ON 0-COMMUTATIVE B-ALGEBRAS

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Abstract. In this paper we show that if \( X \) is a 0-commutative B-algebra, then
\[
(x * a) * (y * b) = (b * a) * (y * x).
\]
Using this property we show that the class of \( p \)-semisimple BCI-algebras is equivalent to the class of 0-commutative B-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 4], Q. P. Hu and X. Li studied a huge class of abstract algebras, so called BCH-algebras. J. Neggers and H. S. Kim ([14]) introduced the notion of \( d \)-algebras which is another generalization of BCK-algebras, and also they introduced the notion of \( B \)-algebras ([15, 16]), i.e., (I) \( x * x = 0 \), (II) \( x * 0 = x \), and (III) \( x * y * z = x * (z * (0 * y)) \) for any \( x, y, z \) in a \( B \)-algebra \( X \). It is known that the \( B \)-algebra is equivalent in some sense to a group. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim ([12]) introduced a new notion, called a \( BH \)-algebra, which is a generalization of \( BCH/BCI/BCK \)-algebras, i.e., (I) \( x * x = 0 \), (II) \( x * 0 = x \), and (IV) \( x * y = 0 \) and \( y * x = 0 \) imply \( x = y \) for any \( x, y \) in a \( BH \)-algebra \( X \). A. Walendziak obtained the another equivalent axioms for a \( B \)-algebra ([17]). H. S. Kim and J. Neggers ([11]) introduced the notion of (pre-)Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have the same order 2, i.e., a Boolean group. Recently, C. B. Kim and H. S. Kim ([10]) introduced the notion of a \( BM \)-algebra which is a specialization of \( B \)-algebras, and they proved the followings: the class of \( BM \)-algebras is a proper subclass of \( B \)-algebras, and also show that a \( BM \)-algebra is equivalent to a 0-commutative \( B \)-Algebra. Moreover, they showed that the class of Coxeter algebras is a proper subclass of \( BM \)-algebras. In this paper, we show that if \( X \) is a 0-commutative \( B \)-algebra, then \( (x * a) * (y * b) = (b * a) * (y * x) \). Through the use of this property, we show that the class of \( p \)-semisimple BCI-algebras is equivalent to the class of 0-commutative \( B \)-algebras.

2. Preliminaries

A \( B \)-algebra ([15]) is a non-empty set \( X \) with a constant 0 and a binary operation “\(*\)” satisfying the following axioms:

(A1) \( x * x = 0 \),

(A2) \( x * 0 = x \),

(A3) \( (x * y) * z = x * (z * (0 * y)) \),

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for any \( x, y, z \in X \).

**Proposition 2.1.** ([15]) If \((X; *, 0)\) is a \(B\)-algebra, then

(i) \((x * y) * (0 * y) = x\),

(ii) \(x * (y * z) = (x * (0 * z)) * y\),

(iii) \(x * y = 0\) implies \(x = y\),

(iv) \(0 * (0 * x) = x\),

for any \(x, y, z \in X\).

Recently, A. Walendziak obtained the equivalent axiomatizations for \(B\)-algebras ([17]), and he proved that the congruence lattice of \(B\)-algebras is isomorphic to the lattice of their normal subalgebras ([18]).

**Theorem 2.2.** ([17]) \((X; *, 0)\) is a \(B\)-algebra if and only if it satisfies the following axioms:

(A1) \(x * x = 0\),

(W2) \(0 * (0 * x) = x\),

(W3) \((x * z) * (y * z) = x * y\),

for any \(x, y, z \in X\).

**Lemma 2.3.** ([17]) If \((X; *, 0)\) is a \(B\)-algebra, then \(0 * (x * y) = y * x\) for any \(x, y \in X\).

A \(B\)-algebra \((X; *, 0)\) is said to be 0-commutative ([2]) if \(x * (0 * y) = y * (0 * x)\) for any \(x, y \in X\).

**Proposition 2.4.** ([15]) If \((X; *, 0)\) is a 0-commutative \(B\)-algebra, then \((0 * x) * (0 * y) = y * x\), for any \(x, y \in X\).

**Theorem 2.5.** ([2, 15]) In any \(B\)-algebra, the left and the right cancellation laws hold.

An algebraic system \((X; *, 0)\) is said to be a \(BCI\)-algebra ([5, 6]) if it satisfies the following conditions:

(B1) \((x * y) * (x * z) \leq z * y\),

(B2) \(x * (x * y) \leq y\),

(B3) \(x \leq x\),

(B4) \(x \leq y, y \leq x\) imply \(x = y\),

where \(x \leq y\) is defined by \(x * y = 0\). A \(BCI\)-algebra \(X\) is said to be a \(BCK\)-algebra if \(0 \leq x\) for all \(x \in X\).

For any \(BCI\)-algebra \(X\), the set \(X_+ := \{x \in X \mid 0 \leq x\}\) is called a \(BCK\)-part of \(X\). A \(BCI\)-algebra \(X\) is said to be \(p\)-semisimple if \(X_+ = \{0\}\) (see [1]).

**Theorem 2.6.** ([19]) Let \((X; *, 0)\) be a \(BCI\)-algebra. Then the following are equivalent.

1. \(X\) is \(p\)-semisimple,
2. \(0 * x = 0\) implies \(x = 0\),
(3) \( x \ast (x \ast y) = y, \)
(4) \( x \ast (y \ast z) = z \ast (y \ast x), \)
(5) \( (x \ast y) \ast (z \ast u) = (x \ast z) \ast (y \ast u), \)
(6) \( x \ast (0 \ast z) = z \ast (0 \ast x), \)

for any \( x, y, z, u \in X. \)

3. 0-commutative \( B \)-algebras and \( p \)-semisimple \( BCI \)-algebras

From the following theorem, we can obtain an interesting observation for a 0-commutative \( B \)-algebra.

**Theorem 3.1.** If \((X; \ast, 0)\) is a 0-commutative \( B \)-algebra, then

\[
(x \ast a) \ast (y \ast b) = (b \ast a) \ast (y \ast x)
\]

(1)

for any \( x, y, a, b \in X. \)

**Proof.** For any \( x, y, a, b \in X, \) we obtain

\[
(x \ast a) \ast (y \ast b) = x \ast [(y \ast b) \ast (0 \ast a)] \quad \text{[(A3)]}
\]
\[
= x \ast [y \ast \{(0 \ast a) \ast (0 \ast b)\}] \quad \text{[(A3)]}
\]
\[
= x \ast (y \ast (b \ast a)) \quad \text{[Proposition 2.4]}
\]
\[
= [x \ast (0 \ast (b \ast a))] \ast y \quad \text{[Proposition 2.1-(ii)]}
\]
\[
= [(b \ast a) \ast (0 \ast x)] \ast y \quad \text{[0-commutative]}
\]
\[
= (b \ast a) \ast [y \ast (0 \ast x)] \quad \text{[(A3)]}
\]
\[
= (b \ast a) \ast (y \ast x) \quad \text{[Proposition 2.1-(iv)]}
\]

proving the theorem.

**Corollary 3.2.** If \((X; \ast, 0)\) is a 0-commutative \( B \)-algebra, then

\[
(x \ast z) \ast (y \ast z) = x \ast y
\]

for any \( x, y, z \in X. \)

**Proof.** If we let \( x := a \) in (1), then by Lemma 2.3, \( b \ast y = 0 \ast (y \ast b) = (a \ast a) \ast (y \ast b) = (b \ast a) \ast (y \ast a). \)

**Remark.** In fact, the condition “0-commutative” need not to be necessary, since \( (x \ast z) \ast (y \ast z) = x \ast ((y \ast z) \ast (0 \ast z)) = x \ast y \) by Proposition 2.1-(i). A. Walendziak ([17]) gave another proof for it.

**Corollary 3.3.** If \((X; \ast, 0)\) is a 0-commutative \( B \)-algebra, then

\[
(z \ast y) \ast (z \ast x) = x \ast y
\]

for any \( x, y, z \in X. \)
Proof. By applying Theorem 3.1, we obtain \( x \ast y = (x \ast y) \ast (z \ast z) = (z \ast y) \ast (z \ast x) \) for any \( x, y, z \in X \).

**Corollary 3.4.** If \((X; \ast, 0)\) is a 0-commutative \(\mathcal{B}\)-algebra, then
\[
(x 
\ast a) \ast y = (0 \ast a) \ast (y \ast x)
\]
for any \( x, y, a \in X \).

**Proof.** If we let \( b := 0 \) in (1), then \( (x \ast a) \ast y = (x \ast a) \ast (y \ast 0) = (0 \ast a) \ast (y \ast x) \).

**Corollary 3.5.** If \((X; \ast, 0)\) is a 0-commutative \(\mathcal{B}\)-algebra, then
\[
x \ast (y \ast b) = b \ast (y \ast x)
\]
for any \( x, y, b \in X \).

**Proof.** If we let \( a := 0 \) in (1), then \( x \ast (y \ast b) = (x \ast 0) \ast (y \ast b) = (b \ast 0) \ast (y \ast x) = b \ast (y \ast x) \).

**Theorem 3.6.** If \((X; \ast, 0)\) is a 0-commutative \(\mathcal{B}\)-algebra, then
\[
(x \ast y) \ast z = (x \ast z) \ast y
\]
for any \( x, y, z \in X \).

**Proof.** For any \( x, y, z \in X \), we obtain
\[
(x \ast y) \ast z = x \ast [z \ast (0 \ast y)] \quad \text{[(A3)]}
\]
\[
= x \ast [y \ast (0 \ast z)] \quad \text{[0-commutative]}
\]
\[
= (x \ast z) \ast y, \quad \text{[(A3)]}
\]
proving the theorem.

**Proposition 3.7.** Let \((X; \ast, 0)\) be a 0-commutative \(\mathcal{B}\)-algebra. Then \((X; \leq)\) is a partially ordered set, where \( x \leq y \) if and only if \( x \ast y = 0 \).

**Proof.** It is straightforward by Proposition 2.1-(iii).
Theorem 3.9. If \((X; *, 0)\) is a 0-commutative \(B\)-algebra, then
\[
[x * (x * y)] * y = 0
\]
for any \(x, y, z \in X\).

Proof. By applying Theorem 3.6, we obtain
\[
[x * (x * y)] * y = (x * y) * (x * y) = 0.
\]

Theorem 3.10. Every 0-commutative \(B\)-algebra is a \(BCI\)-algebra.

Proof. It follows from Proposition 3.7 and Theorems 3.8 and 3.9.

The converse of Theorem 3.10 need not to be true in general.

Example 3.11. Let \(X := \{0, 1, 2, 3\}\) be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then it is a \(BCI\)-algebra ([7]), but not a 0-commutative \(B\)-algebra, since \(3 * (0 * 2) = 0 \neq 2 * (0 * 3)\).

Theorem 3.12. Every 0-commutative \(B\)-algebra is a \(p\)-semisimple \(BCI\)-algebra.

Proof. It follows from Theorems 2.6 and 3.10.

Theorem 3.13. Every \(p\)-semisimple \(BCI\)-algebra is a 0-commutative \(B\)-algebra.

Proof. It is enough to show (A3). For any \(x, y, z \in X\), we have
\[
x * (z * (0 * y)) = (x * 0) * (z * (0 * y)) = (x * z) * (0 * (0 * y)) = (x * z) * y,
\]
proving the theorem.

By Theorems 3.12 and 3.13, we conclude that the class of \(p\)-semisimple \(BCI\)-algebras is equivalent to the class of 0-commutative \(B\)-algebras in some sense (refer [15, p. 27-28]). Since it is well known that every abelian group is equivalent to a \(p\)-semisimple \(BCI\)-algebra, we conclude that:

\[
\text{abelian groups} \iff \text{\(p\)-semisimple \(BCI\)-algebras} \iff \text{0-commutative \(B\)-algebras}
\]
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