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T-FUZZY SUBHYPERNEAR-RINGS OF HYPERNEAR-RINGS

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ABSTRACT. Using a t-norm T, the notion of T-fuzzy subhypernear-rings (for short TFS-rings) of hypernear-rings is introduced and some of their properties are investigated. Also we study the structure of TFS-rings under direct product.

1. INTRODUCTION

The theory of hyperstructures has been introduced by Matry in 1934 during the 8^{th} congress of the Scandinavian Mathematicians [10]. Marty introduced the notion of a hypergroup and then many researchers have been worked on this new field of modern algebra and developed it. A comprehensive review of the theory of hyperstructures appear in [2] and [14]. The notion of the hyperfield and hyperring was studied by Krasner [9]. In [3], Dasic has introduced the notion of hypernear-rings generalizing the concept of near-ring [11]. In [7], Gontineac defined the zero-symmetric part and the constant part of a hypernear-ring and introduced a structure theorem and other properties of hypernear-rings. Davvaz in [5] introduced the notion of an H_v-near ring generalizing the notion of hypernear-ring.

The concept of fuzzy sets was introduced by Zadeh [15]. It was first applied to the theory of groups by Rosenfeld [12]. Rosenfeld has introduced fuzzy subgroups of a group and many researchers are engaged in extending the concept. In [1], Anthony and sherwood redefined a fuzzy subgroup of a group using the concept of a triangular norm, also see [6]. This notion was introduced by Schweizer and Sklar [13], in order to generalize the ordinary triangle inequality in a metric space to the more general probabilistic metric spaces.

In [4], Davvaz has introduced the concept of fuzzy subhypernear-rings and fuzzy hyperideals of a hypernear-ring which are a generalization of the concept of a fuzzy subnear-rings and fuzzy ideals in a near-ring. Now, in this paper, using a t-norm T, the notion of T-fuzzy subhypernear-rings (for short TFS-rings) of hypernear-rings is introduced and some of their properties are investigated. Also we study the structure of TFS-rings under direct product.

2. PRELIMINARIES

We now review some basic definitions for the sake of completeness. These definitions are taken primarily from [3,4,7,13].

Definition 2.1. Let H be a non-empty set. A *hyperoperation* $*$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H.

Definition 2.2. A *hypernear-ring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

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1) $(R, +)$ is a *quasi canonical hypergroup* (not necessarily commutative), i.e., in $(R, +)$ the following hold:

a) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;
b) There is $0 \in R$ such that $x + 0 = 0 + x = x$ for all $x \in R$;
c) For every $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$, (we shall write $-x$ for x' and we call it the opposite of x);
d) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

2) With respect to the multiplication, $(R, \cdot)$ is a semigroup having absorbing element 0 i.e., $x \cdot 0 = 0$ for all $x \in R$.
3) The multiplication is distributive with respect to the hyperoperation $+$ on the left side i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

If $x \in R$ and A, B are subsets of R, then by $A + B$, $A + x$ and $x + B$ we mean

$$A + B = \bigcup_{\substack{a \in A \\ b \in B}} a + b, \;\; A + x = A + \{x\}, \;\; x + B = \{x\} + B.$$

Note that for all $x, y \in R$, we have $-(-x) = x, 0 = -0, -(x + y) = -y - x$ and $x(-y) = -xy$.

Definition 2.3. Let $(R, +, \cdot)$ be a hypernear-ring. A non-empty subset H of R is called a *subhypernear-ring* if

(1) $(H, +)$ is a subhypergroup of $(R, +)$, i.e., $a, b \in H$ implies $a + b \subseteq H$, and $a \in H$ implies $-a \in H$,
(2) $ab \in H$, for all $a, b \in H$.

Now we give examples of hypernear-rings and of subhypernear-rings in hypernear-rings as follows.

Example 2.4. Let $R = \{0, a, b\}$ be a set with a hyperoperation "+" and a binary operation " $\cdot$ " as follows:

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	a	b

Then $(R, +, \cdot)$ is a hypernear-ring and $\{0\}$ and R are subhypernear-rings of R.

Example 2.5. [8]. Let $R = \{0, a, b, c\}$ be a set with a hyperoperation " $+$ " and a binary operation " $\cdot$ " as follows:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0, a, c\}$	$\{b, c\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, a, b\}$

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Then $(R, +, \cdot)$ is a hypernear-ring and $\{0\}, \{0, a\}$ and R are subhypernear-rings of R.

Definition 2.6. Let R and S be hypernear-rings, the map $f : R \to S$ is called a *homomorphism* hypernear-rings if for all $x, y \in R$, the following relations hold:

$$f(x + y) = f(x) + f(y), \quad f(0) = 0 \;\text{ and }\; f(xy) = f(x)f(y).$$

¿From the above definition we get $f(-x) = -f(x)$ for all $x \in R$.

A *fuzzy set* μ in a nonempty set X is a function $\mu : X \to [0,1]$ and $\mathrm{Im}(\mu)$ denote the *image set* of μ. Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set

$$X_\mu^t := \{x \in X | \mu(x) \geq t\}$$

is called a *level subset* of μ.

In [4], Davvaz introduced the concept of a fuzzy subhypernear-ring of a hypernear-ring which is a generalization of the concept of a fuzzy subnear-ring in a near-ring as follows.

Definition 2.7. Let $(R, +, \cdot)$ be a hypernear-ring. A fuzzy set μ in R is called a *fuzzy subhypernear-ring* of R if it satisfies

(F1) $\min\{\mu(x), \mu(y)\} \leq \inf_{\alpha \in x+y} \mu(\alpha)$,
(F2) $\mu(x) \leq \mu(-x)$,
(F3) $\min\{\mu(x), \mu(y)\} \leq \mu(xy)$

for all $x, y \in R$.

Definition 2.8. By a *t-norm* T, we mean a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(T1) $T(x, 1) = x$,
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$,
(T3) $T(x, y) = T(y, x)$,
(T4) $T(x, T(y, z)) = T(T(x, y), z)$

for all $x, y \in R$.

Here are some examples of t-norms:

1) $T_0(x, y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwize,} \end{cases}$
2) $T_1(x, y) = \max\{0, x + y - 1\}$,
3) $T_2(x, y) = \frac{xy}{2-(x+y-xy)}$,
4) $T_3(x, y) = xy$,
5) $T_4(x, y) = \frac{xy}{x+y-xy}$,
6) $T_5(x, y) = \min\{x, y\}$.

Every t-norm T has a useful property:

$$T(\alpha, \beta) \leq \min\{\alpha, \beta\} \text{ for all } \alpha, \beta \in [0,1].$$

3. T-fuzzy subhypernear-rings

In what follows, let R denote a hypernear-ring unless otherwise specified. We first consider the T-fuzzification of subhypernear-rings in hypernear-rings as follows.

Definition 3.1. Let T be a t-norm. A fuzzy set μ in R is called a *T-fuzzy subhypernear-ring* (for short, *TFS-ring*) of R if it satisfies

(TF1) $T(\mu(x), \mu(y)) \leq \inf_{\alpha \in x+y} \mu(\alpha)$,
(TF2) $\mu(x) \leq \mu(-x)$,
(TF3) $T(\mu(x), \mu(y)) \leq \mu(xy)$

for all $x, y \in R$.

Example 3.2. Consider the hypernear-ring R in Example 2.4, we define a fuzzy set $\mu : R \longrightarrow [0,1]$ by $\mu(a) = \mu(b) = 1/2$ and $\mu(0) = 1$. Then we have:

(x,y)	$(0,0)$	$(0,a)$	$(0,b)$	$(a,0)$	(a,a)	(a,b)	$(b,0)$	(b,a)	(b,b)
$\inf_{z\in x+y}\mu(z)$	1	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
$\mu(xy)$	1	1	1	1	1/2	1/2	1	1/2	1/2
$T_0(\mu(x),\mu(y))$	1	1/2	1/2	1/2	0	0	1/2	0	0
$T_1(\mu(x),\mu(y))$	0	1/2	1/2	1/2	0	0	1/2	0	0
$T_2(\mu(x),\mu(y))$	1	1/2	1/2	1/2	1/5	1/5	1/2	1/5	1/5
$T_3(\mu(x),\mu(y))$	1	1/2	1/2	1/2	1/4	1/4	1/2	1/4	1/4
$T_4(\mu(x),\mu(y))$	1	1/2	1/2	1/2	1/3	1/3	1/2	1/3	1/3
$T_5(\mu(x),\mu(y))$	1	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2

The above table show that μ is a T_0FS-ring, T_1FS-ring, T_2FS-ring, T_3FS-ring, T_4FS-ring and T_5FS-ring. If we consider a fuzzy set $\lambda : R \longrightarrow [0,1]$ by $\lambda(a) < \lambda(b) < \lambda(0)$, then λ is not a T_5FS-ring of R, because $\inf_{x\in b+b}\{\lambda(x)\} = \lambda(a)$ and $\min\{\lambda(b),\lambda(b)\} = \lambda(b)$.

Example 3.3. Consider the hypernear-ring R in Example 2.5, we define a fuzzy set μ in R by

$$\mu(0) = 0.7, \mu(a) = 0.5 \text{ and } \mu(b) = \mu(c) = 0.3.$$

Routine calculations give that μ is a T_1FS-ring of R. If we consider a fuzzy set μ in R by

$$\mu(0) = 0.4, \mu(a) = 0.8 \text{ and } \mu(b) = \mu(c) = 0.3.$$

Routine calculations give that μ is a T_3FS-ring of R, but μ is not a T_1FS-ring of R since $\inf\limits_{\alpha\in a+a}\mu(\alpha) = 0.4 \not\geq 0.6 = \max\{0.8+0.8-1, 0\}$.

Theorem 3.4. *Let $I \subseteq R$. Then I is a subhypernear-ring of R if and only if χ_I is a TFS-ring of R.*

Proof. Assume that I is a subhypernear-ring of R. Let $x, y \in R$. If $x, y \in I$ then $x + y \subseteq I$ and $xy \in I$. Thus we have

$$\inf_{\alpha\in x+y}\chi_I(\alpha) = 1 = T(\chi_I(x),\chi_I(y)) \text{ and } \chi_I(xy) = 1 = T(\chi_I(x),\chi_I(y)).$$

Otherwise, we have

$$\inf_{\alpha\in x+y}\chi_I(\alpha) \geq 0 = T(\chi_I(x),\chi_I(y)) \text{ and } \chi_I(xy) \geq 0 = T(\chi_I(x),\chi_I(y)).$$

Let $x \in R$. If $x \in I$ then $-x \in I$ and so we have $\chi_I(x) = \chi_I(-x)$. If $x \notin I$ then $\chi_I(x) = 0 \leq \chi_I(-x)$. Therefore χ_I is a TFS-ring of R.

Conversely, assume that χ_I is a TFS-ring of R. Let $x, y \in I$. Then $\chi_I(x) = 1$ and $\chi_I(y) = 1$. Thus for any $z \in x + y$, we have

$$\chi_I(z) \geq \inf_{\alpha\in x+y}\chi_I(\alpha) \geq T(\chi_I(x),\chi_I(y)) = 1, \text{ and } \chi_I(xy) \geq T(\chi_I(x),\chi_I(y)) = 1.$$

Hence we get $z \in I$, i.e., $x + y \subseteq I$, and $xy \in I$. Let $x \in I$. Then $\chi_I(x) = 1$. Thus by (TF2) we have $1 = \chi_I(x) \leq \chi_I(-x)$. Hence $-x \in I$. Therefore I is a subhypernear-ring of R □

Proposition 3.5. *If $\{\mu_i | i \in \Lambda\}$ is a family of TFS-rings of R, then so is $\bigcap\limits_{i\in\Lambda}\mu_i$.*

Proof. Let $\{\mu_i | i \in \Lambda\}$ is a family of TFS-rings of R and $x, y \in R$. Then we have

$$\begin{aligned} \inf_{\alpha\in x+y}(\bigcap_{i\in\Lambda}\mu_i)(\alpha) &= \inf_{\alpha\in x+y}\{\inf_{i\in\Lambda}\mu_i(\alpha)\} = \inf_{i\in\Lambda}\{\inf_{\alpha\in x+y}\mu_i(\alpha)\} \geq \inf_{i\in\Lambda}\{T(\mu_i(x),\mu_i(y))\} \\ &\geq T(\inf_{i\in\Lambda}\mu_i(x), \inf_{i\in\Lambda}\mu_i(y)) \geq T(\bigcap_{i\in\Lambda}\mu_i(x), \bigcap_{i\in\Lambda}\mu_i(y)). \end{aligned}$$

For all $x \in R$, since $\mu_i(x) \leq \mu_i(-x)$ for $i \in \Lambda$, we have $\bigcap_{i\in\Lambda} \mu_i(x) \leq \bigcap_{i\in\Lambda} \mu_i(-x)$. For every $x, y \in R$, we have

$$(\bigcap_{i\in\Lambda} \mu_i)(xy) = \inf_{i\in\Lambda} \mu_i(xy) \geq \inf_{i\in\Lambda} \{T(\mu_i(x), \mu_i(y))\} \geq T(\inf_{i\in\Lambda} \mu_i(x), \inf_{i\in\Lambda} \mu_i(y)).$$

Hence $\bigcap_{i\in\Lambda} \mu_i$ is a TFS-ring of R. □

Proposition 3.6. *Let T be a t-norm and μ be a fuzzy set of R. If R^t_μ is a subhypernear-ring of R for all $t \in Im(\mu)$, then μ is a TFS-ring of R.*

Proof. Let $x, y \in R$ be such that $\mu(x) = t$ and $\mu(y) = s$ for some $s, t \in Im(\mu)$. Without loss of generality we may assume that $s \geq t$. Then $\mu(y) = s \geq t$, and so $x, y \in R^t_\mu$. Since R^t_μ is a subhypernear-ring, we get $x + y \subseteq R^t_\mu$ and $xy \in R^t_\mu$. Thus we have

$$\inf_{\alpha\in x+y} \mu(\alpha) \geq t = \min\{s, t\} = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$.

Now let $x \in R$ be such that $\mu(x) > \mu(-x)$. Putting $x_0 = \frac{1}{2}\{\mu(x) + \mu(-x)\}$, then $\mu(-x) < x_0 < \mu(x)$, and so $x \in R^{x_0}_\mu$ but $-x \notin R^{x_0}_\mu$. This leads to a contradiction.

Therefore μ is a TFS-ring of R. □

Proposition 3.7. *Let T be a t-norm and H be a subhypernear-ring of R. Then there exists a TFS-ring μ of R such that $R^t_\mu = H$ for some $t \in (0, 1]$.*

Proof. Let μ be a fuzzy set in R defined by

$$\mu(x) := \begin{cases} t & \text{if } x \in H, \\ 0 & \text{otherwise,} \end{cases}$$

where t is a fixed number in $(0, 1]$. Let $x, y \in R$. If $x \in R\backslash H$ or $y \in R\backslash H$, then $\mu(x) = 0$ or $\mu(y) = 0$ and so we have

$$\inf_{\alpha\in x+y} \mu(\alpha) \geq 0 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$. If $x, y \in H$, then we have

$$\inf_{\alpha\in x+y} \mu(\alpha) \geq t = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$.

Let $x \in R$. If $x \in R\backslash H$, then $\mu(x) = 0$ and so we have $\mu(-x) \geq 0 = \mu(x)$. If $x \in H$ then we have $\mu(-x) \geq t = \mu(x)$.

Therefore μ is a TFS-ring of R. It is clear that $R^t_\mu = H$. □

Theorem 3.8. *Let T be a t-norm and μ be a fuzzy set of R with $Im(\mu) = \{t_1, t_2, \cdots, t_n\}$, where $t_i < t_j$ whenever $i > j$. Suppose that there exists a chain of subhypernear-rings of R:*

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = R$$

such that $\mu(H^*_k) = t_k$, where $H^*_k = H_k \backslash H_{k-1}, H_{-1} = \emptyset$ for $k = 0, 1, \cdots, n$. Then μ is a TFS-ring of R.

Proof. Let $x, y \in R$. If x and y belong to the same H^*_k, then we have $\mu(x) = \mu(y) = t_k, x + y \subseteq H_k$ and $xy \in H_k$. Thus we get

$$\inf_{\alpha\in x+y} \mu(\alpha) \geq t_k = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$. If $x \in H_i^*$ and $y \in H_j^*$ for every $i \neq j$. Without loss of generality, we may assume that $i \geq j$. Then we have $\mu(x) = t_i < t_j = \mu(y), x + y \subseteq H_i$ and $xy \in H_i$. It follows that

$$\inf_{\alpha \in x+y} \mu(\alpha) \geq t_i = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq T(\mu(x), \mu(y))$.

Let $x \in R$. Then there exists H_k such that $x \in H_k^*$ for some $k \in \{0, 1, \cdots, n\}$. Thus we have $\mu(x) = t_k = \mu(-x)$.

Therefore μ is a TFS-ring of R. □

For a t-norm T on $[0, 1]$, denote by Δ_T the set of element $\alpha \in [0, 1]$ such that $T(\alpha, \alpha) = \alpha$, i.e., $\Delta_T := \{\alpha \in [0, 1] | T(\alpha, \alpha) = \alpha\}$.

A fuzzy set μ in a set X is said to satisfy *imaginable property* if $Im(\mu) \subseteq \Delta_T$.

Definition 3.9. A TFS-ring is said to be *imaginable* if it satisfies the imaginable property.

Proposition 3.10. *For a subhypernear-ring H of R, let μ be a fuzzy set in R given by*

$$\mu(x) := \begin{cases} s & \text{if } x \in H, \\ t & \text{otherwise} \end{cases}$$

for all $s, t \in [0, 1]$ with $s > t$. Then μ is a T_1FS-ring of R. In particular, if $s = 1$ and $t = 0$ then μ is imaginable.

Proof. Let $x, y \in R$. If $x, y \in H$ then we get $x + y \subseteq H$ and $xy \in H$ since H is a subhypernear-ring of R, and so

$$T_1(\mu(x), \mu(y)) = \max\{s + s - 1, 0\} \leq s = \inf_{\alpha \in x+y} \mu(\alpha)$$

and $T_1(\mu(x), \mu(y)) \leq \mu(xy)$. If $x \in H$ and $y \notin H$ (or, $x \notin H$ and $y \in H$). Then $\mu(x) = s > t = \mu(y)$ (or, $\mu(x) = t < s = \mu(y)$). It follows that

$$T_1(\mu(x), \mu(y)) = \max\{s + t - 1, 0\} \leq t \leq \inf_{\alpha \in x+y} \mu(\alpha)$$

and $T_1(\mu(x), \mu(y)) \leq \mu(xy)$. If $x \notin H$ and $y \notin H$. Then $\mu(x) = t = \mu(y)$ and so we have

$$T_1(\mu(x), \mu(y)) = max\{t + t - 1, 0\} \leq t \leq \inf_{\alpha \in x+y} \mu(\alpha)$$

and $T_1(\mu(x), \mu(y)) \leq \mu(xy)$.

Let $x \in R$. If $x \in H$ then $-x \in H$ and so we have $\mu(x) = s \leq \mu(-x)$. If $x \notin H$ then we get $\mu(x) = t \leq \mu(-x)$.

Therefore μ is a T_1FS-ring of R. Obviously μ is imaginable when $s = 1$ and $t = 0$. □

Proposition 3.11. *Let T be a t-norm and μ be an imaginable TFS-ring of R. Then $\mu(0) \geq \mu(x)$ for all $x \in R$.*

Proof. For every $x \in R$ we have $0 \in x - x$ and so

$$\mu(0) \geq \inf_{z \in x-x} \mu(z) \geq T(\mu(x), \mu(-x)) = T(\mu(x), \mu(x)) = \mu(x).$$

□

Theorem 3.12. *Let T be a t-norm. Then every imaginable TFS-ring of R is a fuzzy subhypernear-ring of R.*

Proof. Let μ be an imaginable TFS-ring of R. Since μ satisfies the imaginable property, we have

$$\begin{aligned}\min\{\mu(x),\mu(y)\} &= T(\min\{\mu(x),\mu(y)\},\min\{\mu(x),\mu(y)\})\\ &\le T(\mu(x),\mu(y)) \le \min\{\mu(x),\mu(y)\}\end{aligned}$$

for all $x, y \in R$. It follows that $\inf\limits_{\alpha \in x+y} \mu(\alpha) \ge T(\mu(x),\mu(y)) = \min\{\mu(x),\mu(y)\}$ and $\mu(xy) \ge \min\{\mu(x),\mu(y)\}$ for all $x, y \in R$. Therefore μ is a fuzzy subhypernear-ring of R. □

Theorem 3.13. *Let μ be a TFS-ring of R and let $t \in [0,1]$. Then*
 (i) *if $t = 1$ then R^t_μ is either empry or a subhypernear-ring of R,*
 (ii) *if $T = min$, then R^t_μ is either empry or a subhypernear-ring of R.*

Proof. (i) Assume that $t = 1$ and let $x, y \in R^t_\mu$. Then we have

$$\inf_{\alpha \in x+y} \mu(\alpha) \ge T(\mu(x),\mu(y)) = T(1,1) = 1 = t,$$

and $\mu(xy) \ge t$. Thus $\alpha \in R^t_\mu$ and so we get $x + y \subseteq R^t_\mu$, and $xy \in R^t_\mu$.

Let $x \in R^t_\mu$. Then since μ is a TFS-ring of R, we have $\mu(-x) \ge \mu(x) \ge t$. Thus we get $-x \in R^t_\mu$.

Therefore R^t_μ is a subhypernear-ring of R whence $t = 1$.

(ii) Similar to the proof of (i). □

Theorem 3.14. *Let T be a t-norm and let μ be an imaginable fuzzy set in R. If each non-empty level subset R^t_μ of μ is a subhypernear-ring of R, then μ is an imaginable TFS-ring of R.*

Proof. For $t \in [0,1]$, suppose that R^t_μ is a non-empty set and a a subhypernear-ring of R. Then we have $\inf\limits_{\alpha \in x+y} \mu(\alpha) \ge \min\{\mu(x),\mu(y)\}$ for all $x, y \in R$. Indeed, if not then there exist $x_0, y_0 \in R$ such that $\inf\limits_{a \in x_0+y_0} \mu(a) < \min\{\mu(x_0),\mu(y_0)\}$. Taking

$$s_0 := \frac{1}{2}\{\inf_{a \in x_0+y_0} \mu(a) + \min\{\mu(x_0),\mu(y_0)\}\},$$

then we get $\inf\limits_{a \in x_0+y_0} \mu(a) < s_0 < \min\{\mu(x_0),\mu(y_0)\}$ and thus $x_0, y_0 \in R^{s_0}_\mu$ and $x_0+y_0 \not\subseteq R^{s_0}_\mu$. This is a contradiction. Hence we have

$$\inf_{\alpha \in x+y} \mu(\alpha) \ge \min\{\mu(x),\mu(y)\} \ge T\{\mu(x),\mu(y)\}$$

for all $x, y \in R$.

Now if (TF2) is not true, then $\mu(x_0y_0) < \min\{\mu(x_0),\mu(y_0)\}$ for some $x_0, y_0 \in R$. Taking

$$s_0 := \frac{1}{2}\{\mu(x_0y_0) + \min\{\mu(x_0),\mu(y_0)\}\},$$

then we get $\mu(x_0y_0) < s_0 < \min\{\mu(x_0),\mu(y_0)\}$ and thus $x_0, y_0 \in R^{s_0}_\mu$ and $x_0y_0 \in R^{s_0}_\mu$. This is a contradiction. Hence we have

$$\mu(xy) \ge \min\{\mu(x),\mu(y)\} \ge T(\mu(x),\mu(y))$$

for all $x, y \in R$.

Finally, if (TF3) is not true, then $\mu(x_0) > \mu(-x_0)$ for some $x_0 \in R$. Taking

$$s_0 := \frac{1}{2}\{\mu(x_0) + \mu(-x_0)\},$$

then we get $\mu(x_0) > s_0 > \mu(-x_0)$ and thus $x_0 \in R^{s_0}_\mu$ and $-x_0 \notin R^{s_0}_\mu$. It is a contradiction.

Therefore μ is an imaginable TFS-ring of R. □

Let $f : R \to S$ be a mapping of hypernear-rings. For a fuzzy set μ in S, the *inverse image* of μ under f, denoted by $f^{-1}(\mu)$, is defined by $f^{-1}(\mu)(x) := \mu(f(x))$ for all $x \in R$.

Proposition 3.15. *Let T be a t-norm and let $f : R \to S$ be a homomorphism of hypernear-rings. If μ is a TFS-ring of S, then $f^{-1}(\mu)$ is a TFS-ring of R.*

Proof. Assume that μ is a TFS-ring of S. Let $x, y \in R$. Then we get

$$\begin{aligned}\inf_{\alpha\in x+y} f^{-1}(\mu)(\alpha) &= \inf_{f(\alpha)\in f(x)+f(y)} \mu(f(\alpha)) \geq T(\mu(f(x)), \mu(f(y))) \\ &= T(f^{-1}(\mu)(x), f^{-1}(\mu)(y)),\end{aligned}$$

and

$$f^{-1}(\mu)(xy) = \mu(f(x)f(y)) \geq T(\mu(f(x)), \mu(f(y))) = T(f^{-1}(\mu)(x), f^{-1}(\mu)(y)).$$

Also, we have $f^{-1}(\mu)(x) = \mu(f(x)) \leq \mu(-f(x)) = \mu(f(-x)) = f^{-1}(\mu)(-x)$ for all $x \in R$. Therefore $f^{-1}(\mu)$ is a TFS-ring of R. □

4. Direct product of TFS-rings

Definition 4.1. Let T be a t-norm and let μ and ν be fuzzy sets in R. Then the *T-product* of μ and ν, written $[\mu \cdot \nu]_T$, is defined by $[\mu \cdot \nu]_T(x) := T(\mu(x), \nu(x))$ for all $x \in R$.

Proposition 4.2. *Let T be a t-norm and let μ and ν be TFS-rings in R. If T^* is a t-norm which dominates, i.e., $T^*(T(\alpha,\beta), T(\gamma,\delta)) \geq T(T^*(\alpha,\gamma), T^*(\beta,\delta))$ for all $\alpha,\beta,\gamma,\delta \in [0,1]$, then T^*-product of μ and ν, $[\mu\cdot\nu]^*_T$ is a TFS-ring of R.*

Proof. Let $x, y \in R$. Then we have

$$\begin{aligned}\inf_{\alpha\in x+y} [\mu\cdot\nu]^*_T(\alpha) &= \inf_{\alpha\in x+y} T^*(\mu(\alpha), \nu(\alpha)) \geq T^*(\inf_{\alpha\in x+y} \mu(\alpha), \inf_{\alpha\in x+y} \nu(\alpha)) \\ &\geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &\geq T(T^*(\mu(x), \nu(x)), T^*(\mu(y), \nu(y)) = T([\mu\cdot\nu]^*_T(x), [\mu\cdot\nu]^*_T(y)),\end{aligned}$$

and

$$\begin{aligned}[\mu\cdot\nu]^*_T(xy) &= T^*(\mu(xy), \nu(xy)) \geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &\geq T(T^*(\mu(x), \nu(x)), T^*(\mu(y), \nu(y)) = T([\mu\cdot\nu]^*_T(x), [\mu\cdot\nu]^*_T(y)).\end{aligned}$$

Also, we get $[\mu\cdot\nu]^*_T(x) = T^*(\mu(x), \nu(x)) \leq T^*(\mu(-x), \nu(-x)) = [\mu\cdot\nu]^*_T(-x)$ for all $x \in R$. Therefore $[\mu\cdot\nu]^*_T$ is a TFS-ring of R. □

Let $f : R \to S$ be a homomorphism of hypernear-rings, and let T and T^* be t-norms such that T^* dominates T. If μ and ν are TFS-rings in S, then $[\mu\cdot\nu]^*_T$ is a TFS-ring of S. By 3.15, the inverse images $f^{-1}(\mu), f^{-1}(\nu)$ and $f^{-1}([\mu\cdot\nu]^*_T)$ are TFS-rings of R. The next theorem provides that the relation between $f^{-1}([\mu\cdot\nu]^*_T)$ and the T^*-product $[f^{-1}(\mu)\cdot f^{-1}(\nu)]^*_T$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 4.3. *Let $f : R \to S$ be a homomorphism of hypernear-rings, and let T and T^* be t-norms such that T^* dominates T. Let μ and ν be TFS-rings in S. If $[\mu\cdot\nu]^*_T$ is T^*-product of μ and ν, and $[f^{-1}(\mu)\cdot f^{-1}(\nu)]^*_T$ is the T^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$ then $f^{-1}([\mu\cdot\nu]^*_T) = [f^{-1}(\mu)\cdot f^{-1}(\nu)]^*_T$.*

Proof. Let $x \in R$. Then we have

$$\begin{aligned}f^{-1}([\mu\cdot\nu]^*_T)(x) &= [\mu\cdot\nu]^*_T(f(x)) = T^*(\mu(f(x)), \nu(f(x))) \\ &= T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x)) = [f^{-1}(\mu)\cdot f^{-1}(\nu)]^*_T.\end{aligned}$$

□

Let R_1 and R_2 be two hypernear-rings, then for all (x_1, y_1) and (x_2, y_2) in $R_1 \times R_2$ we define

$$(x_1, y_1) + (x_2, y_2) = \{(x, y) \mid x \in x_1 + x_2,\ y \in y_1 + y_2\}$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2,\ y_1 y_2).$$

Clearly $R_1 \times R_2$ is a hypernear-ring and we call this hypernear-ring the direct product of R_1 and R_2.

Definition 4.4. Let T be a t-norm and let μ_1 and μ_2 be fuzzy sets on hypernear-rings R_1 and R_2 respectively. Then μ defined on $R_1 \times R_2$ by the formula

$$\mu(x, y) = T(\mu_1(x),\ \mu_2(y))$$

is a fuzzy set on $R_1 \times R_2$ which is defined by $\mu_1 \times \mu_2$.

Theorem 4.5. *Let T be a t-norm and let $R = R_1 \times R_2$ be the direct product of hypernear-rings R_1 and R_2. If μ_1 and μ_2 are TFS-rings of R_1 and R_2 respectively, then $\mu = \mu_1 \times \mu_2$ is a TFS-ring of R.*

Proof. Let (x_1, y_1) and (x_2, y_2) be two arbitrary elements of $R_1 \times R_2$. For every $(x, y) \in (x_1, y_1) + (x_2, y_2)$ we have

$$\begin{aligned}
(\mu_1 \times \mu_2)(x, y) &= T(\mu_1(x), \mu_2(y)) \\
&\geq T(T(\mu_1(x_1), \mu_1(x_2)),\ T(\mu_2(y_1), \mu_2(y_2))) \\
&= T(T(T(\mu_1(x_1), \mu_1(x_2)), \mu_2(y_1)), \mu_2(y_2)) \\
&= T(T(\mu_2(y_1), T(\mu_1(x_1), \mu_1(x_2)), \mu_2(y_2)) \\
&= T(T(T(\mu_2(y_1), \mu_1(x_1)), \mu_1(x_2)), \mu_2(y_2)) \\
&= T(\mu_2(y_2),\ T(\mu_1(x_2),\ T(\mu_2(y_1), \mu_1(x_1)))) \\
&= T(T(\mu_1(x_1), \mu_2(y_1)),\ T(\mu_1(x_2), \mu_2(y_2))) \\
&= T((\mu_1 \times \mu_2)(x_1, y_1),\ (\mu_1 \times \mu_2)(x_2, y_2)).
\end{aligned}$$

Hence $\inf\limits_{(x,y) \in (x_1,y_1)+(x_2,y_2)} (\mu_1 \times \mu_2)(x, y) \geq T((\mu_1 \times \mu_2)(x_1, y_1),\ (\mu_1 \times \mu_2)(x_2, y_2))$. Similarly we obtain

$$\begin{aligned}
(\mu_1 \times \mu_2)((x_1, y_1) \cdot (x_2, y_2)) &= (\mu_1 \times \mu_2)(x_1 x_2, y_1 y_2) \\
&= T(\mu_1(x_1 x_2),\ \mu_2(y_1 y_2)) \\
&\geq T(T(\mu_1(x_1), \mu_1(x_2)),\ T(\mu_2(y_1), \mu_2(y_2))) \\
&\vdots \\
&= T((\mu_1 \times \mu_2)(x_1, y_1),\ (\mu_1 \times \mu_2)(x_2, y_2)).
\end{aligned}$$

Also, we have

$$(\mu_1 \times \mu_2)(x, y) = T(\mu_1(x), \mu_2(y)) \leq T(\mu_1(-x), \mu_2(-y)) = (\mu_1 \times \mu_2)(-x, -y).$$

Therefore $\mu_1 \times \mu_2$ is a TFS-ring of $R_1 \times R_2$. □

Theorem 4.6. *Let T be a t-norm and let μ_1 and μ_2 be fuzzy sets of the hypernear-rings R_1 and R_2 respectively. If $\mu_1 \times \mu_2$ is an imaginable TFS-ring of $R_1 \times R_2$, then at least one of the following two statements must hold:*

(1) $\mu_2(0) \geq \mu_1(x)$ *for all* $x \in R_1$,
(2) $\mu_1(0) \geq \mu_2(y)$ *for all* $y \in R_2$.

Proof. Suppose that $\mu_1 \times \mu_2$ is an imaginable TFS-ring of $R_1 \times R_2$. By contraposition, suppose that none of the statements (1) and (2) holds. Then there exist $x_0 \in R_1$ and $y_0 \in R_2$ such that

$$\mu_1(x_0) > \mu_2(0) \ \text{ and } \ \mu_2(y_0) > \mu_1(0).$$

Now, we have

$$\begin{aligned}(\mu_1 \times \mu_2)(x_0, y_0) &= T(\mu_1(x_0), \mu_2(y_0)) > T(\mu_2(0), \mu_1(0)) \\ &= T(\mu_1(0), \mu_2(0)) = (\mu_1 \times \mu_2)(0, 0).\end{aligned}$$

But, by Proposition 3.11, always we have $(\mu_1 \times \mu_2)(0, 0) \geq (\mu_1 \times \mu_2)(x_0, y_0)$. □

Theorem 4.7. *Let T be a t-norm. Let μ_1, μ_2 and $\mu_1 \times \mu_2$ be fuzzy sets of the hypernear-rings R_1, R_2 and $R_1 \times R_2$ respectively, such that satisfy imaginable property. If $\mu_1 \times \mu_2$ is a TFS-ring of $R_1 \times R_2$, then μ_1 is a TFS-ring of R_1 or μ_2 is a TFSring of R_2.*

Proof. Since $\mu_1 \times \mu_2$ is an imaginable TFS-ring of $R_1 \times R_2$, by Theorem 4.6, we assume that $\mu_1(x) \leq \mu_2(0)$ for all $x \in R_1$, and we show that μ_1 is a TFS-ring of R_1. Let x and y be two arbitrary elements of R_1. For every $z \in x + y$ we have

$$\begin{aligned}\mu_1(z) &= T(\mu_1(z), 1) \geq T(\mu_1(z), \mu_2(0)) \\ &= (\mu_1 \times \mu_2)(z, 0) \\ &\geq \inf_{(z,0)\in(x,0)+(y,0)} (\mu_1 \times \mu_2)(z, 0) \\ &\geq T((\mu_1 \times \mu_2)(x, 0),\ (\mu_1 \times \mu_2)(y, 0)) \\ &= T(T(\mu_1(x), \mu_2(0)),\ T(\mu_1(y), \mu_2(0))) \\ &\geq T(T(\mu_1(x), \mu_1(x)),\ T(\mu_1(y), \mu_1(x))) \\ &= T(\mu_1(x),\ T(\mu_1(y), \mu_1(x))) \\ &= T(\mu_1(x),\ T(\mu_1(x), \mu_1(y))) \\ &= T(T(\mu_1(x), \mu_1(x)), \mu_1(y)) \\ &= T(\mu_1(x), \mu_1(y)).\end{aligned}$$

Therefore $\inf_{z\in x+y} \mu_1(z) \geq T(\mu_1(x), \mu_1(y))$. Similarly, we obtain

$$\begin{aligned}\mu_1(xy) &= T(\mu_1(xy), 1) \geq T(\mu_1(xy), \mu_2(0)) \\ &= (\mu_1 \times \mu_2)(xy, 0) = (\mu_1 \times \mu_2)((x, 0) \cdot (y, 0)) \\ &\geq T((\mu_1 \times \mu_2)(x, 0),\ (\mu_1 \times \mu_2)(y, 0)) \\ &\vdots \\ &= T(\mu_1(x), \mu_1(y)).\end{aligned}$$

Also we have

$$\begin{aligned}\mu_1(-x) &= T(\mu_1(-x), 1) \geq T(\mu_1(-x), \mu_2(0)) = (\mu_1 \times \mu_2)(-x, 0) = (\mu_1 \times \mu_2)(-(x, 0)) \\ &\geq (\mu_1 \times \mu_2)(x, 0) = T(\mu_1(x), \mu_2(0)) \geq T(\mu_1(x), \mu_1(x)) = \mu_1(x).\end{aligned}$$

□

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