ON SENSIBLE FUZZY SUBALGEBRAS OF BCK-ALGEBRAS WITH RESPECT TO A s-NORM

YONG UK CHO, YOUNG BAE JUN* AND EUN HWAN ROH

Received November 21, 2004

ABSTRACT. The notion of fuzzy subalgebras with respect to a s-norm is introduced, and some related results are investigated. A kind of product, called *S-product*, and the direct product of fuzzy subalgebras of BCK-algebras via a s-norm S are also defined, and some properties of the S-product and the direct product of fuzzy subalgebras of BCK-algebras with respect to a s-norm are discussed.

1. Introduction

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers. L. A. Zadeh [7] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, O. G. Xi [5] applied this concept to BCK-algebras, and he introduced the notion of fuzzy subalgebras(ideals) of BCK-algebras with respect to minimum, and since then Y. B. Jun et al. studied fuzzy subalgebras and fuzzy ideals (see [1, 2, 3, 4]). In the present paper, we will redefine the fuzzy subalgebra of BCK-algebras with respect to a s-norm S, and obtain some related results. We will define a kind of product, called *S-product*, and the direct product of fuzzy subalgebras of BCK-algebras via a s-norm S. We will investigate some properties of the S-product and the direct product of fuzzy subalgebras of BCK-algebras with respect to a s-norm.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is said to be a *BCK-algebra* if it satisfies:
(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$,
for all $x, y, z \in X$. Define a binary relation "$\leq$" on X by letting $x \leq y$ if and only if $x * y = 0$. Then $(X; \leq)$ is a partially ordered set with the least element 0. A subset S of a BCK-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A mapping $\theta : X \to X'$ of BCK-algebras is called a *homomorphism* if $\theta(x * y) = \theta(x) * \theta(y)$ for all $x, y \in X$. In what follows, let X denote a BCK-algebra unless otherwise specified. A *fuzzy*

2000 *Mathematics Subject Classification.* 06F35, 03G25, 94D05.
Key words and phrases. (Continuous) s-norm, (sensible) fuzzy subalgebra with respect to a s-norm, S-product, anti fuzzy subalgebra.
The author Y. B. Jun is an Executive Research Worker of Educational Research Institute in GSNU.
*Corresponding author. Tel.: +82 55 751 5674; Fax : +82 55 751 6117 (Y. B. Jun)

set in X is a function $\mu : X \to [0,1]$. Let μ be a fuzzy set in X. For $\alpha \in [0,1]$, the set

$$L(\mu;\alpha) = \{x \in X \mid \mu(x) \le \alpha\}$$

is called a *lower level set* of μ.

A fuzzy set μ in X is called a *fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\,(\mu(x * y) \ge \min\{\mu(x), \mu(y)\}),$$

and is called an *anti fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\,(\mu(x * y) \le \max\{\mu(x), \mu(y)\}).$$

3. Sensible fuzzy subalgebras with respect to a s-norm

Definition 3.1. ([6]) A binary operation S on $[0,1]$ is called a *s-norm* if

(S1) $S(\alpha, 0) = \alpha$,
(S2) $S(\alpha, \beta) \le S(\alpha, \gamma)$ whenever $\beta \le \gamma$,
(S3) $S(\alpha, \beta) = S(\beta, \alpha)$,
(S4) $S(\alpha, S(\beta, \gamma)) = S(S(\alpha, \beta), \gamma)$,

for all $\alpha, \beta, \gamma \in [0,1]$.

For a s-norm S, note that $\max(\alpha, \beta) \le S(\alpha, \beta)$ for all $\alpha, \beta \in [0,1]$.

Definition 3.2. Let S be a s-norm. A fuzzy set μ in X is said to be *sensible* if $\mathrm{Im}(\mu) \subseteq \Omega_S$, where $\Omega_S := \{\alpha \in [0,1] \mid S(\alpha, \alpha) = \alpha\}$.

Definition 3.3. Let S be a s-norm. A function $\mu : X \to [0,1]$ is called a *fuzzy subalgebra* of X with respect to S if $\mu(x * y) \le S(\mu(x), \mu(y))$ for all $x, y \in X$. If a fuzzy subalgebra μ of X with respect to S is sensible, we say that μ is a *sensible fuzzy subalgebra* of X with respect to S.

Example 3.4. Let S_0 be a s-norm defined by $S_0(\alpha, 0) = \alpha = S_0(0, \alpha)$ and $S_0(\alpha, \beta) = 1$ if $\alpha \ne 0 \ne \beta$ for all $\alpha, \beta \in [0,1]$. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = \alpha_0$, $\mu(a) = \mu(b) = \alpha_1$ and $\mu(c) = \alpha_2$, where $\alpha_0, \alpha_1, \alpha_2 \in [0,1]$ with $\alpha_0 < \alpha_1 < \alpha_2$. Routine calculations give that μ is a fuzzy subalgebra of X with respect to S_0, which is not sensible.

Example 3.5. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with Cayley table as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Let S_m be a s-norm defined by $S_m(\alpha, \beta) = \min(\alpha + \beta, 1)$ for all $\alpha, \beta \in [0,1]$. Define a fuzzy set μ in X by

$$\mu(x) := \begin{cases} 0 & \text{if } x \in \{0, a, b\}, \\ 1 & \text{otherwise}, \end{cases}$$

By routine calculations we know that $\mu(x*y) \leq S_m(\mu(x), \mu(y))$ for all $x \in X$, and $\mathrm{Im}(\mu) \subseteq \Omega_{S_m}$. Hence μ is a sensible fuzzy subalgebra of X with respect to S_m.

Proposition 3.6. *Let S_m be the s-norm in Example 3.5 and let A be a subalgebra of X. Then a fuzzy set μ in X defined by*

$$\mu(x) := \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise} \end{cases}$$

is a sensible fuzzy subalgebra of X with respect to S_m.

Proof. Let $x, y \in X$. If $x \notin A$ or $y \notin A$, then $\mu(x) = 1$ or $\mu(y) = 1$ and so $S_m(\mu(x), \mu(y)) = 1 \geq \mu(x*y)$. Assume that $x \in A$ and $y \in A$. Then $x*y \in A$ and thus $\mu(x*y) = 0 \leq S_m(\mu(x), \mu(y))$. Obviously $\mathrm{Im}(\mu) \subseteq \Omega_{S_m}$, ending the proof. □

Proposition 3.7. *Let S be a s-norm. If μ is a sensible fuzzy subalgebra of X with respect to S, then $\mu(0) \leq \mu(x)$ for all $x \in X$.*

Proof. Since $x*x = 0$ for all $x \in X$, it follows that

$$\mu(0) = \mu(x*x) \leq S(\mu(x), \mu(x)) = \mu(x)$$

for all $x \in X$. □ □

Proposition 3.8. *Let S be a s-norm. Every sensible fuzzy subalgebra of X with respect to S is an anti fuzzy subalgebra of X.*

Proof. Let μ be a sensible fuzzy subalgebra of X with respect to S. Then

$$\mu(x*y) \leq S(\mu(x), \mu(y)) \text{ for all } x, y \in X.$$

Since μ is sensible, we have

$$\begin{aligned} \max(\mu(x), \mu(y)) &= S(\max(\mu(x), \mu(y)), \max(\mu(x), \mu(y)) \\ &\geq S(\mu(x), \mu(y)) \geq \max(\mu(x), \mu(y)) \end{aligned}$$

by using (S2) and (S3). It follows that $\mu(x*y) \leq S(\mu(x), \mu(y)) = \max(\mu(x), \mu(y))$ so that μ is an anti fuzzy subalgebra of X. □

Theorem 3.9. *Let μ be a fuzzy subalgebra of X with respect to a s-norm S and let $\alpha \in [0, 1]$. Then*

(i) *if $\alpha = 0$ then the lower level set $L(\mu; \alpha)$ of μ is either empty or a subalgebra of X.*

(ii) *if $S = \max$ then the lower level set $L(\mu; \alpha)$ of μ is either empty or a subalgebra of X, and moreover $\mu(0) \leq \mu(x)$ for all $x \in X$.*

Proof. (i) Let $x, y \in L(\mu; \alpha)$. Then $\mu(x) \leq \alpha = 0$ and $\mu(y) \leq \alpha = 0$. It follows from Definitions 3.1 and 3.3 that

$$\mu(x*y) \leq S(\mu(x), \mu(y)) \leq S(0, 0) = 0$$

so that $x*y \in L(\mu; \alpha)$. Hence $L(\mu; \alpha)$ is a subalgebra of X whenever $\alpha = 0$.

(ii) Assume that $S = \max$ and let $x, y \in L(\mu; \alpha)$. Then

$$\mu(x*y) \leq S(\mu(x), \mu(y)) = \max(\mu(x), \mu(y)) \leq \max(\alpha, \alpha) = \alpha$$

for all $\alpha \in [0, 1]$, which implies that $x*y \in L(\mu; \alpha)$. Thus $L(\mu; \alpha)$ is a subalgebra of X. Moreover, since $x*x = 0$ for all $x \in X$, we have

$$\mu(0) = \mu(x*x) \leq S(\mu(x), \mu(x)) = \max(\mu(x), \mu(x)) = \mu(x).$$

This completes the proof. □ □

Theorem 3.10. *Let S be a s-norm and let μ be a fuzzy set in X. If the non-empty lower level set $L(\mu;\alpha)$ of μ is a subalgebra of X, then μ is a fuzzy subalgebra of X with respect to S.*

Proof. Assume that there exist $x_0, y_0 \in X$ such that $\mu(x_0 * y_0) > S(\mu(x_0), \mu(y_0))$. Taking $\alpha_0 := \frac{1}{2}(\mu(x_0 * y_0) + S(\mu(x_0), \mu(y_0)))$, then

$$\max(\mu(x_0), \mu(y_0)) \leq S(\mu(x_0), \mu(y_0)) < \alpha_0 < \mu(x_0 * y_0).$$

It follows that $x_0, y_0 \in L(\mu;\alpha_0)$ and $x_0 * y_0 \notin L(\mu;\alpha_0)$. This is a contradiction and hence μ satisfies the inequality $\mu(x * y) \leq S(\mu(x), \mu(y))$ for all $x, y \in X$. □

Theorem 3.11. *Let S be a s-norm satisfying $S(\alpha,\alpha) = \alpha$ for all $\alpha \in [0,1]$. If a fuzzy set μ in X is a fuzzy subalgebra of X with respect to S, then the non-empty lower level set $L(\mu;\alpha)$ of μ is a subalgebra of X for every $\alpha \in [0,1]$.*

Proof. Let $x, y \in L(\mu;\alpha)$ for $\alpha \in [0,1]$. Using (S2) and (S3), we have

$$\mu(x * y) \leq S(\mu(x), \mu(y)) \leq S(\mu(x), \alpha) = S(\alpha, \mu(x)) \leq S(\alpha, \alpha) = \alpha,$$

which means that $x * y \in L(\mu;\alpha)$. Hence $L(\mu;\alpha)$ is a subalgebra of X. □

Theorem 3.12. *Let S be a s-norm and let μ be a fuzzy subalgebra of X with respect to S. If there is a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} S(\mu(x_n), \mu(x_n)) = 0,$$

then $\mu(0) = 0$.

Proof. For any $x \in X$, we get $\mu(0) = \mu(x * x) \leq S(\mu(x), \mu(x))$. Therefore $\mu(0) \leq S(\mu(x_n), \mu(x_n))$ for each $n \in \mathbf{N}$, and so $0 \leq \mu(0) \leq \lim_{n\to\infty} S(\mu(x_n), \mu(x_n)) = 0$. It follows that $\mu(0) = 0$. □

If μ is a fuzzy set in X and θ is a mapping from X into itself, we define a mapping $\mu[\theta] : X \to [0,1]$ by $\mu[\theta](x) = \mu(\theta(x))$ for all $x \in X$.

Proposition 3.13. *Let S be a s-norm. If μ is a fuzzy subalgebra of X with respect to S and θ is an endomorphism of X, then $\mu[\theta]$ is a fuzzy subalgebra of X with respect to S.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\mu[\theta](x * y) &= \mu(\theta(x * y)) = \mu(\theta(x) * \theta(y)) \\ &\leq S(\mu(\theta(x)), \mu(\theta(y))) = S(\mu[\theta](x), \mu[\theta](y)).\end{aligned}$$

Hence $\mu[\theta]$ is a fuzzy subalgebra of X with respect to S. □

Let f be a mapping defined on X. If ν is a fuzzy set in $f(X)$ then the fuzzy set $f^{-1}(\nu)$ in X defined by $[f^{-1}(\nu)](x) = \nu(f(x))$ for all $x \in X$ is called the *preimage* of ν under f.

Theorem 3.14. *Let S be a s-norm. An onto homomorphic preimage of a fuzzy subalgebra with respect to S is a fuzzy subalgebra with respect to S.*

Proof. Let $f : X \to X'$ be an onto homomorphism of BCK-algebras and let ν be a fuzzy subalgebra of X' with respect to S. Then

$$\begin{aligned}[f^{-1}(\nu)](x * y) &= \nu(f(x * y)) = \nu(f(x) * f(y)) \\ &\leq S(\nu(f(x)), \nu(f(y))) = S([f^{-1}(\nu)](x), [f^{-1}(\nu)](y))\end{aligned}$$

for all $x, y \in X$. Hence $f^{-1}(\nu)$ is a fuzzy subalgebra of X with respect to S. □

If μ is a fuzzy set in X and f is a mapping defined on X. The fuzzy set μ^f in $f(X)$ defined by $\mu^f(y) = \inf_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(X)$ is called the *anti image* of μ under f.

Definition 3.15. Definition 3.15 A s-norm S on $[0,1]$ is said to be *continuous* if S is a continuous function from $[0,1] \times [0,1]$ to $[0,1]$ with respect to the usual topology.

Theorem 3.16. *Let S be a continuous s-norm and let f be a homomorphism on X. If μ is a fuzzy subalgebra of X with respect to S, then the anti image of μ under f is a fuzzy subalgebra of $f(X)$ with respect to S.*

Proof. Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1 * y_2)$, where $y_1, y_2 \in f(X)$. Consider the set

$$A_1 * A_2 := \{x \in X \mid x = a_1 * a_2 \text{ for some } a_1 \in A_1 \text{ and } a_2 \in A_2\}.$$

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ and so

$$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2,$$

i.e., $x \in f^{-1}(y_1 * y_2) = A_{12}$. Thus $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} \mu^f(y_1 * y_2) &= \inf_{x \in f^{-1}(y_1 * y_2)} \mu(x) = \inf_{x \in A_{12}} \mu(x) \leq \inf_{x \in A_1 * A_2} \mu(x) \\ &\leq \inf_{x_1 \in A_1, x_2 \in A_2} \mu(x_1 * x_2) \leq \inf_{x_1 \in A_1, x_2 \in A_2} S(\mu(x_1), \mu(x_2)). \end{aligned}$$

Now S is continuous, and therefore if ε is any positive number, then there exists a number $\delta > 0$ such that

$$S(x_1^*, x_2^*) \leq S(\inf_{x_1 \in A_1} \mu(x_1), \inf_{x_2 \in A_2} \mu(x_2)) + \varepsilon$$

whenever $x_1^* \leq \inf_{x_1 \in A_1} \mu(x_1) + \delta$ and $x_2^* \leq \inf_{x_2 \in A_2} \mu(x_2) + \delta$. Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\mu(a_1) \leq \inf_{x_1 \in A_1} \mu(x_1) + \delta$ and $\mu(a_2) \leq \inf_{x_2 \in A_2} \mu(x_2) + \delta$. Then

$$S(\mu(a_1), \mu(a_2)) \leq S(\inf_{x_1 \in A_1} \mu(x_1), \inf_{x_2 \in A_2} \mu(x_2)) + \varepsilon.$$

Consequently

$$\begin{aligned} \mu^f(y_1 * y_2) &\leq \inf_{x_1 \in A_1, x_2 \in A_2} S(\mu(x_1), \mu(x_2)) \\ &\leq S(\inf_{x_1 \in A_1} \mu(x_1), \inf_{x_2 \in A_2} \mu(x_2)) = S(\mu^f(y_1), \mu^f(y_2)), \end{aligned}$$

and so μ^f is a fuzzy subalgebra of $f(X)$ with respect to S. □

Lemma 3.17. *Let S be a s-norm. Then*

$$S(S(\alpha, \beta), S(\gamma, \delta)) = S(S(\alpha, \gamma), S(\beta, \delta))$$

for all $\alpha, \beta, \gamma, \delta \in [0,1]$.

Proof. Using (S3) and (S4), it is straightforward. □ □

Theorem 3.18. *Let S be a s-norm. Let X_1 and X_2 be BCK-algebras and $X = X_1 \times X_2$ be the direct product BCK-algebra of X_1 and X_2. Let μ_1 be a fuzzy subalgebra of X_1 with respect to S and μ_2 a fuzzy subalgebra of X_2 with respect to S. Then $\mu = \mu_1 \times \mu_2$ is a fuzzy subalgebra of X with respect to S defined by*

$$\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = S(\mu_1(x_1), \mu_2(x_2)).$$

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of the BCK-algebra $X = X_1 \times X_2$. Then

$$\begin{aligned} &\mu(x * y) = \mu((x_1, x_2) * (y_1, y_2)) = \mu(x_1 * y_1, x_2 * y_2) \\ &\quad = S(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2)) \leq S(S(\mu_1(x_1), \mu_1(y_1)), S(\mu_2(x_2), \mu_2(y_2))) \\ &\quad = S(S(\mu_1(x_1), \mu_2(x_2)), S(\mu_1(y_1), \mu_2(y_2))) = S(\mu(x_1, x_2), \mu(y_1, y_2)) = S(\mu(x), \mu(y)). \end{aligned}$$

Hence μ is a fuzzy subalgebra of X with respect to S. □

Now we will generalize the idea to the product of n fuzzy subalgebras. We first need to generalize the domain of s-norm S to $\prod_{i=1}^{n}[0,1]$ as follows:

Definition 3.19. Definition 3.19 The function $S_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ is defined by

$$S_n(\alpha_1, \alpha_2, \cdots, \alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n))$$

for all $1 \le i \le n$, where $n \ge 2$, $S_2 = S$ and $S_1 = \text{id}$ (identity).

Lemma 3.20. *For a s-norm S and every $\alpha_i, \beta_i \in [0,1]$ where $1 \le i \le n$ and $n \ge 2$, we have*

$$S_n(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2), \cdots, S(\alpha_n, \beta_n)) = S(S_n(\alpha_1, \alpha_2, \cdots, \alpha_n), S_n(\beta_1, \beta_2, \cdots, \beta_n)).$$

Proof. It can be checked by induction on n. □

Theorem 3.21. *Let S be a s-norm and let $\{X_i\}_{i=1}^n$ be the finite collection of BCK-algebras and $X = \prod_{i=1}^{n} X_i$ the direct product BCK-algebra of $\{X_i\}$. Let μ_i be a fuzzy subalgebra of X_i with respect to S, where $1 \le i \le n$. Then $\mu = \prod_{i=1}^{n} \mu_i$ defined by*

$$\mu(x_1, x_2, \cdots, x_n) = (\prod_{i=1}^{n} \mu_i)(x_1, x_2, \cdots, x_n) = S_n(\mu_1(x_1), \mu_2(x_2), \cdots, \mu_n(x_n))$$

is a fuzzy subalgebra of the BCK-algebra X with respect to S.

Proof. Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be any elements of X. Then

$$\begin{aligned}\mu(x * y) &= \mu(x_1 * y_1, x_2 * y_2, \cdots, x_n * y_n) = S_n(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2), \cdots, \mu_n(x_n * y_n)) \\ &\le S_n(S(\mu_1(x_1), \mu_1(y_1)), S(\mu_2(x_2), \mu_2(y_2)), \cdots, S(\mu_n(x_n), \mu_n(y_n))) \\ &= S(S_n(\mu_1(x_1), \mu_2(x_2), \cdots, \mu_n(x_n)), S_n(\mu_1(y_1), \mu_2(y_2), \cdots, \mu_n(y_n))) \\ &= S(\mu(x_1, x_2, \cdots, x_n), \mu(y_1, y_2, \cdots, y_n)) = S(\mu(x), \mu(y)).\end{aligned}$$

Hence μ is a fuzzy subalgebra of X with respect to S. □

Definition 3.22. Definition 3.22 Let S be a s-norm. Let μ and ν be fuzzy sets in X. Then the *S-product* of μ and ν, written as $[\mu \cdot \nu]_S$, is defined by $[\mu \cdot \nu]_S(x) = S(\mu(x), \nu(x))$ for all $x \in X$.

Theorem 3.23. *Let μ and ν be fuzzy subalgebras of X with respect to a s-norm S. Let S^* be a s-norm which dominates S, i.e.,*

$$S(S^*(\alpha, \beta), S^*(\gamma, \delta)) \ge S^*(S(\alpha, \gamma), S(\beta, \delta))$$

for all $\alpha, \beta, \gamma, \delta \in [0,1]$. Then S^*-product of μ and ν, $[\mu \cdot \nu]_{S^*}$, is a fuzzy subalgebra of X with respect to S.

Proof. For any $x, y \in X$ we have

$$\begin{aligned}[\mu \cdot \nu]_{S^*}(x * y) &= S^*(\mu(x * y), \nu(x * y)) \le S^*(S(\mu(x), \mu(y)), S(\nu(x), \nu(y))) \\ &\le S(S^*(\mu(x), \nu(x)), S^*(\mu(y), \nu(y))) = S([\mu \cdot \nu]_{S^*}(x), [\mu \cdot \nu]_{S^*}(y)).\end{aligned}$$

Hence $[\mu \cdot \nu]_{S^*}$ is a fuzzy subalgebra of X with respect to S. □

Let $f : X \to X'$ be an onto homomorphism of BCK-algebras. If μ and ν are fuzzy subalgebras of X' with respect to a s-norm S, then the S^*-product of μ and ν, $[\mu \cdot \nu]_{S^*}$, is a fuzzy subalgebra of X' with respect to S whenever S^* dominates S (see [Theorem 3.23]). Then by Theorem 3.14, the preimages $f^{-1}(\mu)$, $f^{-1}(\nu)$ and $f^{-1}([\mu \cdot \nu]_{S^*})$ are fuzzy subalgebras of X with respect to S. The next theorem provides that the relation between $f^{-1}([\mu \cdot \nu]_{S^*})$ and the S^*-product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{S^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 3.24. *Let $f : X \to X'$ be an onto homomorphism of BCK-algebras. Let μ and ν be fuzzy subalgebras of X' with respect to a s-norm S. Let S^* be a s-norm which dominates S. If $[\mu \cdot \nu]_{S^*}$ is the S^*-product of μ and ν and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{S^*}$ is the S^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then*

$$f^{-1}([\mu \cdot \nu]_{S^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{S^*}.$$

Proof. For any $x \in X$ we get

$$\begin{aligned}[f^{-1}([\mu \cdot \nu]_{S^*})](x) &= [\mu \cdot \nu]_{S^*}(f(x)) = S^*(\mu(f(x)), \nu(f(x))) \\ &= S^*([f^{-1}(\mu)](x), [f^{-1}(\nu)](x)) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{S^*}(x),\end{aligned}$$

ending the proof. □

Let μ and ν be fuzzy subalgebras of X with respect to a continuous s-norm S. Let f be a homomorphism on X and let S^* be a s-norm which dominates S. Then the S^*-product $[\mu \cdot \nu]_{S^*}$ of μ and ν is a fuzzy subalgebra of X with respect to S (see [Theorem 3.23]). Using Theorem 3.16, μ^f, ν^f and $([\mu \cdot \nu]_{S^*})^f$ are fuzzy subalgebras of $f(X)$ with respect to S. It follows from Theorem 3.23 that the S^*-product $[\mu^f \cdot \nu^f]_{S^*}$ of μ^f and ν^f is a fuzzy subalgebra of $f(X)$ with respect to S. Now for each $y \in f(X)$, we have

$$\begin{aligned}([\mu \cdot \nu]_{S^*})^f(y) &= \inf_{x \in f^{-1}(y)} [\mu \cdot \nu]_{S^*}(x) = \inf_{x \in f^{-1}(y)} S^*(\mu(x), \nu(x)) \\ &\geq S^*(\inf_{x \in f^{-1}(y)} \mu(x), \inf_{x \in f^{-1}(y)} \nu(x)) = S^*(\mu^f(y), \nu^f(y)) = [\mu^f \cdot \nu^f]_{S^*}(y).\end{aligned}$$

Hence we have the following theorem.

Theorem 3.25. *Let μ and ν be fuzzy subalgebras of X with respect to a continuous s-norm S and let f be a homomorphism on X. If a s-norm S^* dominates S, then $([\mu \cdot \nu]_{S^*})^f$, the image of the S^*-product of μ and ν under f, and $[\mu^f \cdot \nu^f]_{S^*}$, the S^*-product of μ^f and ν^f satisfy the inclusion:*

$$[\mu^f \cdot \nu^f]_{S^*} \subseteq ([\mu \cdot \nu]_{S^*})^f.$$

References

[1] S. M. Hong and Y. B. Jun, *Fuzzy and level subalgebras of BCK(BCI)-algebras,* Pusan Kyongnam Math. J. (presently, East Asian Math. J.) **7(2)** (1991), 185–190.

[2] S. M. Hong and Y. B. Jun, *Anti fuzzy ideals in BCK-algebras,* Kyungpook Math. J. **38(1)** (1998), 145–150.

[3] Y. B. Jun, S. M. Hong, S. J. Kim and S. Z. Song, *Fuzzy ideals and fuzzy subalgebras of BCK-algebras,* J. Fuzzy Math. **7(2)** (1999), 411–418.

[4] Y. B. Jun, S. M. Hong and E. H. Roh, *Fuzzy characteristic subalgebras/ideals of a BCK-algebra,* Pusan Kyongnam Math. J. (presently, East Asian Math. J.) **9(1)** (1993), 127–132.

[5] O. G. Xi, *Fuzzy BCK-algebras,* Math. Japon. **36(5)** (1991), 935–942.

[6] Y. Yu, J. N. Mordeson and S. C. Cheng, *Elements of L-algebra,* Lecture Notes in Fuzzy Math. and Computer Science, Creighton Univ., Omaha, Nebraska 68178, USA (1994).

[7] L. A. Zadeh, *Fuzzy sets,* Inform. and Control. **8** (1965), 338–353.

Y. U. Cho, Department of Mathematics Education, College of Education, Silla University, Pusan 617-736, Korea

E-mail address: yucho@silla.ac.kr

Y. B. Jun, Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea

E-mail address: ybjun@gsnu.ac.kr icedoor@hotmail.com

E. H. Roh, Department of Mathematics Education, Chinju National University of Education, Chinju 660-756, Korea
E-mail address: ehroh@cue.ac.kr