NUMERICAL COMPUTATIONS AND PATTERN FORMATION FOR ADSORBATE-INDUCED PHASE TRANSITION MODEL

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Abstract. In Hildebrand et al., *Self-Organized Chemical Nanoscale Microreactors*, Phys. Rev. Lett. **83** (1999), 1475-1478, they presented a model for describing the process of pattern formation in the catalytic oxidation of CO molecules on a Pt(110) surface. This paper is concerned with numerical computations for their model. We find out various types of pattern solutions which are classified into three classes, i. e. transient, evolutionary and stationary patterns.

1 Introduction In 1990, Jakubith, Rotermund, Engel, Oertzen and Ertl [10] (see also Ertl [6]) found out that, in the catalytic oxidation of CO molecules on a Pt(110) surface, adsorbed CO molecules and O atoms form various types of spatio-temporal patterns, such as propagating wave, spiral, target, stripe and chaotic patterns.

To understand these remarkable phenomena from macroscopic point of view, Hildebrand et al. [8, 9] and Hildebrand [7] presented a simple kinetic model of the surface reaction coupled to a structural phase transition of the surface:

$$
(1.1)\qquad
\begin{cases}
\dfrac{\partial u}{\partial t} = a\Delta u + c\nabla\cdot\{u(1-u)\nabla\chi(\rho)\} & \\
\qquad\qquad - fe^{\alpha\chi(\rho)}u - gu + h(1-u) & \text{in } \ \Omega\times(0,\infty),\\
\dfrac{\partial \rho}{\partial t} = b\Delta\rho + d\rho(\rho+u-1)(1-\rho) - \zeta(\rho-\tfrac{1}{2}) & \text{in } \ \Omega\times(0,\infty),\\
\dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \ \partial\Omega\times(0,\infty),\\
u(x,0)=u_0(x), \quad \rho(x,0)=\rho_0(x) & \text{in } \ \Omega.
\end{cases}
$$

Here, Ω denotes a Pt surface on which the patterns are performed. In this paper, Ω is a square domain $[0,L]\times[0,L]$. The unknown functions $u(x,t)$ and $\rho(x,t)$ denote the adsorbate coverage rate of the surface by CO molecules and the structural state of surface at a position $x\in\Omega$ and time $t\in[0,\infty)$ respectively. $d\rho(\rho+u-1)(1-\rho)$ shows that the surface structure has two stable states. $c\nabla\cdot\{u(1-u)\nabla\chi(\rho)\}$ shows the advection of u over Ω induced by the gradient of the local chemical potential $\chi(\rho)$ with mobility $1-u$. $fe^{\alpha(\rho)}$ denotes the desorption rate of the molecules depending on $\chi(\rho)$ and h the adsorption rate. $\chi(\rho)$ denotes a chemical potential function of ρ, the suggested form of $\chi(\rho)$ is

$$\chi(\rho) = -\rho^2(3-2\rho).$$

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g denotes the desorption rate of the molecules by the effect of a chemical reaction. a and b are diffusion constants of u and ρ respectively.

But the nature of ζ is rather different from others. As a matter of fact, the term $-\zeta(\rho - \frac{1}{2})$ has been incorporated from the view point of thermodynamics (see [7]). If $\zeta = 0$, then $\rho \equiv 0$, $u \equiv \frac{h}{f+g+h}$ and $\rho \equiv 1$, $u \equiv \frac{h}{fe^{-\alpha}+g+h}$ are homogeneous stationary solutions and both homogeneous solutions are always stable. To the contrary, if $\zeta \neq 0$, $\rho \equiv 0$ or $\rho \equiv 1$ is no longer stationary solution. Furthermore, for a suitable case, there exists no stable homogeneous stationary solution. In this sense ζ may be regarded as bifurcation parameter (see [19]).

All the constants $a, b, c, d, f, g, h, \alpha$ and ζ are assumed to be positive. On the unknown functions $u(x, t)$ and $\rho(x, t)$, the Neumann boundary conditions are imposed except at the vertices of Ω. $u_0(x)$ and $\rho_0(x)$ are initial functions satisfying $0 \le u_0(x) \le 1$ and $0 \le \rho_0(x) \le 1$ in Ω. These conditions naturally imply that the solution u, ρ also satisfies the same conditions.

It is already known that a unique global solution to (1.1) can be constructed for any pair of initial functions $0 \le u_0 \le 1$ and $0 \le \rho_0 \le 1$ such that $u_0 \in H^1(\Omega)$ and $\rho_0 \in H^2(\Omega)$ with $\Delta\rho_0 \in H^1(\Omega)$ and $\frac{\partial \rho_0}{\partial n} = 0$ on $\partial\Omega$. The global solution also satisfies the conditions $0 \le u \le 1$ and $0 \le \rho \le 1$. From the Cauchy problem (1.1) we can then define a dynamical system with a certain phase space $\mathcal{X}$ in an infinite-dimensional universal space X. Furthermore, as shown in [12, 13], the dynamical system possesses exponential attractors $\mathcal{M}$.

The exponential attractor is a notion of attractors (see [16, 20]) which was introduced by Eden et al. [17]. In fact, $\mathcal{M}$ is a compact subset of X enjoying the following properties: 1) $\mathcal{A} \subset \mathcal{M} \subset \mathcal{X}$, where $\mathcal{A}$ is the global attractor; 2) $\mathcal{M}$ has a finite fractal dimension; 3) $\mathcal{M}$ is positively invariant in the sense that every trajectory starting from $\mathcal{M}$ remains in $\mathcal{M}$ for any positive time; and 4) $\mathcal{M}$ attracts every trajectory in $\mathcal{X}$ in an exponential rate, that is

$$h(S(t)\mathcal{X}, \mathcal{M}) \le Me^{-\delta t} \qquad \text{for all } 0 < t < \infty, \tag{1.2}$$

where $h(\cdot, \cdot)$ denotes the Hausdorff pseudodistance, $\delta > 0$ is an exponent and $M > 0$ is a constant.

In general, exponential attractors possess more robustness than the global attractors in the sense that they attract every trajectory in an exponential rate. It is also known that they attract even approximate solutions to any discretized problem in their neighborhood in exponential rates and continue trapping the approximate solutions in the neighborhood for ever, see [1], and that, in a certain sense, exponential attractors depend on a parameter continuously (although the global attractor depends only upper continuously), see [5]. In addition, exponential attractors have a richer content than that of the global attractor, because the global attractor consists only of the states in a final stage. On the other hand, exponential attractors are not uniquely determined in general. As a matter of fact, if there exists an exponential attractor then there exists a family of exponential attractors in such a way that in any neighborhood of the global attractor one can find an exponential attractor. As mentioned, exponential attractors have a finite fractal dimension. Consequently any solution in exponential attractors possesses only a finite number of the freedom of behavior. By this fact the existence of exponential attractors is considered to have a close connection with the pattern formation, see Haken [18]. In [3] a general strategy is studied in order to construct the exponential attractors for dynamical systems determined from nonlinear diffusion systems. In consideration of the important role of exponential attractors in numerical computations and in pattern formation, we shall review briefly the construction of exponential attractors for our system ([12]) and related results.

This paper is concerned with numerical computations for the problem (1.1). We shall report that various types of pattern solutions have been found out. According to their temporal ranges, we may classify them into three classes: transient, evolutionary and stationary pattern solutions. As transient pattern solutions, we find out two kinds of target solutions, namely target solution with continuous rings and target solution with perforated rings. As evolutionary pattern solutions, we find out network solution and labyrinth-like solution. As stationary pattern solutions, we find out honeycomb, hexagon, checkerboard and stripe solutions. Some of them show very good accordance with the experimental results [10] and [6]. We know that each pattern solution is observed only when the family of coefficients a, b, c, d, f, g, α and ζ are chosen properly.

As the solutions for the system (1.1) satisfy the conditions $0 \le u \le 1$ and $0 \le \rho \le 1$, we are concerned with the numerical solutions satisfying also such conditions. However, to obtain such numerical solutions it costs a lot of computations by the repercussion of advection term. To reduce the computation time, we reconcile ourselves to handle some favorable cases only.

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2 Exponential attractors In this section we review briefly some known results for the system (1.1) which have been deduced by mathematical analysis. For the details, see the references [12, 13]. In these papers we handled only the case when $\zeta = 0$ for simplicity. But it is very easy to verify that these methods are available for the general case $\zeta > 0$ also.

Let Ω be a bounded convex domain in $\mathbb{R}^2$. For $0 \le s \le 2$, $H^s(\Omega)$ denotes the usual Sobolev space. For $\frac{3}{2} < s \le 2$, $H^s_N(\Omega)$ is a closed subspace of $H^s(\Omega)$ consisting of functions in $H^2(\Omega)$ which satisfy the Neumann boundary conditions on the boundary $\partial\Omega$. Furthermore, for $2 \le s < \frac{7}{2}$, $\mathcal{H}^s_N(\Omega)$ is defined as the space of all functions $u \in H^2_N(\Omega)$ such that $\Delta u \in H^{s-2}(\Omega)$, and for $\frac{7}{2} < s \le 4$, $\mathcal{H}^s_N(\Omega)$ as the space of all functions $u \in H^2_N(\Omega)$ such that $\Delta u \in H^{s-2}_N(\Omega)$. When Ω is a sufficiently smooth (indeed $\mathcal{C}^4$) domain, $\mathcal{H}^s_N(\Omega)$ is a subspace of $H^s(\Omega)$ for every $2 < s \le 4$. But, as Ω may not be smooth now, we have to distinguish $\mathcal{H}^s_N(\Omega)$ and $H^s_N(\Omega)$.

We set a space of initial functions by

$$\mathcal{K} = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix};\ u_0 \in H^1(\Omega) \quad \text{and} \quad \rho_0 \in \mathcal{H}^3_N(\Omega) \quad \text{with} \quad 0 \le u_0,\, \rho_0 \le 1 \right\}. \tag{2.1}$$

For each pair of u_0 and ρ_0 in $\mathcal{K}$, we can construct by the use of the theory of abstract parabolic evolution equations (see [15]) a unique local solution to (1.1) in the function space:

$$\begin{cases} u \in \mathcal{C}^1((0,T];\, L^2(\Omega)) \cap \mathcal{C}([0,T];\, H^1(\Omega)), \ \ \sqrt{t}\, u \in \mathcal{C}([0,T];\, H^2_N(\Omega)), \\ \rho \in \mathcal{C}^1((0,T];\, H^2_N(\Omega)) \cap \mathcal{C}([0,T];\, \mathcal{H}^3_N(\Omega)), \ \ \sqrt{t}\, \rho \in \mathcal{C}([0,T];\, \mathcal{H}^4_N(\Omega)), \end{cases} \tag{2.2}$$

where $T > 0$ depends on the initial functions u_0 and ρ_0. The local solution also satisfies the conditions $0 \le u \le 1$ and $0 \le \rho \le 1$.

Moreover, we can establish a priori estimates for every local solution in the space (2.2) (see [13, 12]). Consequently we construct for each pair of u_0 and ρ_0 in $\mathcal{K}$, a unique global solution to (1.1) in the function space:

$$\begin{cases} u \in \mathcal{C}^1((0,\infty);\, L^2(\Omega)) \cap \mathcal{C}([0,\infty);\, H^1(\Omega)), \ \ \sqrt{t}\, u \in \mathcal{C}([0,\infty);\, H^2_N(\Omega)), \\ \rho \in \mathcal{C}^1((0,\infty);\, H^2_N(\Omega)) \cap \mathcal{C}([0,\infty);\, \mathcal{H}^3_N(\Omega)), \ \ \sqrt{t}\, \rho \in \mathcal{C}([0,\infty);\, \mathcal{H}^4_N(\Omega)). \end{cases}$$

Of course, the global solution also satisfies the conditions $0 \le u \le 1$ and $0 \le \rho \le 1$.

On the basis of these results, we next define a dynamical system in a Banach space. Indeed we set a universal space

$$X = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in L^2(\Omega) \quad \text{and} \quad \rho \in H^2_N(\Omega) \right\}. \tag{2.3}$$

Obviously, $\mathcal{K} \subset X$. For each $U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in \mathcal{K}$, $S(t)U_0$ is defined by

$$S(t)U_0 = S(t) \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} = \begin{pmatrix} u(t) \\ \rho(t) \end{pmatrix}, \qquad 0 \le t < \infty,$$

where $u(t)$, $\rho(t)$ is the global solution to (1.1) satisfying $u(0) = u_0$ and $\rho(0) = \rho_0$. Then, $S(t)$ is a nonlinear operator from $\mathcal{K}$ into itself enjoying the semigroup property $S(t)S(s) = S(t+s)$ and $S(0) = 1$. Moreover, $S(t)U_0$ is shown to be continuous in (t, U_0) with respect to the topology of $[0, \infty) \times X$. In this way, a dynamical system $(S(t), \mathcal{K}, X)$ with the universal space X is determined from the Cauchy problem (1.1).

All the solutions to (1.1) enjoy strong decay estimates (see [12]). That is, there exists a universal constant $C > 0$ such that the following statement is true. For each bounded set B of $\mathcal{K}$, there is a time $t_B > 0$ depending on B such that

$$\sup_{t \ge t_B} \sup_{U_0 \in B} \|S(t)U_0\|_Z \le C,$$

where Z is a second Banach space such that

$$Z = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H^{\frac{1}{2}}(\Omega) \quad \text{and} \quad \rho \in \mathcal{H}^{\frac{5}{2}}_N(\Omega) \right\}$$

and $\|\cdot\|_Z$ denotes its norm. From these estimates an absorbing and positively invariant compact set is constructed. There exists a compact set $\mathcal{X}$ of X such that

1. $\mathcal{X}$ is a compact subset of X such that $\mathcal{X} \subset \mathcal{K}$;
2. $\mathcal{X}$ is a positively invariant set of $S(t)$, i. e. $S(t)\mathcal{X} \subset \mathcal{X}$ for every $t \ge 0$;
3. $\mathcal{X}$ is an absorbing set in the sense that, for any bounded set B of $\mathcal{K}$, there is a time t_B dependent on B such that $S(t)B \subset \mathcal{X}$ for every $t \ge t_B$.

These properties of $\mathcal{X}$ mean that $(S(t), \mathcal{X}, X)$ also becomes a dynamical system in the universal space X and the asymptotic behavior of trajectories of the original system $(S(t), \mathcal{K}, X)$ is reduced to that of $(S(t), \mathcal{X}, X)$.

On the new phase space $\mathcal{X}$, the semigroup $S(t)$ enjoys the compact Lipschitz condition introduced by Efendiev et al. [4]. That is, for any fixed time $t^* > 0$, the following Lipschitz condition

$$\|S(t^*)U_0 - S(t^*)V_0\|_Z \le L^* \|U_0 - V_0\|_X, \quad U_0, V_0 \in \mathcal{X}$$

holds with some constant $L^* > 0$ (see [12]). Note that the embedding $Z \to X$ is compact. According to [4], we can derive from such a compact Lipschitz condition existence of a family of exponential attractors for the system $(S(t), \mathcal{X}, X)$.

In any neighborhood of the global attractor $\mathcal{A} = \bigcap_{0 \le t < \infty} S(t)\mathcal{X}$, there exists an exponential attractor. In a certain sense, the exponential attractors depend on the coefficients $a \sim \zeta$ of the system continuously.

3 Stability of exponential attractor under approximations This section is devoted to reviewing some known stability result of the exponential attractors under approximations. For the details, see the references [1, 11].

Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{X}$ be a subset of X which is a metric space with the induced distance $d(U,V) = \|U-V\|_X$, $U, V \in \mathcal{X}$. Let $S(t)$ be a nonlinear semigroup acting on $\mathcal{X}$. So $(S(t), \mathcal{X}, X)$ is a dynamical system.

Let ξ, $0 < \xi \le \xi_0$, be a parameter of approximation, $\xi_0 >$ being a fixed small number. Let X_ξ, $0 < \xi \le \xi_0$, be a family of finite-dimensional spaces such that $X_\xi \to X$ as $\xi \to 0$.

Let, for each $0 < \xi \le \xi_0$, an approximating dynamical system $(S_\xi(t), \mathcal{X}_\xi, X_\xi)$ is defined, $\mathcal{X}_\xi$ being a phase space in X_ξ and $S_\xi(t)$ being a semigroup acting on $\mathcal{X}_\xi$.

For the system $(S(t), \mathcal{X}, X)$ and for the approximating systems $(S_\xi(t), \mathcal{X}_\xi, X_\xi)$, we assume the following conditions:

1. $\mathcal{X}$ is a compact set of X;
2. $(S(t), \mathcal{X}, X)$ possesses an exponential attractor $\mathcal{M}$ with the estimate (1.2);
3. $(S_\xi(t), \mathcal{X}_\xi, X_\xi)$, $0 < \xi \le \xi_0$, are dynamical systems;
4. $\mathcal{X}$ comprehends the union $\bigcup_{0<\xi\le\xi_0} \mathcal{X}_\xi$;
5. $S_\xi(t)$ approximates $S(t)$ on each phase space $\mathcal{X}_\xi$ with order $O(\xi)$ in the following sense

$$Q_\xi(t;T) = \sup_{U_\xi \in \mathcal{X}_\xi} \|S_\xi(t)U_\xi - S(t)U_\xi\|_X \le C_T t^{(\nu-1)/2} O(\xi), \qquad 0 < t \le T$$

   for any finite $0 < T < \infty$ with some constant C_T independent of ξ and some exponent $0 < \nu \le 1$.

As shown in [1], we deduce under these conditions that the estimate

$$d(S_\xi(t)U_\xi, \mathcal{M}) \le \begin{cases} Me^{-\delta t} + C_T t^{(\nu-1)/2} O(\xi) & \text{for } 0 < t \le T, \\ Me^{-\delta T} + C_T O(\xi) & \text{for } T \le t < \infty \end{cases}$$

is valid for any $U_\xi \in \mathcal{X}_\xi$ with arbitrarily fixed time $0 < T < \infty$. Here, $d(U, B)$ denotes the distance of a point $U \in \mathcal{X}$ and a set $B \subset \mathcal{X}$.

From this result we verify the following fact. Let $\varepsilon > 0$ be any positive number. Set $T = \frac{1}{\delta} \log \frac{2M}{\varepsilon}$, then $Me^{-\delta T} = \frac{\varepsilon}{2}$. In addition, take ξ sufficiently small in such a way that $O(\xi) \le \frac{\varepsilon}{2C_T}$. Then it follows that $d(S_\xi(t)U_\xi, \mathcal{M}) \le \varepsilon$ for any $t \in [T, \infty)$. That is, any approximate solution $S_\xi(t)U_\xi$ is trapped in the ε-neighborhood of $\mathcal{M}$ for all $t \ge T$.

4 Numerical Computations In this section we shall present some numerical results for the system (1.1),

Various types of pattern solutions are indeed found out. According to their temporal range we may classify them into three classes, which are transient, evolutionary and stationary patterns.

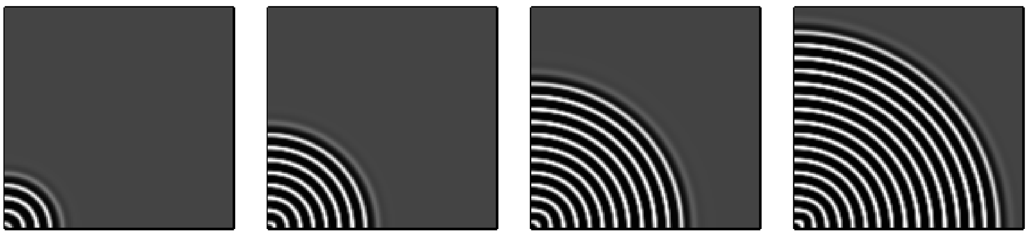
Throughout this section, the size of Ω is taken $L = 4$ or 8. The coefficients of system (1.1) is fixed as $a = 1,\ b = 2^{-8},\ c = \alpha,\ d = 1,\ f = 512,\ g = \mu h,\ h = 32$, where μ is a ratio of g to h. But the three coefficients ζ, α and μ are variable. As mentioned above, the coefficients g and h denote the desorption and adsorption rate of the molecules respectively. So, the larger μ is, the more molecules desorb, and the coverage rate tends to decrease. To the contrary, the smaller μ is, the fewer molecules desorb, and the coverage rate tends to increase. The family of homogeneous stationary solution for the system (1.1) are denoted by $(\overline{u}_i, \overline{\rho}_i)$ $(i = 1, 2, \cdots, n)$, where $0 \le \overline{u}_i,\ \overline{\rho}_i \le 1$ and $\overline{\rho}_i < \overline{\rho}_j$ $(i < j)$. The integer n is a number of homogeneous stationary solutions.

4.1 Transient Patterns Two types of target pattern solutions are found out as the transient patterns. One is a target pattern solution with continuous rings (Fig. 1). The other is a target pattern solution with perforated rings (Fig. 2). These two target pattern solutions are observed temporary for a while after the initial time. They seem to have similarities with numerical results due to Aida and Yagi [2] where they handled a model describing the process of pattern formation by Salmonella typhimurium (see [14]).

First, one is obtained by the following coefficients and initial functions. The coefficients ζ, α and μ are fixed as $\zeta = 1$, $\alpha = 56$ and $\mu = 1.8$. In this case there is a unique homogeneous stationary solution, $(\overline{u}_1, \overline{\rho}_1)$ $(n = 1)$. This solution $(\overline{u}_1, \overline{\rho}_1)$ is unstable. By means of this, initial functions are defined as $u_0(x) = \overline{u}_1$ and $\rho_0(x) = \overline{\rho}_1 + \delta_1(x)$. Here, $\delta_1(x)$ is a small perturbation which is rotational symmetric near the origin and is zero far from the origin.

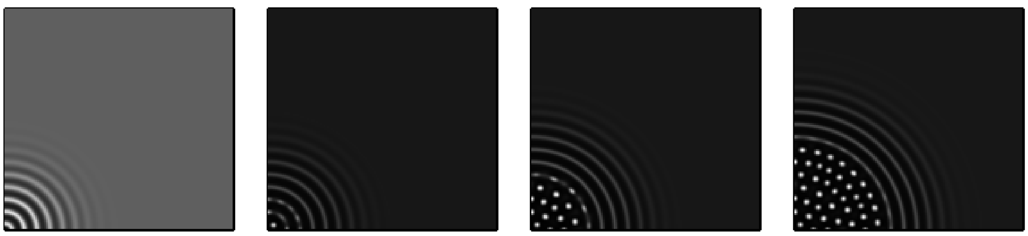
Next, the other is obtained by the following coefficients and initial functions. The coefficients ζ, α and μ are fixed as $\zeta = 1$, $\alpha = 56$ and $\mu = 4$. In this case also there is a unique homogeneous stationary solution, $(\overline{u}_1, \overline{\rho}_1)$ $(n = 1)$, and this solution is unstable. Then initial functions $u_0(x)$ and $\rho_0(x)$ are defined in the same way as before.

These two target pattern solutions exhibit the following behaviors in a short range respectively. In the former, once a ring is formed, new rings are formed outside of that. This process is repeated until outermost ring hits the boundary of Ω. In the latter, the rings are formed in the same way. But the rings are perforated in turn from inner to outer.



(a) $t = 6$ (b) $t = 12$ (c) $t = 18$ (d) $t = 24$

Fig. 1: Target pattern with continuous rings: White indicates high coverage rate and black opposite. $(L = 8,\ \zeta = 1,\ \alpha = 56,\ \mu = 1.8)$.



(a) $t = 56$ (b) $t = 72$ (c) $t = 88$ (d) $t = 104$

Fig. 2: Target pattern with perforated rings: White indicates high coverage rate and black opposite. $(L = 8,\ \zeta = 1,\ \alpha = 56,\ \mu = 4)$.

Ultimately the perforated rings are formed in the whole Ω.

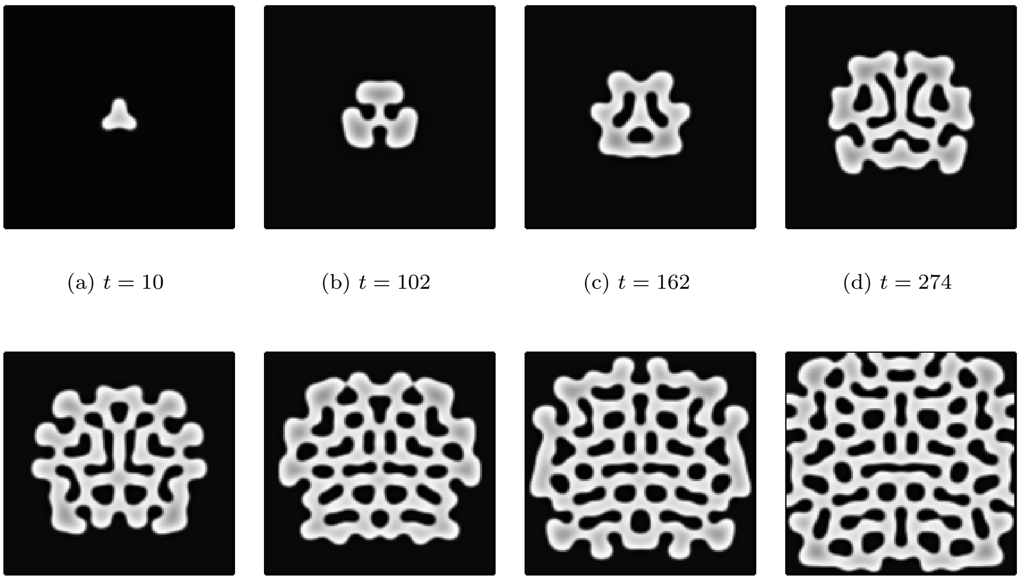
When the coefficient μ is large, the coverage rate of molecules tends to be small. So the rings each become thin. For sufficiently large μ, the rings become too thin to be stable, and are perforated.

4.2 Evolutionary Patterns Two types of pattern solutions are found out as the evolutionary patterns. One is a network pattern solution (Fig. 3). The other is a labyrinth-like pattern solution (Fig. 4). These pattern solutions evolve with some features for a long range. Furthermore their behaviors are irreversible for the time.

First, one is obtained by the following coefficients and initial functions. The coefficients ζ, α and μ are fixed as $\zeta = 0.25$, $\alpha = 16$ and $\mu = 0.7$. In this case there are three homogeneous stationary solutions, $(\overline{u}_i, \overline{\rho_i})$ $(i = 1, 2, 3)$. The solutions $(\overline{u}_i, \overline{\rho}_i)$ $(i = 1, 3)$ are stable but the other is unstable. By means of them, initial functions are defined as $u_0(x) = \overline{u}_1$ and $\rho_0(x) = \overline{\rho}_1 + (\overline{\rho}_3 - \overline{\rho}_1)\delta_2(x)$. Here, the function $\delta_2(x)$ is 1 on a small triangular region at the center of Ω and is zero outside the triangular region.

Next, the other is obtained by the following coefficients and initial functions. The coefficients ζ, α and μ are fixed as $\zeta = 0.25$, $\alpha = 16$ and $\mu = 0.9$. In this case also there are three homogeneous stationary solutions, $(\overline{u}_i, \overline{\rho_i})$ $(i = 1, 2, 3)$, and the solution $(\overline{u}_1, \overline{\rho}_1)$ is stable but the others are unstable. By means of them, initial functions $u_0(x)$ and $\rho_0(x)$ are defined in the same way as before.

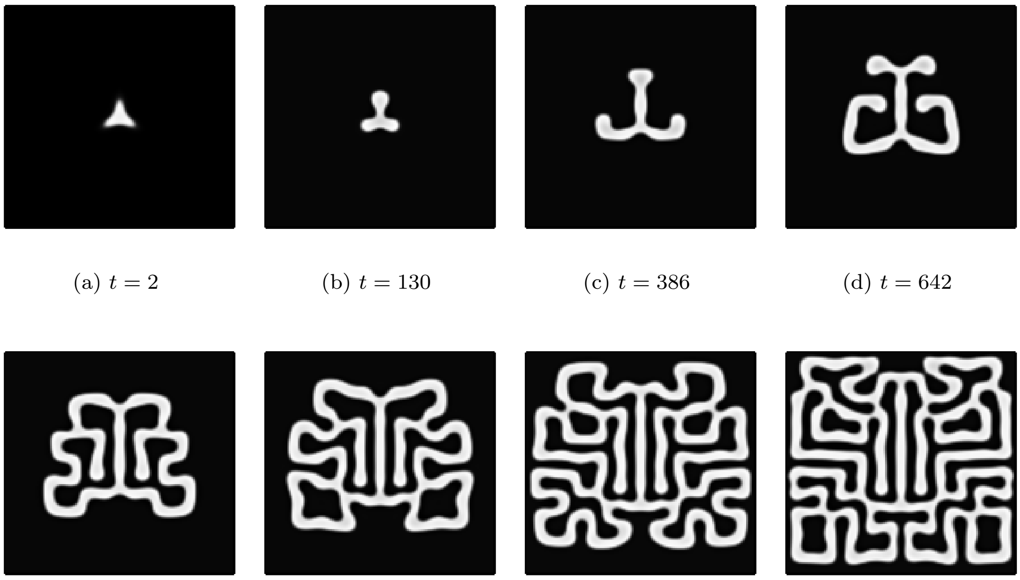
The network and the labyrinth-like pattern solution exhibit the following behaviors respectively. In the former, the white region spreads over Ω producing many growth points and bifurcating often at the growth points, and forms a network patterns consisting of many nodes and branches. In the latter, the white region also spreads over Ω but possessing only



(a) $t = 10$ (b) $t = 102$ (c) $t = 162$ (d) $t = 274$

(e) $t = 338$ (f) $t = 402$ (g) $t = 466$ (h) $t = 558$

Fig. 3: Network pattern solution: White indicates high coverage rate and black opposite. $(L = 8,\ \zeta = 0.25,\ \alpha = 16,\ \mu = 0.7)$



(a) $t = 2$ (b) $t = 130$ (c) $t = 386$ (d) $t = 642$

(e) $t = 898$ (f) $t = 1154$ (g) $t = 1410$ (h) $t = 1666$

Fig. 4: Labyrinth-like pattern solution: White indicates high coverage rate and black opposite. $(L = 8,\ \zeta = 0.25,\ \alpha = 16,\ \mu = 0.9)$

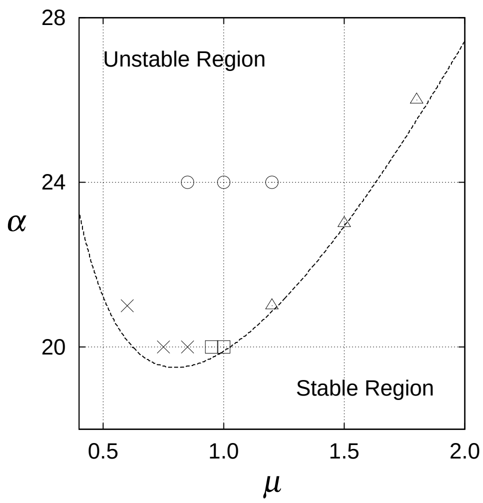
a few growth points and rarely bifurcating, and forms a labyrinth-like pattern.

When the coefficient μ is large, the molecules adsorbed on the surface tend to desorb through reaction. So the white region becomes thin. For sufficiently large μ, the white region becomes too thin to produce the growth points.

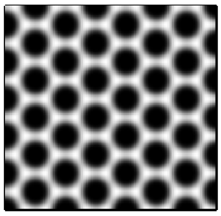
4.3 Stationary Patterns Four types of pattern solutions are found out as the stationary patterns. They are honeycomb, hexagon, checkerboard and stripe pattern solutions. Such patterns are obtained by choosing the coefficients ζ, α and μ properly. Especially μ is chosen carefully.

The coefficient ζ is fixed as $\zeta = 1$. In this case, for any coefficients α and μ there is a unique homogeneous stationary solution, $(\overline{u}_1, \overline{\rho}_1)$ $(n = 1)$. We inspect its stability by using the linearized stability analysis. As a result, stable and unstable regions of solution $(\overline{u}_1, \overline{\rho}_1)$ are obtained as in Fig. 5(a). Initial functions $u_0(x)$ and $\rho_0(x)$ are defined as $u_0(x) = \overline{u}_1$ and $\rho_0(x) = \overline{\rho}_1 + \varepsilon\delta_3(x)$. Here, ε is a sufficiently small positive constant and $\delta_3(x)$ is a function which has its values in $[-\frac{1}{2}, \frac{1}{2}]$ randomly.

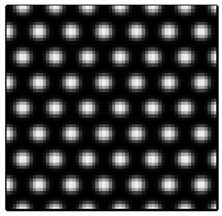
We must notice that all these stationary pattern solutions are obtained when the unique homogeneous stationary solution $(\overline{u}_1, \overline{\rho}_1)$ is unstable. Furthermore, it may be noticed that the honeycomb and hexagon pattern solutions are obtained when μ is chosen as $0 < \mu < 1$ and $1 < \mu$ respectively. To the contrary, the checkerboard pattern solution is obtained when μ is nearly equal to 1. The stripe pattern solution is obtained when α is sufficiently large no matter how μ is chosen. Typical examples of these pattern solutions are given by Fig. 5(b)-5(e).



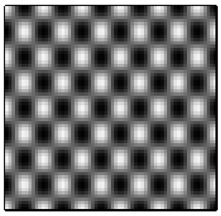
(a) Stable and unstable region of $(\overline{u}_1, \overline{\rho}_1)$



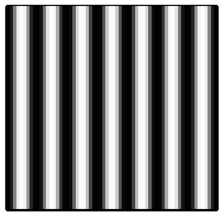
(b) Honeycomb



(c) Hexagon



(d) Checkerboard



(e) Stripe

Fig. 5: Stationary pattern solutions: In (a), the signs of cross, triangle, square and circle indicate the point where honeycomb, hexagon, checkerboard and stripe are observed respectively. In (b)-(e), white indicates high coverage rate and black opposite. $L = 4$.

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