T-FUZZY SUBHYPERNEAR-RINGS OF HYPERNEAR-RINGS

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Received October 16, 2004

Abstract. Using a t-norm $T$, the notion of $T$-fuzzy subhypernear-rings (for short $TFS$-rings) of hypernear-rings is introduced and some of their properties are investigated. Also we study the structure of $TFS$-rings under direct product.

1. Introduction

The theory of hyperstructures has been introduced by Matry in 1934 during the 8th congress of the Scandinavian Mathematicians [10]. Marty introduced the notion of a hypergroup and then many researchers have been worked on this new field of modern algebra and developed it. A comprehensive review of the theory of hyperstructures appear in [2] and [14]. The notion of the hyperfield and hyperring was studied by Krasner [9]. In [3], Dasic has introduced the notion of hypernear-rings generalizing the concept of near-ring [11]. In [7], Gontineac defined the zero-symmetric part and the constant part of a hypernear-ring and introduced a structure theorem and other properties of hypernear-rings. Davvaz in [5] introduced the notion of an $H_v$-near ring generalizing the notion of hypernear-ring.

The concept of fuzzy sets was introduced by Zadeh [15]. It was first applied to the theory of groups by Rosenfeld [12]. Rosenfeld has introduced fuzzy subgroups of a group and many researchers are engaged in extending the concept. In [1], Anthony and sherwood redefined a fuzzy subgroup of a group using the concept of a triangular norm, also see [6]. This notion was introduced by Schweizer and Sklar [13], in order to generalize the ordinary triangle inequality in a metric space to the more general probabilistic metric spaces.

In [4], Davvaz has introduced the concept of fuzzy subhypernear-rings and fuzzy hyperideals of a hypernear-ring which are a generalization of the concept of a fuzzy subnear-rings and fuzzy ideals in a near-ring. Now, in this paper, using a t-norm $T$, the notion of $T$-fuzzy subhypernear-rings (for short $TFS$-rings) of hypernear-rings is introduced and some of their properties are investigated. Also we study the structure of $TFS$-rings under direct product.

2. Preliminaries

We now review some basic definitions for the sake of completeness. These definitions are taken primarily from [3,4,7,13].

Definition 2.1. Let $H$ be a non-empty set. A hyperoperation $*$ on $H$ is a mapping of $H \times H$ into the family of non-empty subsets of $H$.

Definition 2.2. A hypernear-ring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

2000 Mathematics Subject Classification: 03F55, 06F05, 20M12, 03B52, 16Y99.
Key words and phrases: hypernear-ring, t-norm, (imaginable) $T$-fuzzy subhypernear-ring, $T$-product.

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* Supported by the research fund of Chinju National University of Education, 2004.
1) \((R, +)\) is a quasi canonical hypergroup (not necessarily commutative), i.e., in \((R, +)\) the following hold:

a) \(x + (y + z) = (x + y) + z\) for all \(x, y, z \in R\);

b) There is \(0 \in R\) such that \(x + 0 = 0 + x = x\) for all \(x \in R\);

c) For every \(x \in R\) there exists one and only one \(x' \in R\) such that \(0 \in x + x'\), (we shall write \(-x\) for \(x'\) and we call it the opposite of \(x\));

d) \(z \in x + y\) implies \(y \in -x + z\) and \(x \in z - y\).

2) With respect to the multiplication, \((R, \cdot)\) is a semigroup having absorbing element \(0\) i.e., \(x \cdot 0 = 0\) for all \(x \in R\).

3) The multiplication is distributive with respect to the hyperoperation \(+\) on the left side i.e., \(x \cdot (y + z) = x \cdot y + x \cdot z\) for all \(x, y, z \in R\).

If \(x \in R\) and \(A, B\) are subsets of \(R\), then by \(A + B, A + x\) and \(x + B\) we mean

\[
A + B = \bigcup_{a \in A, b \in B} a + b, \quad A + x = A + \{x\}, \quad x + B = \{x\} + B.
\]

Note that for all \(x, y \in R\), we have \(-(-x) = x, 0 = -0, -(x + y) = -y - x\) and \(x(-y) = -xy\).

**Definition 2.3.** Let \((R, +, \cdot)\) be a hypernear-ring. A non-empty subset \(H\) of \(R\) is called a subhypernear-ring if

1) \((H, +)\) is a subhypergroup of \((R, +)\), i.e., \(a, b \in H\) implies \(a + b \subseteq H\), and \(a \in H\) implies \(-a \in H\),

2) \(ab \in H\), for all \(a, b \in H\).

Now we give examples of hypernear-rings and of subhypernear-rings in hypernear-rings as follows.

**Example 2.4.** Let \(R = \{0, a, b\}\) be a set with a hyperoperation \(\"+\"\) and a binary operation \(\"\cdot\"\) as follows:

\[
\begin{array}{ccc}
\oplus & 0 & a & b \\
0 & \{0\} & \{a\} & \{b\} \\
a & \{a\} & \{0, a, b\} & \{a, b\} \\
b & \{b\} & \{a, b\} & \{0, a, b\}
\end{array}
\quad
\begin{array}{ccc}
\cdot & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & a & b \\
b & 0 & a & b
\end{array}
\]

Then \((R, +, \cdot)\) is a hypernear-ring and \(\{0\}\) and \(R\) are subhypernear-rings of \(R\).

**Example 2.5.** \([8]\). Let \(R = \{0, a, b, c\}\) be a set with a hyperoperation \(\"+\"\) and a binary operation \(\"\cdot\"\) as follows:

\[
\begin{array}{ccc}
\oplus & 0 & a & b & c \\
0 & \{0\} & \{a\} & \{b\} & \{c\} \\
a & \{a\} & \{0, a\} & \{b\} & \{c\} \\
b & \{b\} & \{0, a, c\} & \{b, c\} & \{c\} \\
c & \{c\} & \{b, c\} & \{0, a, b\} & \{0, a, c\}
\end{array}
\quad
\begin{array}{ccc}
\cdot & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & 0 & a & b & c \\
b & 0 & a & b & c \\
c & a & 0 & a & b & c
\end{array}
\]

Then \((R, +, \cdot)\) is a hypernear-ring and \(\{0\}, \{0, a\}\) and \(R\) are subhypernear-rings of \(R\).

**Definition 2.6.** Let \(R\) and \(S\) be hypernear-rings, the map \(f : R \to S\) is called a homomorphism hypernear-rings if for all \(x, y \in R\), the following relations hold:

\[
f(x + y) = f(x) + f(y), \quad f(0) = 0 \quad \text{and} \quad f(xy) = f(x)f(y).
\]

From the above definition we get \(f(-x) = -f(x)\) for all \(x \in R\).
A fuzzy set \( \mu \) in a nonempty set \( X \) is a function \( \mu : X \to [0,1] \) and \( \text{Im}(\mu) \) denote the image set of \( \mu \). Let \( \mu \) be a fuzzy set in a set \( X \). For \( t \in [0,1] \), the set
\[
X^t_\mu := \{ x \in X | \mu(x) \geq t \}
\]
is called a level subset of \( \mu \).

In [4], Davvaž introduced the concept of a fuzzy subhypernear-ring of a hypernear-ring which is a generalization of the concept of a fuzzy subnear-ring in a near-ring as follows.

**Definition 2.7.** Let \((R,+,\cdot)\) be a hypernear-ring. A fuzzy set \( \mu \) in \( R \) is called a fuzzy subhypernear-ring of \( R \) if it satisfies
\[
\begin{align*}
(F1) & \quad \min\{\mu(x),\mu(y)\} \leq \inf_{\alpha \in x+y} \mu(\alpha), \\
(F2) & \quad \mu(x) \leq \mu(-x), \\
(F3) & \quad \min\{\mu(x),\mu(y)\} \leq \mu(xy)
\end{align*}
\]
for all \( x, y \in R \).

**Definition 2.8.** By a t-norm \( T \), we mean a function \( T : [0,1] \times [0,1] \to [0,1] \) satisfying the following conditions:
\[
\begin{align*}
(T1) & \quad T(x,1) = x, \\
(T2) & \quad T(x,y) \leq T(x,z) \text{ if } y \leq z, \\
(T3) & \quad T(x,y) = T(y,x), \\
(T4) & \quad T(x, T(y,z)) = T(T(x,y),z)
\end{align*}
\]
for all \( x, y \in R \).

Here are some examples of t-norms:
\[
\begin{align*}
1) & \quad T_0(x,y) = \begin{cases} x & \text{if } y = 1, \\
 & y \text{ if } x = 1, \\
 & 0 \text{ otherwise,} \end{cases} \\
2) & \quad T_1(x,y) = \max\{0,x+y-1\}, \\
3) & \quad T_2(x,y) = \frac{xy}{x+y-x-y}, \\
4) & \quad T_3(x,y) = xy, \\
5) & \quad T_4(x,y) = \frac{xy}{x+y-xy}, \\
6) & \quad T_5(x,y) = \min\{x,y\}.
\end{align*}
\]
Every t-norm \( T \) has a useful property:
\[
T(\alpha, \beta) \leq \min\{\alpha, \beta\} \text{ for all } \alpha, \beta \in [0,1].
\]

3. T-fuzzy subhypernear-rings

In what follows, let \( R \) denote a hypernear-ring unless otherwise specified. We first consider the \( T \)-fuzzification of subhypernear-rings in hypernear-rings as follows.

**Definition 3.1.** Let \( T \) be a t-norm. A fuzzy set \( \mu \) in \( R \) is called a T-fuzzy subhypernear-ring (for short, TFS-ring) of \( R \) if it satisfies
\[
\begin{align*}
(TF1) & \quad T(\mu(x),\mu(y)) \leq \inf_{\alpha \in x+y} \mu(\alpha), \\
(TF2) & \quad \mu(x) \leq \mu(-x), \\
(TF3) & \quad T(\mu(x),\mu(y)) \leq \mu(xy)
\end{align*}
\]
for all \( x, y \in R \).

**Example 3.2.** Consider the hypernear-ring \( R \) in Example 2.4, we define a fuzzy set \( \mu : R \to [0,1] \) by \( \mu(a) = \mu(b) = 1/2 \) and \( \mu(0) = 1 \). Then we have:
Theorem 3.4. If \( \mu \) is a \( T_0 \)FS-ring, \( T_1 \)FS-ring, \( T_2 \)FS-ring, \( T_3 \)FS-ring, \( T_4 \)FS-ring and \( T_5 \)FS-ring. If we consider a fuzzy set \( \lambda : R \rightarrow [0,1] \) by \( \lambda(a) < \lambda(b) < \lambda(0) \), then \( \lambda \) is not a \( T_3 \)FS-ring of \( R \), because \( \inf_{x \in b+b}\{\lambda(x)\} = \lambda(a) \) and \( \min\{\lambda(b), \lambda(b)\} = \lambda(b) \).

Example 3.3. Consider the hypernear-ring \( R \) in Example 2.5, we define a fuzzy set \( \mu \) in \( R \) by

\[
\mu(0) = 0.7, \mu(a) = 0.5 \quad \text{and} \quad \mu(b) = \mu(c) = 0.3.
\]

Routine calculations give that \( \mu \) is a \( T_1 \)FS-ring of \( R \). If we consider a fuzzy set \( \mu \) in \( R \) by

\[
\mu(0) = 0.4, \mu(a) = 0.8 \quad \text{and} \quad \mu(b) = \mu(c) = 0.3.
\]

Routine calculations give that \( \mu \) is a \( T_3 \)FS-ring of \( R \), but \( \mu \) is not a \( T_1 \)FS-ring of \( R \) since \( \inf_{\alpha \in a+a} \mu(\alpha) = 0.4 \geq 0.6 = \max\{0.8 + 0.8 - 1, 0\} \).

Theorem 3.4. Let \( I \subseteq R \). Then \( I \) is a subhypernear-ring of \( R \) if and only if \( \chi_I \) is a TFS-ring of \( R \).

Proof. Assume that \( I \) is a subhypernear-ring of \( R \). Let \( x, y \in R \). If \( x, y \in I \) then \( x + y \subseteq I \) and \( xy \in I \). Thus we have

\[
\inf_{\alpha \in x+y} \chi_I(\alpha) = 1 = T(\chi_I(x), \chi_I(y)) \quad \text{and} \quad \chi_I(xy) = 1 = T(\chi_I(x), \chi_I(y)).
\]

Otherwise, we have

\[
\inf_{\alpha \in x+y} \chi_I(\alpha) \geq 0 = T(\chi_I(x), \chi_I(y)) \quad \text{and} \quad \chi_I(xy) \geq 0 = T(\chi_I(x), \chi_I(y)).
\]

Let \( x \in R \). If \( x \in I \) then \( -x \in I \) and so we have \( \chi_I(x) = \chi_I(-x) \). If \( x \notin I \) then \( \chi_I(x) = 0 \leq \chi_I(-x) \). Therefore \( \chi_I \) is a TFS-ring of \( R \).

Conversely, assume that \( \chi_I \) is a TFS-ring of \( R \). Let \( x, y \in I \). Then \( \chi_I(x) = 1 \) and \( \chi_I(y) = 1 \). Thus for any \( z \in x + y \), we have

\[
\chi_I(z) \geq \inf_{\alpha \in x+y} \chi_I(\alpha) \geq T(\chi_I(x), \chi_I(y)) = 1, \quad \text{and} \quad \chi_I(xy) \geq T(\chi_I(x), \chi_I(y)) = 1.
\]

Hence we get \( z \in I \), i.e., \( x + y \subseteq I \), and \( xy \in I \). Let \( x \in I \). Then \( \chi_I(x) = 1 \). Thus by (TF2) we have \( 1 = \chi_I(x) \leq \chi_I(-x) \). Hence \( -x \in I \). Therefore \( I \) is a subhypernear-ring of \( R \).

Proposition 3.5. If \( \{\mu_i | i \in \Lambda\} \) is a family of TFS-rings of \( R \), then so is \( \bigcap_{i \in \Lambda} \mu_i \).

Proof. Let \( \{\mu_i | i \in \Lambda\} \) is a family of TFS-rings of \( R \) and \( x, y \in R \). Then we have

\[
\inf_{\alpha \in x+y} \left( \bigcap_{i \in \Lambda} \mu_i(\alpha) \right) = \inf_{\alpha \in x+y} \inf_{i \in \Lambda} \mu_i(\alpha) = \inf_{i \in \Lambda} \inf_{\alpha \in x+y} \mu_i(\alpha) \geq \inf_{i \in \Lambda} \left( \inf_{i \in \Lambda} \mu_i(x), \inf_{i \in \Lambda} \mu_i(y) \right) \geq \inf_{i \in \Lambda} \left( \bigcap_{i \in \Lambda} \mu_i(x), \bigcap_{i \in \Lambda} \mu_i(y) \right).
\]
For all $x \in R$, since $\mu_i(x) \leq \mu_i(-x)$ for $i \in \Lambda$, we have $\bigcap_{i \in \Lambda} \mu_i(x) \leq \bigcap_{i \in \Lambda} \mu_i(-x)$. For every $x, y \in R$, we have

$$\left( \bigcap_{i \in \Lambda} \mu_i(xy) = \inf_{i \in \Lambda} \mu_i(xy) \geq \inf_{i \in \Lambda} \{ T(\mu_i(x), \mu_i(y)) \} \geq T(\inf_{i \in \Lambda} \mu_i(x), \inf_{i \in \Lambda} \mu_i(y)).$$

Hence $\bigcap_{i \in \Lambda} \mu_i$ is a TFS-ring of $R$. \hfill \Box

**Proposition 3.6.** Let $T$ be a $t$-norm and $\mu$ be a fuzzy set of $R$. If $R_\mu^t$ is a subhypernear-ring of $R$ for all $t \in \text{Im}(\mu)$, then $\mu$ is a TFS-ring of $R$.

**Proof.** Let $x, y \in R$ be such that $\mu(x) = t$ and $\mu(y) = s$ for some $s, t \in \text{Im}(\mu)$. Without loss of generality we may assume that $s \geq t$. Then $\mu(y) = s \geq t$, and so $x, y \in R_\mu^t$. Since $R_\mu^t$ is a subhypernear-ring, we get $x + y \subseteq R_\mu^t$ and $xy \in R_\mu^t$. Thus we have

$$\inf_{\alpha \in x+y} \mu(\alpha) \geq t = \min\{s, t\} = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$.

Now let $x \in R$ be such that $\mu(x) > \mu(-x)$.

Putting $x_0 = \frac{1}{2}(\mu(x) + \mu(-x))$, then $\mu(-x) < x_0 < \mu(x)$, and so $x \in R_{\mu_0}^t$ but $-x \notin R_{\mu_0}^t$. This leads to a contradiction.

Therefore $\mu$ is a TFS-ring of $R$. \hfill \Box

**Proposition 3.7.** Let $T$ be a $t$-norm and $H$ be a subhypernear-ring of $R$. Then there exists a TFS-ring $\mu$ of $R$ such that $R_\mu^t = H$ for some $t \in (0, 1]$.

**Proof.** Let $\mu$ be a fuzzy set in $R$ defined by

$$\mu(x) := \begin{cases} t & \text{if } x \in H, \\ 0 & \text{otherwise}, \end{cases}$$

where $t$ is a fixed number in $(0, 1]$. Let $x, y \in R$. If $x \in R \setminus H$ or $y \in R \setminus H$, then $\mu(x) = 0$ or $\mu(y) = 0$ and so we have

$$\inf_{\alpha \in x+y} \mu(\alpha) \geq 0 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$. If $x, y \in H$, then we have

$$\inf_{\alpha \in x+y} \mu(\alpha) \geq t = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$.

Let $x \in R$. If $x \in R \setminus H$, then $\mu(x) = 0$ and so we have $\mu(-x) \geq 0 = \mu(x)$. If $x \in H$ then we have $\mu(-x) \geq t = \mu(x)$.

Therefore $\mu$ is a TFS-ring of $R$. It is clear that $R_{\mu}^t = H$. \hfill \Box

**Theorem 3.8.** Let $T$ be a $t$-norm and $\mu$ be a fuzzy set of $R$ with $\text{Im}(\mu) = \{t_1, t_2, \cdots, t_n\}$, where $t_i < t_j$ whenever $i > j$. Suppose that there exists a chain of subhypernear-rings of $R$:

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = R$$

such that $\mu(H_k) = t_k$, where $H_k = H_k \setminus H_{k-1}, H_{-1} = \emptyset$ for $k = 0, 1, \cdots, n$. Then $\mu$ is a TFS-ring of $R$.

**Proof.** Let $x, y \in R$. If $x$ and $y$ belong to the same $H_k$, then we have $\mu(x) = \mu(y) = t_k, x + y \subseteq H_k$ and $xy \in H_k$. Thus we get

$$\inf_{\alpha \in x+y} \mu(\alpha) \geq t_k = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$
and \( \mu(xy) \geq T(\mu(x), \mu(y)) \). If \( x \in H_i^+ \) and \( y \in H_j^+ \) for every \( i \neq j \). Without loss of generality, we may assume that \( i \geq j \). Then we have \( \mu(x) = t_i < t_j = \mu(y) \), \( x + y \subseteq H_i \) and \( xy \in H_i \). It follows that

\[
\inf_{\alpha \in x+y} \mu(\alpha) \geq t_i = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))
\]

and \( \mu(xy) \geq T(\mu(x), \mu(y)) \).

Let \( x \in R \). Then there exists \( H_k \) such that \( x \in H_k^+ \) for some \( k \in \{0, 1, \ldots, n\} \). Thus we have \( \mu(x) = t_k = \mu(-x) \).

Therefore \( \mu \) is a TFS-ring of \( R \).

For a t-norm \( T \) on \([0, 1]\), denote by \( \alpha_T \) the set of element \( \alpha \in [0, 1] \) such that \( T(\alpha, \alpha) = \alpha \), i.e., \( \alpha_T := \{ \alpha \in [0, 1] \mid T(\alpha, \alpha) = \alpha \} \).

A fuzzy set \( \mu \) in a set \( X \) is said to satisfy imaginable property if \( \text{Im}(\mu) \subseteq \alpha_T \).

**Definition 3.9.** A TFS-ring is said to be imaginable if it satisfies the imaginable property.

**Proposition 3.10.** For a subhypernear-ring \( H \) of \( R \), let \( \mu \) be a fuzzy set in \( R \) given by

\[
\mu(x) := \begin{cases} 
  s & \text{if } x \in H, \\
  t & \text{otherwise}
\end{cases}
\]

for all \( s, t \in [0, 1] \) with \( s > t \). Then \( \mu \) is a \( T_1 \)-FS-ring of \( R \). In particular, if \( s = 1 \) and \( t = 0 \) then \( \mu \) is imaginable.

**Proof.** Let \( x, y \in R \). If \( x, y \in H \) then we get \( x + y \subseteq H \) and \( xy \in H \) since \( H \) is a subhypernear-ring of \( R \), and so

\[
T_1(\mu(x), \mu(y)) = \max\{s + s - 1, 0\} \leq s = \inf_{\alpha \in x+y} \mu(\alpha)
\]

and \( T_1(\mu(x), \mu(y)) \leq \mu(xy) \). If \( x \in H \) and \( y \not\in H \) (or, \( x \not\in H \) and \( y \in H \)). Then \( \mu(x) = s > t = \mu(y) \) (or, \( \mu(x) = t < s = \mu(y) \)). It follows that

\[
T_1(\mu(x), \mu(y)) = \max\{s + t - 1, 0\} \leq t = \inf_{\alpha \in x+y} \mu(\alpha)
\]

and \( T_1(\mu(x), \mu(y)) \leq \mu(xy) \). If \( x \not\in H \) and \( y \not\in H \). Then \( \mu(x) = t = \mu(y) \) and so we have

\[
T_1(\mu(x), \mu(y)) = \max\{t + t - 1, 0\} \leq t = \inf_{\alpha \in x+y} \mu(\alpha)
\]

and \( T_1(\mu(x), \mu(y)) \leq \mu(xy) \).

Let \( x \in R \). If \( x \in H \) then \( -x \in H \) and so we have \( \mu(x) = s \leq \mu(-x) \). If \( x \not\in H \) then we get \( \mu(x) = t \leq \mu(-x) \).

Therefore \( \mu \) is a \( T_1 \)-FS-ring of \( R \). Obviously \( \mu \) is imaginable when \( s = 1 \) and \( t = 0 \).

**Proposition 3.11.** Let \( T \) be a t-norm and \( \mu \) be an imaginable TFS-ring of \( R \). Then \( \mu(0) \geq \mu(x) \) for all \( x \in R \).

**Proof.** For every \( x \in R \) we have \( 0 \in x - x \) and so

\[
\mu(0) \geq \inf_{z \in x-x} \mu(z) \geq T(\mu(x), \mu(-x)) = T(\mu(x), \mu(x)) = \mu(x).
\]
Proof. Let $\mu$ be an imaginable TFS-ring of $R$. Since $\mu$ satisfies the imaginable property, we have
\[ \min\{\mu(x), \mu(y)\} = T(\min\{\mu(x), \mu(y)\}, \min\{\mu(x), \mu(y)\}) \leq T(\mu(x), \mu(y)) \leq \min\{\mu(x), \mu(y)\} \]
for all $x, y \in R$. It follows that $\inf_{\alpha \in x+y} \mu(\alpha) \geq T(\mu(x), \mu(y)) = \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$. Therefore $\mu$ is a fuzzy subhypernear-ring of $R$. \hfill \square

**Theorem 3.13.** Let $\mu$ be a TFS-ring of $R$ and let $t \in [0, 1]$. Then
(i) if $t = 1$ then $R^t_\mu$ is either empty or a subhypernear-ring of $R$,
(ii) if $T = \min$, then $R^t_\mu$ is either empty or a subhypernear-ring of $R$.

**Proof.** (i) Assume that $t = 1$ and let $x, y \in R^t_\mu$. Then we have
\[ \inf_{\alpha \in x+y} \mu(\alpha) \geq T(\mu(x), \mu(y)) = T(1, 1) = 1 = t, \]
and $\mu(xy) \geq t$. Thus $\alpha \in R^t_\mu$ and so we get $x + y \subseteq R^t_\mu$, and $xy \in R^t_\mu$.

Let $x \in R^t_\mu$. Then since $\mu$ is a TFS-ring of $R$, we have $\mu(-x) \geq \mu(x) \geq t$. Thus we get $-x \in R^t_\mu$.

Therefore $R^t_\mu$ is a subhypernear-ring of $R$ whence $t = 1$.

(ii) Similar to the proof of (i). \hfill \square

**Theorem 3.14.** Let $T$ be a t-norm and let $\mu$ be an imaginable fuzzy set in $R$. If each non-empty level subset $R^\alpha_\mu$ of $\mu$ is a subhypernear-ring of $R$, then $\mu$ is an imaginable TFS-ring of $R$.

**Proof.** For $t \in [0, 1]$, suppose that $R^t_\mu$ is a non-empty set and a a subhypernear-ring of $R$. Then we have $\inf_{\alpha \in x+y} \mu(\alpha) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$. Indeed, if not then there exist $x_0, y_0 \in R$ such that $\inf_{\alpha \in x_0+y_0} \mu(a) < \min\{\mu(x_0), \mu(y_0)\}$. Taking
\[ s_0 := \frac{1}{2} \{ \inf_{\alpha \in x_0+y_0} \mu(a) + \min\{\mu(x_0), \mu(y_0)\} \}, \]
then we get $\inf_{\alpha \in x_0+y_0} \mu(a) < s_0 < \min\{\mu(x_0), \mu(y_0)\}$ and thus $x_0, y_0 \in R^{s_0}_\mu$ and $x_0+y_0 \not\subseteq R^{s_0}_\mu$.

This is a contradiction. Hence we have
\[ \inf_{\alpha \in x+y} \mu(\alpha) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y)) \]
for all $x, y \in R$.

Now if (TF2) is not true, then $\mu(x_0y_0) < \min\{\mu(x_0), \mu(y_0)\}$ for some $x_0, y_0 \in R$. Taking
\[ s_0 := \frac{1}{2} \{ \mu(x_0y_0) + \min\{\mu(x_0), \mu(y_0)\} \}, \]
then we get $\mu(x_0y_0) < s_0 < \min\{\mu(x_0), \mu(y_0)\}$ and thus $x_0, y_0 \in R^{s_0}_\mu$ and $x_0y_0 \not\subseteq R^{s_0}_\mu$. This is a contradiction. Hence we have
\[ \mu(xy) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y)) \]
for all $x, y \in R$.

Finally, if (TF3) is not true, then $\mu(x_0) > \mu(-x_0)$ for some $x_0 \in R$. Taking
\[ s_0 := \frac{1}{2} \{ \mu(x_0) + \mu(-x_0) \}, \]
then we get $\mu(x_0) > s_0 > \mu(-x_0)$ and thus $x_0 \in R^{s_0}_\mu$ and $-x_0 \not\subseteq R^{s_0}_\mu$. It is a contradiction. Therefore $\mu$ is an imaginable TFS-ring of $R$. \hfill \square
Proposition 3.15. Let $T$ be a t-norm and let $f : R \to S$ be a homomorphism of hypernear-rings. If $\mu$ is a TFS-ring of $S$, then $f^{-1}(\mu)$ is a TFS-ring of $R$.

Proof. Assume that $\mu$ is a TFS-ring of $S$. Let $x, y \in R$. Then we get

$$\inf_{\alpha \in \mathbb{R}} f^{-1}(\mu)(\alpha) = \inf_{\alpha \in (x+y)} f(\alpha) \geq T(x, y) = \inf_{\alpha \in (x+y)} f^{-1}(\mu)(\alpha),$$

and

$$f^{-1}(\mu(xy)) = f^{-1}(\mu(f(x)f(y))) = f^{-1}(\mu(x)f(y)) = f^{-1}(\mu(x), f^{-1}(\mu)(y)).$$

Also, we have $f^{-1}(\mu)(x) = f^{-1}(\mu)(1-x) = f^{-1}(\mu)(1-x)$ for all $x \in R$.

Therefore $f^{-1}(\mu)$ is a TFS-ring of $R$. □

4. Direct product of TFS-rings

Definition 4.1. Let $T$ be a t-norm and let $\mu$ and $\nu$ be fuzzy sets in $R$. Then the $T$-product of $\mu$ and $\nu$, written $[\mu \cdot \nu]_T$, is defined by $[\mu \cdot \nu]_T(x) := T(\mu(x), \nu(x))$ for all $x \in R$.

Proposition 4.2. Let $T$ be a t-norm and let $\mu$ and $\nu$ be TFS-rings in $R$. If $T^*$ is a t-norm which dominates $T$, i.e., $T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T^*(\alpha, \gamma, \beta, \delta)$ for all $\alpha, \beta, \gamma, \delta \in [0, 1]$, then $T^*$-product of $\mu$ and $\nu$, $[\mu \cdot \nu]_{T^*}$, is a TFS-ring of $R$.

Proof. Let $x, y \in R$. Then we have

$$\inf_{\alpha \in \mathbb{R}} [\mu \cdot \nu]_{T^*}(\alpha) = \inf_{\alpha \in \mathbb{R}} T^*(\mu(\alpha), \nu(\alpha)) \geq T^*(\inf_{\alpha \in \mathbb{R}} \mu(\alpha), \nu(\alpha)) \geq T^*(\mu(x), \nu(x)) \geq T^*((\mu(x), \nu(x)) = T^*(\mu(x), \nu(x)), [\mu \cdot \nu]_{T^*}(y)),$$

and

$$[\mu \cdot \nu]_{T^*}(xy) = T^*(\mu(xy), \nu(xy)) \geq T^*(\mu(x), \nu(x)) \geq T^*(\mu(\mu(x), \nu(\nu(y))) \geq T^*(\mu(x), \nu(x)) = T^*(\mu(x), \nu(x)), [\mu \cdot \nu]_{T^*}(y)).$$

Also, we get $[\mu \cdot \nu]_{T^*}(x) \leq T^*(\mu(-x), \nu(-x)) = [\mu \cdot \nu]_{T^*}(-x)$ for all $x \in R$.

Therefore $[\mu \cdot \nu]_{T^*}$ is a TFS-ring of $R$. □

Let $f : R \to S$ be a homomorphism of hypernear-rings, and let $T$ and $T^*$ be t-norms such that $T^*$ dominates $T$. If $\mu$ and $\nu$ are TFS-rings in $S$, then $[\mu \cdot \nu]_{T^*}$ is a TFS-ring of $R$. By 3.15, the inverse images $f^{-1}(\mu), f^{-1}(\nu)$ and $f^{-1}([\mu \cdot \nu]_{T^*})$ are TFS-rings of $R$.

The next theorem provides that the relation between $f^{-1}([\mu \cdot \nu]_{T^*})$ and the $T^*$-product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 4.3. Let $f : R \to S$ be a homomorphism of hypernear-rings, and let $T$ and $T^*$ be t-norms such that $T^*$ dominates $T$. Let $\mu$ and $\nu$ be TFS-rings in $S$. If $[\mu \cdot \nu]_{T^*}$ is the $T^*$-product of $\mu$ and $\nu$, and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ is the $T^*$-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$ then $f^{-1}([\mu \cdot \nu]_{T^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$.

Proof. Let $x \in R$. Then we have

$$f^{-1}([\mu \cdot \nu]_{T^*})(x) = [\mu \cdot \nu]_{T^*}(f(x)) = T^*(\mu(f(x)), \nu(f(x))) = T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x)) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}.□$$
Let $R_1$ and $R_2$ be two hypernear-rings, then for all $(x_1, y_1)$ and $(x_2, y_2)$ in $R_1 \times R_2$ we define

$$(x_1, y_1) + (x_2, y_2) = \{(x, y) \mid x \in x_1 + x_2, \ y \in y_1 + y_2\}$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2).$$

Clearly $R_1 \times R_2$ is a hypernear-ring and we call this hypernear-ring the direct product of $R_1$ and $R_2$.

**Definition 4.4.** Let $T$ be a t-norm and let $\mu_1$ and $\mu_2$ be fuzzy sets on hypernear-rings $R_1$ and $R_2$ respectively. Then $\mu$ defined on $R_1 \times R_2$ by the formula

$$\mu(x, y) = T(\mu_1(x), \mu_2(y))$$

is a fuzzy set on $R_1 \times R_2$ which is defined by $\mu_1 \times \mu_2$.

**Theorem 4.5.** Let $T$ be a t-norm and let $R = R_1 \times R_2$ be the direct product of hypernear-rings $R_1$ and $R_2$. If $\mu_1$ and $\mu_2$ are TFS-rings of $R_1$ and $R_2$ respectively, then $\mu = \mu_1 \times \mu_2$ is a TFS-ring of $R$.

**Proof.** Let $(x_1, y_1)$ and $(x_2, y_2)$ be two arbitrary elements of $R_1 \times R_2$. For every $(x, y) \in (x_1, y_1) + (x_2, y_2)$ we have

$$(\mu_1 \times \mu_2)(x, y) = T(\mu_1(x), \mu_2(y))$$

$$\geq T(T(\mu_1(x_1), \mu_1(x_2)), T(\mu_2(y_1), \mu_2(y_2)))$$

$$= T(T(\mu_1(x_1), \mu_1(x_2)), T(\mu_2(y_1), \mu_2(y_2)))$$

$$= T(T(\mu_1(x_1), \mu_1(x_2)), T(\mu_2(y_1), \mu_2(y_2)))$$

$$= T(T(\mu_2(y_1), \mu_1(x_1)), T(\mu_2(y_1), \mu_1(x_2)))$$

$$= T(T(\mu_1(x_1), \mu_2(y_1)), T(\mu_1(y_1), \mu_2(y_1)))$$

$$= T((\mu_1 \times \mu_2)(x_1, y_1), (\mu_1 \times \mu_2)(x_2, y_2)).$$

Hence

$$\inf_{(x, y) \in (x_1, y_1) + (x_2, y_2)} (\mu_1 \times \mu_2)(x, y) \geq T((\mu_1 \times \mu_2)(x_1, y_1), (\mu_1 \times \mu_2)(x_2, y_2)).$$

Similarly we obtain

$$(\mu_1 \times \mu_2)((x_1, y_1) \cdot (x_2, y_2)) = (\mu_1 \times \mu_2)(x_1 x_2, y_1 y_2)$$

$$= T((\mu_1(x_1 x_2), \mu_2(y_1 y_2))$$

$$\geq T(T(\mu_1(x_1), \mu_1(x_2)), T(\mu_2(y_1), \mu_2(y_2)))$$

$$\geq T(T(\mu_1(x_1), \mu_1(x_2)), T(\mu_2(y_1), \mu_2(y_2)))$$

$$= T((\mu_1 \times \mu_2)(x_1, y_1), (\mu_1 \times \mu_2)(x_2, y_2)).$$

Also, we have

$$(\mu_1 \times \mu_2)(x, y) = T(\mu_1(x), \mu_2(y)) \leq T(\mu_1(-x), \mu_2(-y)) = (\mu_1 \times \mu_2)(-x, -y).$$

Therefore $\mu_1 \times \mu_2$ is a TFS-ring of $R_1 \times R_2$.

**Theorem 4.6.** Let $T$ be a t-norm and let $\mu_1$ and $\mu_2$ be fuzzy sets of the hypernear-rings $R_1$ and $R_2$ respectively. If $\mu_1 \times \mu_2$ is an imaginarable TFS-ring of $R_1 \times R_2$, then at least one of the following two statements must hold:

1. $\mu_2(0) \geq \mu_1(x)$ for all $x \in R_1$,
2. $\mu_1(0) \geq \mu_2(y)$ for all $y \in R_2$.

**Proof.** Suppose that $\mu_1 \times \mu_2$ is an imaginarable TFS-ring of $R_1 \times R_2$. By contraposition, suppose that none of the statements (1) and (2) holds. Then there exist $x_0 \in R_1$ and $y_0 \in R_2$ such that

$$\mu_1(x_0) > \mu_2(0) \text{ and } \mu_2(y_0) > \mu_1(0).$$
Now, we have
\[(\mu_1 \times \mu_2)(x_0, y_0) = T(\mu_1(x_0), \mu_2(y_0)) > T(\mu_2(0), \mu_1(0)) = T(\mu_1(0), \mu_2(0)) = (\mu_1 \times \mu_2)(0, 0).\]

But, by Proposition 3.11, always we have \((\mu_1 \times \mu_2)(0, 0) \geq (\mu_1 \times \mu_2)(x_0, y_0)\). □

**Theorem 4.7.** Let \(T\) be a t-norm. Let \(\mu_1, \mu_2, \mu_1 \times \mu_2\) be fuzzy sets of the hypernear-rings \(R_1, R_2\) and \(R_1 \times R_2\) respectively, such that satisfy imaginable property. If \(\mu_1 \times \mu_2\) is a TFS-ring of \(R_1 \times R_2\), then \(\mu_1\) is a TFS-ring of \(R_1\) or \(\mu_2\) is a TFS-ring of \(R_2\).

**Proof.** Since \(\mu_1 \times \mu_2\) is an imaginable TFS-ring of \(R_1 \times R_2\), by Theorem 4.6, we assume that \(\mu_1(x) \leq \mu_2(0)\) for all \(x \in R_1\), and we show that \(\mu_1\) is a TFS-ring of \(R_1\). Let \(x\) and \(y\) be two arbitrary elements of \(R_1\). For every \(z \in x + y\) we have
\[
\mu_1(z) = T(\mu_1(z), 1) \geq T(\mu_1(z), \mu_2(0)) = (\mu_1 \times \mu_2)(z, 0) \geq \inf_{(z, 0) \in (x, 0) + (y, 0)} (\mu_1 \times \mu_2)(z, 0) \geq T((\mu_1 \times \mu_2)(x, 0), (\mu_1 \times \mu_2)(y, 0)) = T(T(\mu_1(x), \mu_2(0)), T(\mu_1(y), \mu_2(0))) \geq T(T(\mu_1(x), \mu_1(x)), T(\mu_1(y), \mu_1(x))) = T(\mu_1(x), 1) \geq T(\mu_1(x), \mu_1(y)).
\]

Therefore \(\inf_{z \in x+y} \mu_1(z) \geq T(\mu_1(x), \mu_1(y))\). Similarly, we obtain
\[
\mu_1(xy) = T(\mu_1(xy), 1) \geq T(\mu_1(xy), \mu_2(0)) = (\mu_1 \times \mu_2)(xy, 0) = (\mu_1 \times \mu_2)((x, 0) \cdot (y, 0)) \geq T((\mu_1 \times \mu_2)(x, 0), (\mu_1 \times \mu_2)(y, 0)) \geq T(T(\mu_1(x), \mu_1(x)), \mu_1(y)).
\]

Also we have
\[
\mu_1(-x) = T(\mu_1(-x), 1) \geq T(\mu_1(-x), \mu_2(0)) = (\mu_1 \times \mu_2)(-x, 0) = (\mu_1 \times \mu_2)(-x, 0) = T(\mu_1(x), \mu_2(0)) \geq T(\mu_1(x), \mu_1(x)) = \mu_1(x).
\]
□

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