that for any term t of depth 2 or more, the term function induced by t on $\mathcal{A}_2$ has constant value a_2, so that the algebra satisfies all 2-normal identities. To show that this algebra does not satisfy any non-2-normal identity, it suffices to show that we can separate each of the depth 0 or 1 terms x, y, xx, yy, xy and yx from each other, and from any term of depth 2 or more, using elements of $\mathcal{A}_2$. We can certainly separate any of these terms from a term of depth 2 or more, by using $x = y = a_0$. We can separate x from any of the other five terms by using $x = a_0$, and similarly for y. We can separate xx from each of xy, yx and yy using $x = a_0$ and $y = a_1$. Similarly we can separate yy from xy and yx. Finally, we can separate xy and yx by using $x = a_0$ and $y = b_0$. ■

Lemma 3.2 *Let type $\tau = (2)$, and let $k = 3$. Then any characteristic algebra $\mathcal{A}_3$ for 3-normality must contain at least seven elements.*

Proof: We know from Corollary 2.12 that any characteristic algebra $\mathcal{A}_3$ must contain at least six elements, with two elements at level 0, two at level 1, and one each at levels 2 and 3. Suppose that $\mathcal{B}_3$ is a six element characteristic algebra, with $L_0 = \{a_0, b_0\}$, $L_1 = \{a_1, b_1\}$, $L_2 = \{a_2\}$ and $L_3 = \{a_3\}$. From Construction 2, we need a_0 and b_0 to break the identity $(xx)(yy) \approx (yy)(xx)$, and hence we need $a_0a_0 \neq b_0b_0$ at level 1. Since by Lemma 2.6 we have $a_0a_0 = a_1$, we therefore must have $b_0b_0 = b_1$. From Construction 1, we also need to break $(xy)(yx) \approx (yx)(xy)$ using a_0 and b_0, and hence we must have $a_0b_0 \neq b_0a_0$ as two distinct elements at L_1. This gives two possible cases to consider.

Case 1: $a_0b_0 = a_1$ and $b_0a_0 = b_1$:
Then there are no level 0 elements in $\mathcal{A}_3$ which break the identity $(xy)(yx) \approx (xx)(yy)$.

Case 2: $a_0b_0 = b_1$ and $b_0a_0 = a_1$:
Dually, there are no level 0 elements in $\mathcal{A}_3$ which break the identity $(xy)(yx) \approx (yy)(xx)$.

This shows that having only two elements at level 0 is not enough to break all the necessary identities. ■

Now we produce an example of a seven-element minimal characteristic algebra for 3-normality, for type (2). Our proof that this algebra is characteristic uses the concept of the shape of a term, from Definition 2.4. Note that for type (2), we have the three shapes $x(xx)$, $(xx)x$ and $(xx)(xx)$ for terms of depth 2.

Theorem 3.3 *The seven-element type* (2) *algebra* $\mathcal{A}_3$ *with operation table shown below is a minimal characteristic algebra for* 3*-normality.*

A_3	a_0	b_0	c_0	a_1	b_1	a_2	a_3
a_0	a_1	a_1	a_1	a_2	a_2	a_3	a_3
b_0	a_2	a_1	b_1	a_2	a_3	a_3	a_3
c_0	b_1	a_1	b_1	a_2	a_3	a_3	a_3
a_1	a_2	a_3	a_3	a_2	a_2	a_3	a_3
b_1	a_2	a_2	a_2	a_3	a_2	a_3	a_3
a_2	a_3	a_3	a_3	a_3	a_3	a_3	a_3
a_3	a_3	a_3	a_3	a_3	a_3	a_3	a_3

Proof: For convenience, the operation table shown for our algebra has the elements separated by levels. It follows from the construction that for any binary term t of depth three or more, and any inputs a and b from the algebra, we have $t^{A_3}(a, b) = a_3$. This shows that this algebra satisfies any 3-normal identity of type (2). We must show now that the algebra does not satisfy any non-3-normal identity $s \approx t$, where at least one of the terms s and t has depth less than three. Without loss of generality, we take the depth of s to be less than or equal to the depth of t, and hence less than three.

We first consider the case that s and t have different depths. If s has depth 0 and t has depth one or more, then we can separate s and t by using a_0 for all the variables in the identity. The same substitution works if s has depth one and t has depth more than one. This leaves only the case that s has depth 2 and t has depth three or more. Then s must have one of the shapes $x(xx)$, $(xx)x$ or $(xx)(xx)$, and we can check directly from the table that there is a value to use for x in each of these which gives a result of a_2, while the depth three or more term t must result in a_3.

This leaves only the case that terms s and t have the same depth, where this depth is less than three. Clearly a depth zero identity $x \approx y$ can be broken using any two elements of $\mathcal{A}_3$, so we may take the depth to be one or two. For the depth one case, we note that the subset $\{a_1, b_1, a_2, a_3\}$ is a subalgebra of our algebra which is isomorphic to the algebra $\mathcal{A}_2$ from Theorem 3.1, under an isomorphism which decreases the subscript of each element by one. This means that the depth one identity $s \approx t$ can be broken using the same elements as used in the proof for $k = 2$, again with the subscripts adjusted.

So we suppose that both s and t have depth two. There are three shapes of terms to be considered, namely $x(xx)$, $(xx)x$ and $(xx)(xx)$. By inspection from the table, we see that using $x = b_0$ separates the second of these from the other two, while $x = c_0$ separates the first from the other two. Thus the three shapes are separated from each other. This means that if the two terms s and t have different shapes, then we can break the identity $s \approx t$.

Finally, we must consider the terms within each shape. It will suffice to consider only identities which use at most two variables x and y, since we can break an identity with more than two variables by using some duplicate inputs. Moreover, we can separate any term which contains only one of the two variables from any term containing both; as noted above, there is a choice of inputs which will make a one-variable term come out to a_2, while using a_3 for the other variable guarantees a result of a_3 on the other side. Thus we need consider only terms s and t which have the same shape and each contain both variables x and y. There are six such terms for each of the shapes $x(xx)$ and $(xx)x$, and fourteen such terms for the shape $(xx)(xx)$. A case-by-case examination of all possible pairs of level zero elements to use for x and y verifies that all pairs are separated. ■

4 Embedding One Characteristic Algebra in the Next We saw in Example 2.1 that the two-element zero algebra is a minimal characteristic algebra for 1-normality (ordinary normality). This algebra appears in the lower right corner of the table from Theorem 3.1 for the minimal characteristic algebra $\mathcal{A}_2$ for 2-normality, and as noted in the proof of Theorem 3.3 we have an isomorphic copy of $\mathcal{A}_2$ appearing in the lower right corner of $\mathcal{A}_3$. In this section we show how each characteristic algebra $\mathcal{A}_k$ can be embedded in the next one, $\mathcal{A}_{k+1}$. These next few results hold for any type τ.

Definition 4.1 Let $k \geq 2$, and let $\mathcal{A}_k$ be a characteristic algebra for k-normality. For each $0 \leq p \leq k$, we set

$$S_p \quad = \quad \bigcup_{j=p}^{k} L_j.$$

Lemma 4.2 *Let $k \geq 2$ and $0 \leq p \leq k$. The subset S_p is a subalgebra of $\mathcal{A}_k$.*

Proof: This follows immediately from Lemma 2.9. ■

Lemma 4.3 *Let $k \geq 2$ and let $0 \leq p \leq k$. Then the algebra S_p is characteristic for $(k-p)$-normality.*

Proof: Let t be an n-ary term of depth at least $k-p$. Then for any inputs $x_1, \ldots, x_n$ from the set S_p, the output $t^{A_k}(x_1, \ldots, x_n)$ will have level k (in A_k), so it must be a_k. This means that S_p satisfies any $(k-p)$-normal identity of type τ.

Now let $s \approx t$ be a non-$(k-p)$-normal identity, with s a term whose depth is less than $k-p$ and less than or equal to the depth of t. Let s' be the term constructed from s by replacing each occurrence of each variable x by the full-shape term $FS_i^p(x)$ using any operation symbol f_i, depth p and variable x. Note that the depth of the term s' is p plus the depth of s. Let t' be similarly defined from t. Then the identity $s' \approx t'$ has the property that the depth of s' is less than k and less than or equal to the depth of t'. This identity must then be broken in $\mathcal{A}_k$, so there exists some choice of inputs b_x to use for the variables x in s' and t' so that the outputs are different. But then the elements $(FS_i^p)^{A_k}(b_x)$ are at level p, so are elements of S_p, and they serve to break the identity $s \approx t$ in S_b. ■

This Lemma gives us a way to build a (minimal) characteristic algebra for $(k+1)$-normality, once we have one for k-normality. We begin with the universe set A_k, but relabelled so that elements at level j in A_k are now at level $j+1$. Then we add a set L_0 of new level 0 elements, which must all multiply to give products at level one or higher. This includes the level 0 element a_0, as described in Lemma 2.6, so that $a_0 a_0 = a_1$. We take $A_{k+1} = A_k \cup L_0$.

It follows from this construction that for any term t of depth $k+1$ or more, the output from t for any inputs from the universe set is a_{k+1}. This shows that this set satisfies every

$(k+1)$-normal identity. We must now consider how to ensure that the algebra does not satisfy any non-$(k+1)$-normal identity $s \approx t$, where at least one of the terms s and t has depth less than $k+1$. Without loss of generality, we take the depth of s to be less than or equal to the depth of t, and hence less than $k+1$.

We first consider the case that s and t have different depths. In this case it will suffice to consider only the "shapes", that is, to assume that s and t each contain only one variable x. We need to ensure that for any depths $0 \leq m \neq n \leq k$, we separate any shape at depth m from any shape at depth n, and that we separate any shape at depth m from a term of depth $k+1$. In particular, this means that for any shape of depth k or less, there must be a choice of input to use for the single variable that makes the output different from the absorbing element a_{k+1}.

Next we consider the case that s and t have the same depth, less than or equal to k. If this depth is less than k, then we use the fact that we have embedded a characteristic algebra for k-normality to conclude that there are elements (of depth one or more) which break the identity. This leaves only the case that s and t both have depth k. Here we have to separate all the distinct shapes at depth k from each other.

Then finally we must consider the terms within each shape, at depth k. It will suffice to consider only identities which use at most two variables x and y, since we can break an identity with more than two variables by using some duplicate inputs. Moreover, we can separate any term which contains only one of the two variables from any term containing both; as noted above, there is a choice of inputs which will make a one-variable term come out to a value other than a_{k+1}, while using a_k for the other variable guarantees a result of a_{k+1} on the other side. Thus we need consider only terms s and t which have the same shape and each contain both variables x and y.

Overall, we must ensure that three conditions are met:

1. We must separate any two shapes at different depths less than $k+1$ from each other, and from any term at depth $k+1$.

2. We must separate all the shapes at depth k from each other.

3. For each shape at depth k, we must separate all two-variable terms of that shape from each other.

These three things can always be done with a finite number of terms at level zero. This proves that

Theorem 4.4 *For any $k \geq 1$, there is a finite characteristic algebra for k-normality.*

Now we return to our study of type (2) algebras. Using the embedding technique, we can put tighter restrictions on the size of a minimal characteristic algebra $\mathcal{A}_4$ for $k=4$. Since any such algebra must contain a characteristic algebra for $k=3$, for which the minimum cardinality is 7, and there must be at least two elements in L_0, $|\mathcal{A}_4| \geq 9$. But more level 0 elements are needed to break all non-4-normal identities. There are twenty-one one-variable terms of depth 3, which by Lemma 2.10 can only be separated by inputs from L_0. Assuming there are only two elements at level 3 and above to differentiate these terms (an absorbing element and a pre-absorbing), at least five different inputs are required to separate all these terms. This can be seen by associating with each term a string consisting of its outputs on all level 0 inputs. Saying two terms are separated by the algebra is equivalent to saying they have different strings. With only two letters available, five characters is the shortest length possible to produce twenty-one distinct strings. Thus, L_0 of $\mathcal{A}_4$ must contain at least five elements. This suggests that $|\mathcal{A}_4| \geq 12$.

However, if the embedded $k = 3$ algebra is altered so that it contains two pre-absorbing elements instead of only one, then we would have three elements to differentiate depth 3 terms, and similar reasoning requires that level 0 contain just three elements. This changes the lower bound on the size of a generator: now we have $|\mathcal{A}_4| \geq (7+1)+3 = 11$. Adding three pre-absorbing elements yields a minimum size of twelve, adding four pre-absorbing elements also gives size twelve, and with five the minimum size is thirteen; and the size continues to grow as more pre-absorbing elements are added. Therefore, the smallest possible size for $\mathcal{A}_4$ is eleven, if we have two pre-absorbing elements, or twelve if we have three pre-absorbing elements.

In the next section we shall show that an eleven-element characteristic algebra, using two pre-absorbing elements, is not possible. This will leave our minimum size at twelve or more.

Theorem 4.5 *The twelve-element type* (2) *algebra* $\mathcal{A}_4$ *with operation table shown below is a minimal characteristic algebra for* 4*-normality.*

Proof: That this algebra meets the three requirements described above was verified by a computer program which calculates the output of each of the terms of the twenty-one different possible shapes.

Minimality of this generator follows from the argument above, and the proof in the next section that there is no generator of size eleven. ■

$\mathcal{A}_4$	a_0	b_0	c_0	d_0	e_0	a_1	b_1	c_1	a_2	b_2	a_3	a_4
a_0	a_1	b_1	a_1	a_1	a_1	a_3	a_2	a_3	a_4	a_3	a_4	a_4
b_0	b_1	a_1	a_1	c_1	c_1	a_2	a_2	b_2	a_3	a_4	a_4	a_4
c_0	b_1	a_1	b_1	b_2	b_1	a_3	b_2	b_2	a_3	a_4	a_4	a_4
d_0	a_1	c_1	c_1	b_1	c_1	a_2	a_2	b_2	a_4	a_3	a_4	a_4
e_0	c_1	a_1	c_1	b_1	c_1	b_2	a_2	a_2	a_4	a_3	a_4	a_4
a_1	a_2	a_3	b_2	a_3	a_2	a_2	a_2	a_2	a_3	a_3	a_4	a_4
b_1	a_2	a_2	a_2	b_2	a_3	a_3	a_2	b_2	a_4	a_3	a_4	a_4
c_1	b_2	b_2	b_2	b_2	a_2	b_2	a_2	b_2	a_4	a_3	a_4	a_4
a_2	a_3	a_4	a_4	a_3	a_3	a_3	a_3	a_4	a_3	a_3	a_4	a_4
b_2	a_4	a_3	a_3	a_4	a_3	a_3	a_4	a_4	a_4	a_3	a_4	a_4
a_3	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4
a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4

We have now constructed minimal characteristic algebras, for type (2), for k-normality for $k = 1, 2, 3, 4$. The construction for $k = 4$ was quite lengthy (including the proofs in Section 5) and it appears that it will be quite difficult to find a characteristic algebra for $k = 5$, and to prove that such an algebra is minimal. By Lemma 4.3, such an algebra would contain a subalgebra isomorphic to $\mathcal{A}_4$, of size at least 12. In addition, we would need enough new level 0 elements to separate all the depth 4 shapes. There are 651 such shapes, so if our algebra has exactly one absorbing and one pre-absorbing element, we need at least 10 level 0 elements to give $2^{10} > 651$. This gives a bound of size 22. However a smaller algebra might be possible if the $\mathcal{A}_4$ subalgebra was modified to give more separating elements.

5 Ruling Out Size 11 In this section we show that there cannot be a characteristic algebra for 4-normality which contains eleven elements, two of which are at level 3. We proceed by contradiction, by assuming that such an algebra exists and deducing a series of (eventually contradictory) constraints on its elements. We let $\mathcal{A}_4$ be such an algebra. Note that for this algebra the set L_0 of level 0 elements has size three, while L_3 has size two and L_4 has size one. This leaves five elements left to use for levels 1 and 2.

We begin with some basic properties that any 4-normal algebra containing eleven elements would have to have. First we recall that by Lemma 2.10, any regular identity of depth 3 (that is, any identity between terms that use the same set of variables and are both depth 3) must be broken via a substitution by level 0 elements. As previously noted, we need only worry about breaking all identities between one- and two-variable terms. For any identity where at least one term uses two variables, substituting the same element for both variables will not break the identity. Thus any regular identity between depth 3 terms on two variables must be broken by a pair (x_0, y_0) of two distinct level 0 inputs. Since there are exactly three level 0 elements, there exist six such pairs, which we shall refer to as the six level 0 input pairs.

Lemma 5.1 *In any* 11-*element characteristic* 4-*normal algebra, the squares of level* 0 *elements are all distinct.*

Proof: For the full-shape term of depth 3, there are fourteen terms that use two variables and are composed entirely of the squares xx and yy (for example, $[(xx)(yy)][(yy)(xx)]$ and $[(yy)(yy)][(yy)(xx)]$). If all level 0 squares were equal, then none of the identities between terms of this type could be broken, so the algebra would not be 4-normal. If the squares of any two level 0 elements a_0 and b_0 are equal, then the two level 0 input pairs (a_0, b_0) and (b_0, a_0) do not break any identities between terms of this type. Moreover, if c_0 is the third element of level 0, we have $t^{A_4}(a_0, c_0) = t^{A_4}(b_0, c_0)$ and $t^{A_4}(c_0, a_0) = t^{A_4}(c_0, b_0)$ for all terms t in this set. This effectively leaves only two of the six level 0 input pairs to break all identities of this type; but this is insufficient, since at least three pairs are required to separate fourteen terms. ■

Lemma 5.2 *For any* 11-*element characteristic* 4-*normal algebra,* $a_0 b_0 \in L_1$ *for all* $a_0, b_0 \in L_0$.

Proof: We show first that all squares $a_0 a_0$ of level 0 elements must be level 1 elements. There are twenty-one one-variable depth 3 terms, all of which contain (xx) as a subterm. If the square of any level 0 element was at level 2 or higher, then by Lemma 2.9 using this element for x would output the absorbing element a_4 for all twenty-one terms. This would leave only two level 0 elements left to substitute for x to separate these terms, which is insufficient.

For the non-square products of level 0 elements, we note that there are one hundred and ten terms of depth 3 that contain both (xy) and (yx) as subterms. Any identity between terms of this sort must be broken with a substitution by level 0 elements. If there were two level 0 elements whose product was at level 2 or higher, using this pair as input would force all one hundred and ten terms to output the absorbing element. Thus this input pair would not separate any of these terms. This leaves five level 0 input pairs to separate all of these terms; but with three elements at levels 3 or 4, five pairs can separate at most $3^5 = 81$ terms. ∎

These two Lemmas shows that there are at least three elements at level 1, each of which is a square of a distinct element from level 0. Combining this with the fact from Lemma 2.11 that every level must contain at least two elements, we can now completely describe the level assignments of our algebra. We have three elements at each of levels 0 and 1, two elements at each of levels 2 and 3, and one absorbing element at level 4.

Lemma 5.3 *In any* 11*-element characteristic* 4*-normal algebra, the square of any level* 1 *element must be at level* 2*, and there are two level* 1 *elements whose squares are distinct.*

Proof: Consider the identity $[(xx)(xx)][(yy)(yy)] \approx [(yy)(yy)][(xx)(xx)]$. Because it is regular, this identity must be broken via substitution by level 0 elements, and since by the previous Lemma squares of level 0 elements must be level 1 elements, this implies that there must be at least two different squares of level 1 elements at level 2, to use to separate $(xx)(xx)$ and $(yy)(yy)$.

For the other claim, we show by contradiction that none of the three level 1 elements can have a square which is at level 3 or higher. Let us denote the three level 0 elements by a_0, b_0 and c_0 and their squares by a_1, b_1 and c_1 respectively, and suppose without loss of generality that a_1a_1 is at level 3 or higher. We note that of the twenty-one one-variable depth 3 terms there are nine that contain $(xx)(xx)$ as a subterm: one term with each of x, xx, $x(xx)$, and $(xx)x$ on the left and right, and $[(xx)(xx)][(xx)(xx)]$. By Lemma 2.9, any of these nine terms will have $t^{A_4}(a_0) = a_4$, so a_0 will not help to separate these terms. For each of the nine terms, we consider the output pair $(t^{A_4}(b_0), t^{A_4}(c_0))$. Separating the nine terms is equivalent to producing (at least) nine distinct such output pairs. Since for any t the two outputs from level 0 inputs must be at level 3 or 4, we have three possible values to use and hence exactly nine pairs, and every one of the nine output pairs must occur. But one of these output pairs is the pair (a_4, a_4), and hence there must be a term t for which $t^{A_4}(x_0)$ is a_4 for all x_0 in level 0. But now the algebra satisfies the identity $t \approx s$, where s is any term of depth 4, which violates 4-normality. ∎

Lemma 5.4 *In any* 11*-element characteristic* 4*-normal algebra,* $(aa)a \in L_2$ *and* $a(aa) \in L_2$ *for any* $a \in L_0$*.*

Proof: Of the twenty-one one-variable depth 3 terms, there are nine that contain $x(xx)$ as a subterm and nine with $(xx)x$ as a subterm. The result follows by an argument similar to the one for the second part of Lemma 5.3. ∎

Lemma 5.5 *In any* 11*-element characteristic* 4*-normal algebra, the two level* 2 *elements have different squares and do not commute with each other.*

Proof: To see the first part, consider the identity $[x(xx)][x(xx)] \approx [(xx)x][(xx)x]$. By the previous Lemma, both the left and right sides are squares of level 2 elements; if both squares were equal this identity could not be broken.

To see that the non-square products are distinct, consider again the identity from the first part of the proof of Lemma 5.3. ■

Lemma 5.6 *In any* 11*-element characteristic* 4*-normal algebra, there exists an element* $m \in L_2$ *such that either* $wm = mm$ *for any* $w \in L_2$ *or* $mw = mm$ *for any* $w \in L_2$*.*

Proof: Let p and q be the elements of level 2, and suppose that $pq \notin \{pp, qq\}$ and $qp \notin \{pp, qq\}$. From the previous Lemma $pp \neq qq$ and $pq \neq qp$. Thus, since all products of level 2 elements are at level 3 or higher, we have at least four elements at level 3 or higher. But as noted above $|L_3| + |L_4| = 3$, so it must be the case that either $pq = pp$ or qq, or $qp = pp$ or qq, proving the result. ■

Lemma 5.7 *For any* 11*-element characteristic* 4*-normal algebra, any level* 2 *element with the property of the special element* m *from Lemma 5.6 is the square of exactly one level* 1 *element.*

Proof: Suppose there are two level 1 elements p and q with $pp = qq = m$, where m has the property that $wm = mm$ for all $w \in L_2$. Consider the three terms $[x(xx)][(xx)(xx)]$, $[(xx)x][(xx)(xx)]$, and $[(xx)(xx)][(xx)(xx)]$. From Lemma 5.1, we know $p = p_0p_0$ and $q = q_0q_0$, for two distinct level 0 elements p_0 and q_0. Substituting either of p_0 or q_0 for x then makes our three terms equal. Substituting the remaining level 0 element for x causes all of $(xx)(xx)$, $x(xx)$, and $(xx)x$ to output a level 2 element. Since $|L_2| = 2$, two of these terms must give the same output. If s and t are those two terms, then $s[(xx)(xx)]$ and $t[(xx)(xx)]$ agree on that input, and the identity $s[(xx)(xx)] \approx t[(xx)(xx)]$ is satisfied by the algebra, violating 4-normality.

If $mw = mm$ for all $w \in L_2$ instead, the result follows from a similar argument using the terms $[(xx)(xx)][x(xx)]$, $[(xx)(xx)][(xx)x]$, and $[(xx)(xx)][(xx)(xx)]$. ■

A consequence of Lemmas 5.7 and 5.3 is that the special element m is unique in level 2. We shall henceforth assume that this element m has the property that $wm = mm$ for all $w \in L_2$; the arguments for the other case are analogous.

We showed in Lemma 5.2 that any product of two level 0 elements must be in level 1. Now we consider whether the product of two level 1 elements is always at level 2, or whether some products can "jump" a level, to level 3 or 4. With the results we have so far, we can now rule out the existence of a characteristic 4-normal algebra in which all products between level 1 elements are at level 2. To do this, we need to consider the full terms on two variables. Such terms can be expressed as st, where s and t are terms of depth 2 that use variables x and/or y, taken from the following list of terms:

$$\begin{array}{llll} (xx)(xx), & (yx)(xx), & (xy)(xx), & (xx)(yx), \\ (xx)(xy), & (yy)(xx), & (yx)(yx), & (yx)(xy), \\ (xy)(xy), & (xy)(yx), & (xx)(yy), & (yy)(yx), \\ (yy)(xy), & (yx)(yy), & (xy)(yy), & (yy)(yy). \end{array}$$

Let $s \neq t$ be two of these full depth 2 terms. The regular depth 3 identity $st \approx ts$ must be broken by level 0 elements. This means that for any two such terms $s \neq t$, there must be a level 0 input pair (p_0, q_0) making $s^{A_4}(p_0, q_0) \neq t^{A_4}(p_0, q_0)$. Let t^d denote the dual of t, that is, the term obtained from t by interchanging the variables x and y. If t evaluates to the same output on dual substitutions (that is, if $t^{A_4}(a_0, b_0) = t^{A_4}(b_0, a_0)$ for all a_0,$b_0 \in L_0$), then t and t^d are equal on all level 0 substitutions and $tt^d \approx t^d t$ is satisfied. This violates 4-normality, so cannot happen.

Lemma 5.8 *There does not exist an* 11*-element characteristic algebra for* 4*-normality in which every product of two level* 1 *elements is at level* 2*.*

Proof: Suppose that such an algebra exists. To deduce a contradiction, we consider a sixteen-by-six table of outputs for the algebra, constructed as follows. Each row of the table corresponds to one of the sixteen depth 2 terms listed above; each column of the table corresponds to one level 0 input pair (p_0, q_0). The entry in the row for t and the column for (p_0, q_0) gives the output $t^{A_4}(p_0, q_0)$. By our assumption, each entry in the table is at level 2, so is either the special element m or the remaining level 2 element n. As noted above, for any two terms s and t from the list, there must be a column in which the two terms have a different output.

We first claim that in our algebra no depth 2 term t can evaluate to the element m more than twice; that is, no row of the table can contain more than two m entries. To show this, we define for any term t from the list above of depth 2 terms, the set S_t of terms of the form rt, where r is a depth 2 full-shape term using both variables x and y. This set of terms has fourteen elements, which must be separated from each other using the six level 0 input pairs. By assumption, the output of any such term r on a level 0 input pair has level 2, so is one of the two level 2 elements m and n. With two outputs available, it takes at least four input pairs to ensure $2^4 > 14$ distinct outcomes to separate the fourteen terms. Now suppose that for some level 0 input pair (p_0, q_0) we have $t^{A_4}(p_0, q_0) = m$. Then no matter whether $r^{A_4}(p_0, q_0)$ takes the value n or m, all terms in the set S_t have value $nm = mm = m$ on this input pair. Thus any level 0 input pair which gives t a value of m is essentially useless in separating the fourteen terms from S_t. To ensure at least four useful input pairs, we can have at most two input pairs which give t a value of m.

Using this claim, we consider how many rows are available for use in our table. Based on six level 0 input pairs, there are six different ways that a term can evaluate to m exactly once, and $\binom{6}{2} = 15$ ways to evaluate to m exactly twice. This gives twenty-one possible rows available for our output table. But three of the fifteen ways to have two m entries are instances where a term is equal on dual substitutions, which we noted above cannot occur. Excluding these rows leaves us with eighteen possible rows.

Now, by Lemmas 5.1, 5.2, and 5.3, m occurs as the square of exactly one level 1 element, say m_1, and m_1 in turn is the square of exactly one level 0 element, m_0. Of the six different level 0 input pairs, the term $(xx)(xx)$ evaluates to m exactly twice – on inputs where m_0 is substituted for x – and $(yy)(yy)$ outputs m on the two dual pairs.

We shall say that a term u overlaps a term t if t outputs m exactly once, on a level 0 input pair where u outputs m. If there exist terms s and t such that $(xx)(xx)$ overlaps both s and t, then $s[(xx)(xx)] \approx t[(xx)(xx)]$ will hold in the algebra: both $s[(xx)(xx)]$ and $t[(xx)(xx)]$ will output mm on the inputs where x is replaced by m_0, and output the other square nn for all other inputs. Note that this situation occurs iff $(yy)(yy)$ overlaps s^d and t^d. This means that to preserve 4-normality, $(xx)(xx)$ and $(yy)(yy)$ can each overlap at most one term. Therefore, there can only be four terms that evaluate to m once, two that are not overlapped by either of $(xx)(xx)$ and $(yy)(yy)$, one overlapped by $(xx)(xx)$, and one overlapped by $(yy)(yy)$ (the latter two will be dual terms).

We are now left with exactly $12+4 = 16$ possible arrangements for rows in our output table, to separate our sixteen terms. There must be two terms that are not overlapped by either of $(xx)(xx)$ and $(yy)(yy)$, one term overlapped by $(xx)(xx)$, one overlapped by $(yy)(yy)$, and twelve terms exhausting all ways to output m exactly twice on non-dual substitutions. In particular, there must be a term r that overlaps terms u and v, one of which is overlapped by $(xx)(xx)$, and the other overlapped by neither $(xx)(xx)$ nor $(yy)(yy)$ (note that this means that u and v output m on non-dual substitutions). But then the non-4-normal identity $ur \approx vr$ is satisfied in the algebra, which is a contradiction. ■

Having eliminated algebras in which all products of level 1 elements remain at level 2, we are left with the case of a possible characteristic algebra in which some product of level 1 elements is at level 3 or 4. To rule out this case we show again that such an algebra cannot break all identities between full-shape depth 3 terms, but we use a different approach.

Let us call the four depth 1 terms (xx), (xy), (yx), and (yy) atoms. In this case we consider sets of full-shape depth 3 terms that are composed of two atoms. For instance, the term $[(xx)(xy)][(xy)(xy)]$ uses the two atoms (xx) and (xy). For each choice of two atoms there are sixteen terms, which relate quite naturally to the sixteen full-shape depth 2 terms on two variables displayed above, by associating each atom with one of the variables. In each group, all but one or two terms contain both variables (the only exceptions are those terms only composed of the (xx) atom or the (yy) atom), so most identities between them are regular, and must be broken using level 0 elements. Since every product of two level 0 elements is a level 1 element, and each term in a group relates to one of the full-shape depth 2 terms, breaking any regular identity between two terms in one of these sets of depth 3 terms is equivalent to breaking the underlying identity between depth 2 terms using only substitutions at level 1. To place some additional constraints upon a candidate characteristic algebra, we consider what breaking depth 2 identities in this way requires, always restricting our attention to terms u and v that contain both variables to ensure regularity.

To break the identity $(uu)(uv) \approx (uv)(uv)$ using level 1 elements, it cannot happen that uv outputs either m or an element in L_i for $i \geq 3$. In the first case both terms will output mm because the output of uu must be a level 2 element, and in the second case both terms will output the absorbing element. Thus, uv must output the other level 2 element, n, and uu must output m. Therefore, there must exist a pair of level 1 elements, p_1, q_1, such that $p_1p_1 = m$ and $p_1q_1 = n$.

A similar argument regarding $(uu)(vu) \approx (vu)(vu)$ shows that there must exist level 1 elements r_1, s_1 such that $r_1r_1 = m$ and $s_1r_1 = n$. Since by Lemma 5.7 m is the square of only one level 1 element, we must have $r_1 = p_1$. Note that once these requirements are met, the dual inputs (q_1, p_1) and (s_1, r_1) will break the dual identities $(vv)(vu) \approx (vu)(vu)$ and $(vv)(uv) \approx (uv)(uv)$, respectively. It is also important to realize that unless there is more than one pair of elements satisfying these properties, the pairs (p_1, q_1), (q_1, p_1), (p_1, s_1), and (s_1, p_1) are the only inputs that will break these four identities. Thus, to ensure that all of the related identities between full-shape depth 3 terms in each set are broken, there must be a substitution by level 0 elements so that each pair of atoms induces each of these required level 1 inputs. This means, for example, that to break the related identities between depth 3 terms composed of the (xx) and (xy) atoms, there must be a substitution by level 0 elements for which xx gives p_1 and xy gives q_1, one where xx gives q_1 and xy gives p_1, one where xx gives p_1 and xy gives s_1, and one where xx gives s_1 and xy gives p_1. The same must also be true for every other pair of atoms. From this, we can now prove that there must be a commuting pair of elements which satisfies both of the above requirements.

Lemma 5.9 *Let m and n be the two elements at level 2 in an 11-element characteristic 4-normal algebra. Then there exist elements $p_1, q_1 \in L_1$ such that $p_1 p_1 = m$ and $p_1 q_1 = q_1 p_1 = n$.*

Proof: We know from the discussion above that there is an element p_1 such that $p_1 p_1 = m$, and elements q_1 and s_1 such that $p_1 q_1 = s_1 p_1 = n$. Only the four inputs from the pairs p_1, q_1 and p_1, s_1 can break the four identities given above. If neither of these pairs commute, then $q_1 p_1$ and $p_1 s_1$ equal either m or an element from level 3 or higher, neither of which possibilities will break the identities in question; so each of the four substitutions from both pairs is the only one to break some identity. Therefore, there must be inputs of level 0 elements such that (xx) and (xy) induce all of these inputs. We know that p_1 is the square of exactly one level 0 element, call it p_0, and we let the other two level 0 elements be q_0 and s_0. Then, one of $p_0 q_0$ and $p_0 s_0$ must equal q_1 and the other must equal s_1. To have (yy) and (xy) induce these substitutions, one of $q_0 p_0$ and $s_0 p_0$ must equal q_1 and the other s_1. This makes it impossible for (xy) and (yx) to induce all the necessary inputs, since for all four level 0 inputs where p_0 is substituted for x or y, (xy) and (yx) will only induce inputs involving q_1 and s_1, none of the required ones. The remaining two level 0 substitutions (using q_0 and s_0) can only be used to induce two of the four inputs needed. Thus some identities between depth 3 terms using atoms (xy) and (yx) will remain unbroken in the algebra. Hence, there must be a commuting pair of level 1 elements. ■

Let a_1, b_1, and c_1 be the three level 1 elements in our potential 4-normal algebra. Suppose, without loss of generality, that $a_1 a_1 = m$ and $a_1 b_1 = b_1 a_1 = n$, as required by Lemma 5.9. By assumption, one of the pairs a_1, c_1 or b_1, c_1 has a product (in one order or the other) equal to a level 3 or higher element. Suppose, for instance, that $c_1 a_1$ is at level 3 or 4; the remaining three cases are handled analogously. Then the terms $(uv)(vu)$ and $(vu)(uv)$ both output the absorbing element on the level 1 input pairs (a_1, c_1) and (c_1, a_1). This means that these two input pairs do not break the depth 2 identity $(uv)(vu) \approx (vu)(uv)$. The pairs (a_1, b_1) and (b_1, a_1) also do not break this identity: both terms output nn on these inputs. To break this identity, then, the remaining pair of elements, b_1 and c_1, must have $b_1 c_1 \neq c_1 b_1$ and both products $b_1 c_1$ and $c_1 b_1$ must be at level 2. That is, there must be a pair of level 1 elements which do not commute and whose products are both at level 2. Now breaking the related depth 3 identities depends on each pair of atoms inducing, in addition to (a_1, b_1) and (b_1, a_1), either one of the two inputs obtained from this non-commuting pair, which is impossible.

Now we are ready to eliminate the final possible case.

Lemma 5.10 *There does not exist an* 11*-element characteristic* 4*-normal algebra that contains any pair of level* 1 *elements with products at level* 3 *or higher.*

Proof: We retain the assumptions detailed in the previous paragraph, that $a_1a_1 = m$ and $a_1b_1 = b_1a_1 = n$. As discussed above, in any such algebra every pair of atoms must induce the level 1 inputs (a_1, b_1), (b_1, a_1), and either (w_1, z_1) or (z_1, w_1), where w_1, z_1 is the pair of level 1 elements that do not commute and whose products are both level 2 elements. Thus, w_1, z_1 is either a_1, c_1 or b_1, c_1.

Because there are sixteen terms consisting of the atoms (xy) and (yx), breaking all of the identities between those terms requires at least three different level 0 input pairs in which (xy) and (yx) do not have the same output. Therefore, at level 0 there can be at most one commutative pair of elements, since having two commuting pairs would force us to use two different level 0 input pairs in which xy and yx do not have the same output, which is not possible.

Let a_0, b_0, and c_0 be the three level 0 elements, and let $a_0a_0 = a_1$, $b_0b_0 = b_1$, and $c_0c_0 = c_1$. Note that to ensure that (xx) and (xy) induce (a_1, b_1) and (b_1, a_1), one of a_0b_0 and a_0c_0 must equal b_1 and one of b_0a_0 and b_0c_0 must equal a_1. To ensure that (yy) and (xy) induce the same inputs, one of b_0a_0 and c_0a_0 must equal b_1, and one of a_0b_0 and c_0b_0 must equal a_1. This is equivalent to saying that, in the 3×3 block of the operation table for the algebra dealing with products among level 0 elements, there must be a b_1 in both the row and column where a_1 is a square, and there must be an a_1 in the row and column where b_1 is a square. But it is impossible to have $a_0b_0 = b_0a_0 = a_1$, since then it must also be the case that $a_0c_0 = c_0a_0 = b_1$, so that both a_0, b_0 and a_0, c_0 are commuting pairs of elements, which cannot happen. Likewise, it is impossible that $a_0b_0 = b_0a_0 = b_1$ since that would force $b_0c_0 = c_0b_0 = a_1$.

This leaves three main possibilities: 1) a_0 commutes with c_0 to give b_1, 2) b_0 commutes with c_0 to give a_1, or 3) neither a_0 nor b_0 commutes with c_0. Each of these cases has subcases depending upon whether the pair w_1, z_1 is a_1, c_1 or b_1, c_1. A case by case analysis then shows that in each possible case, there is no way to define multiplication in a candidate algebra so that each pair of atoms induces the inputs necessary to break all the identities among its related set of depth 3 terms. Every algebra with a pair of level 1 elements whose product is an element of level 3 or higher will necessarily satisfy some identity that is not 4-normal. ■

This together with Lemma 5.8 proves the following.

Theorem 5.11 *There does not exist an* 11*-element algebra which is characteristic for* 4*-normality.*

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REFERENCES

[C] I. Chajda, "Normally Presented Varieties", Algebra Universalis 34 (1995), 327 - 335.

[DW1] K. Denecke and S. L. Wismath, "Complexity, Composition and Hypersubstitutions", Int. J. Mathematics and Mathematical Sciences, 2003:15 2003, 959 - 969.

[DW2] K. Denecke and S. L. Wismath, "Valuations of Terms", Algebra Universalis 50 (2003), 107 - 128.

[DW3] K. Denecke and S. L. Wismath, "A Characterization of k-Normal Varieties", Algebra Universalis 51 (2004) no. 4, 395 - 409.

[G1] E. Graczyńska, "On Normal and regular identities and hyperidentities", in: Universal and Applied Algebra, Proceedings of the V Universal Algebra Symposium, Turawa (Poland), 1988, World Scientific, 1989, 107-135.

[G2] E. Graczyńska, "On Normal and regular identities", Algebra Universalis 4 (1974), 14 - 19.

[M] I. I. Mel'nik, " Nilpotent shifts of varieties", (in Russian), Mat. Zametki, Vol. 14 No. 5 (1973), English translation in: Math. Notes 14 (1973), 962-966.

[P1] J. Płonka, "On equational classes of abstract Algebras Defined by Regular Equations", Fund. Math., 64 (1969), 241 - 247.

[P2] J. Płonka, "On Varieties of Algebras Defined by Identities of Some Special Forms", Houston J. of Math., 14 (1988), 253 - 263.

[P3] J. Płonka, "Minimal Characteristic Algebras for Some Properties of Identities", Southeast Asian Bulletin of Mathematics, 25 (2001), 495 - 502.

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