A STUDY ON FUZZY RANDOM PORTFOLIO SELECTION PROBLEMS BASED ON POSSIBILITY AND NECESSITY MEASURES

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Abstract. This paper incorporates fuzzy random variables with a portfolio selection problem based on the single index model. The rate of return on each investment can be represented with a fuzzy random variable. A novel decision-making model based both on possibilistic programming and on stochastic programming. It is shown that the formulated problem is transformed into the deterministic equivalent nonlinear programming problem. The deterministic problem is solved by utilizing the property that the problem is regarded as a convex programming problem including a parameter.

1 Introduction There are two different kinds of decision-making models through mathematical programming in uncertain circumstances. One is stochastic programming and the other is fuzzy programming. The former is a useful tool in stochastic systems and the latter in fuzzy systems. A comparative study between two programs have also been investigated [1], [2]. In real systems, however, fuzzy information and random factor may arise at the same time. Then we are often faced with the case where fuzziness and randomness cannot be separable as information. A fuzzy random variable [3, 4, 5, 6] was defined in order to represent the element containing fuzzy and random information simultaneously. In recent years, some researchers have applied fuzzy random variables to various decision-making problems such as linear programming problem [7, 8] and minimum spanning tree problems [9, 10].

In this article we focus on a single index model of a portfolio selection problem. Previous studies [11, 12] express the rate of return on each investment with a random variable. However, in a case where an expert estimates the value of the rate of return, it includes not only randomness but also fuzziness. Since stochastic programming models cannot take into account of fuzzy information by the expert, it is necessary to introduce a framework of new mathematical programming models to portfolio selection problems. In order to provide such a model, we try to incorporate stochastic programming models with possibilistic programming models. Later, it will be shown that the model has an advantage that an optimal solution of the deterministic equivalent problem is relatively easy to solve in spite that the problem is a nonconvex programming problem.

The rest of this paper is organized as follows: Section 2 roughly explains a single index model of a portfolio selection problem. Section 3 formulates a fuzzy random portfolio selection problem. Section 4 proposes a fuzzy random programming model using the concept of possibility measure and shows the process of transforming the problem including both fuzziness and randomness into the deterministic equivalent problem. Regarding the deterministic problem as a convex programming problem including a parameter, we provide a solution method to obtain an optimal solution. In Section 5, we consider a model using a necessity measure. Finally section 6 summaries this paper.

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2 Single index model A capital asset pricing model (CAPM) introduced by Sharpe [12] explains the relation between risk and return on investment in a stock market. The risk investment is assumed to be linearly decomposed into the risk common to all investments caused by a market and the risk individual to each investment caused by other factors. This model is called as a single index model, because it is expressed through the market only.

$$c_i - r_f = \alpha_i + \beta_i(r_m - r_f) + \epsilon, \quad i = 1, 2, \dots, n,$$

where

c_i	:	the rate of return on investment i.
r_m	:	the rate of return on market portfolio, which is the portfolio of all investments in the market. In the efficient market, r_m is assumed to have a normal distribution $N(\overline{r}_m,\ \sigma_m^2)$.
r_f	:	the rate of return on reckless asset.
β_i	:	beta - value of investment i, which is a measure of the responsiveness of investment i to changes in the market index.
α_i	:	alpha - value of investment i, which is the return on investment i that is independent of changes in the market index.
ϵ	:	a random noise on investment i with a normal distribution $N(0,\ \sigma_{e_i}^2)$, which is independent of the market index.

Replacing $\alpha_i + r_f(1 - \beta_i)$ by α_i, the above equation is simply rewritten by

$$c_i = \alpha_i + \beta_i r_m + \epsilon, \quad i = 1, 2, \dots, n.$$

3 Formulation In this section we consider the case where c_i is represented in the following form.

$$\tilde{C}_i(\omega) = \tilde{A}_i(\omega) + \beta_i r_m(\omega),\ i = 1, 2, \dots, n,$$

where $\tilde{A}_i(\omega)$ is the fuzzy random variable characterized by following membership function.

$$\mu_{\tilde{A}_i(\omega)}(t) = \begin{cases} L\left(\dfrac{\alpha_i(\omega) - t}{\xi_i}\right) & (\alpha_i(\omega) \geq t) \\ R\left(\dfrac{t - \alpha_i(\omega)}{\eta_i}\right) & (\alpha_i(\omega) < t) \end{cases} \tag{1}$$

In (2), $\alpha_i(\omega)$ is a random variable distributed according to normal distribution $N(\bar{\alpha}_i,\ \sigma_i^2)$, and $L(\cdot)$ is a reference function from $\mathbf{R}$ to $\mathbf{R}$ satisfying the following conditions.

1. $L(-t) = L(t)$ for any $t \in R$.
2. $L(t) = 1$ *iff* $t = 0$.
3. $L(\cdot)$ is non-increasing and nonnegative on $[0, +\infty)$.

The function $R(\cdot)$ also satisfies the same condition as $L(\cdot)$.

By using arithmetic operation on L - R fuzzy numbers [13], the membership function of $\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}$ is expressed as

(2)
$$\mu_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(y) = \begin{cases} L\left(\dfrac{\displaystyle\sum_{i=1}^{n}(\alpha_i(\omega)+\beta_i r_m(\omega))x_i - y}{\displaystyle\sum_{i=1}^{n}\xi_i x_i}\right) & \left(\displaystyle\sum_{i=1}^{n}(\alpha_i(\omega)+\beta_i r_m(\omega))x_i \geq y\right) \\ R\left(\dfrac{y - \displaystyle\sum_{i=1}^{n}(\alpha_i(\omega)+\beta_i r_m(\omega))x_i}{\displaystyle\sum_{i=1}^{n}\eta_i x_i}\right) & \left(\displaystyle\sum_{i=1}^{n}(\alpha_i(\omega)+\beta_i r_m(\omega))x_i < y\right). \end{cases}$$

In order to consider the imprecise nature of decision-maker's judgment, we set a fuzzy goal $\tilde{G}$ with the membership function $\mu_{\tilde{G}}(y)$, which is a non-increasing function of y. Let $L(t)$ and $\mu_{\tilde{G}}(y)$ define as follows:

$$L(t) = \begin{cases} 1 & (t=0) \\ l(t) & (0 < t \leq t_L) \\ 0 & (t > t_L), \end{cases} \quad R(t) = \begin{cases} 1 & (t=0) \\ r(t) & (0 < t \leq t_R) \\ 0 & (t > t_R), \end{cases}$$

and

$$\mu_{\tilde{G}}(y) = \begin{cases} 0 & (y \leq y^0) \\ g(y) & (y^0 < y < y^1) \\ 1 & (y \geq y^1), \end{cases}$$

where $l(t)$ and $r(t)$ are continuous decreasing functions and $g(y)$ is a continuous increasing function.

4 Fuzzy random programming model using a possibility measure The degree of possibility that the objective function value satisfies the fuzzy goal $\tilde{G}$ is defined by

$$\Pi_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(\tilde{G}) \triangleq \sup_{y} \min\left\{\mu_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(y),\ \ \mu_{\tilde{G}}(y)\right\}.$$

Let x_i be a rate of allocation to investment i. Then we propose the following problem as a decision-making method under the condition that there are both fuzziness and randomness.

$$\begin{array}{ll} \text{maximize} & h \\ \text{subject to} & \Pr\left(\Pi_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(\tilde{G}) \geq h\right) \geq \theta,\ \displaystyle\sum_{i=1}^{n} x_i = 1 \\ & 0 \leq x_i \leq \gamma_i,\ i = 1, \ldots, n \end{array} \tag{3}$$

where γ_i satisfies $0 < \gamma_i < 1$ and Pr denotes a probability measure. $\Pi_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(\tilde{G}) \geq h$ implies

$$
\begin{aligned}
&\sup_y \min\left\{\mu_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(y),\ \mu_{\tilde{G}}(y)\right\} \geq h \\
&\Longleftrightarrow \exists\, y : \mu_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(y) \geq h,\ \mu_{\tilde{G}}(y) \geq h \\
&\Longleftrightarrow \exists\, y : L\left(\frac{\displaystyle\sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_m(\omega)\}x_j - y}{\displaystyle\sum_{i=1}^{n}\xi_i x_i}\right) \geq h,\ R\left(\frac{y-\displaystyle\sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_m(\omega)\}x_i}{\displaystyle\sum_{i=1}^{n}\eta_i x_i}\right) \geq h, \\
&\qquad \mu_{\tilde{G}}(y) \geq h \\
&\Longleftrightarrow \exists\, y : \sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_m(\omega)-L^*(h)\xi_i\}x_i \leq y \leq \sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_m(\omega)+R^*(h)\eta_i\}x_j, \\
&\qquad y \geq \mu^*_{\tilde{G}}(h) \\
&\Longleftrightarrow \sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_m(\omega)+R^*(h)\xi_i\}x_i \geq \mu^*_{\tilde{G}}(h)
\end{aligned}
$$

where $\mu^*_{\tilde{G}}(h)$ and $L^*(h)$ are pseudo inverse functions defined as follows.

$$
L^*(h) = \begin{cases} t_L & (h=0) \\ l^{-1}(h) & (0<h<1) \\ 0 & (h=1) \end{cases}
$$

$$
\mu^*_{\tilde{G}}(h) = \begin{cases} y^0 & (h=0) \\ g^{-1}(h) & (0<h<1) \\ y^1 & (h=1) \end{cases}
$$

It should be noted that $\mu^*_{\tilde{G}}(h)$ is a monotone non-decreasing continuous function and that $L^*(h)$ is a monotone non-increasing continuous function. Then problem (3) can be transformed into the following form:

$$
(4)\qquad \begin{array}{ll}
\text{maximize} & h \\
\text{subject to} & \Pr\left(\mu^*_{\tilde{G}}(h) \leq \displaystyle\sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i r_m(\omega)+R^*(h)\eta_i\}\,x_i\right) \geq \theta,\ \displaystyle\sum_{i=1}^{n} x_i = 1 \\
& 0 \leq x_i \leq \gamma_i,\ i=1,\ldots,n.
\end{array}
$$

Since $r_m(\omega)$ and $\alpha_i(\omega)$ are random variables having the normal distributions $N(\bar{r}_m,\ \sigma_m^2)$ and $N(\bar{\alpha}_i,\ \sigma_i^2)$, respectively, it is obtained from the nature of normal distribution that

$$
\frac{\displaystyle\sum_{i=1}^{n}\{\beta_i(r_m(\omega)-\bar{r}_m)+\alpha_i(\omega)-\bar{\alpha}_i\}\,x_i}{\sqrt{\displaystyle\sum_{i=1}^{n}\sigma_i^2 x_i^2+\left(\sum_{i=1}^{n}\beta_i x_i\right)^2\sigma_m^2}}
$$

is a random variable having the standard normal distribution $N(0,\ 1)$. Therefore, problem (4) is equivalent to the following deterministic problem:

(5)

$$\begin{aligned} &\text{maximize} \quad h \\ &\text{subject to} \quad \sum_{i=1}^{n} \left\{ \bar{\alpha}_i + \beta_i \bar{r}_m + R^*(h)\eta_i \right\} x_i - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2 + \left(\sum_{i=1}^{n} \beta_i x_i \right)^2 \sigma_m^2} \geq \mu_{\tilde{G}}^*(h) \\ &\qquad\qquad\quad \sum_{i=1}^{n} x_i = 1,\ 0 \leq x_i \leq \gamma_i,\ i = 1, \ldots, n, \end{aligned}$$

where K_θ is a quantile of order θ of the standard normal distribution function F, i.e., $K_\theta = -K_{1-\theta} = F^{-1}(\theta) > 0$ when $\theta > 1/2$.

We denote an optimal solution of problem (5) by $(\boldsymbol{x}^*,\ h^*)$. For simplicity, we rewrite the problem by replacing $R^*(h)$ with q and obtain

(6)

$$\begin{aligned} &\text{minimize} \quad q \\ &\text{subject to} \quad \sum_{i=1}^{n} \left\{ \bar{\alpha}_i + \beta_i \bar{r}_m + q\eta_i \right\} x_i - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2 + \left(\sum_{i=1}^{n} \beta_i x_i \right)^2 \sigma_m^2} \geq \mu_{\tilde{G}}^*(R(q)) \\ &\qquad\qquad\quad \sum_{i=1}^{n} x_i = 1,\ 0 \leq x_i \leq \gamma_i,\ i = 1, \ldots, n. \end{aligned}$$

It is apparent that an optimal solution of the above problem is $(\boldsymbol{x}^*, q^*)$ where q^* equals to $R^*(h^*)$. In order to solve (6), we introduce the following problem with a parameter q.

$$\text{(7)} \quad \begin{aligned} &\text{maximize} \quad \sum_{i=1}^{n} \left\{ \bar{\alpha}_i + \beta_i \bar{r}_m + q\eta_i \right\} x_i - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2 + \left(\sum_{i=1}^{n} \beta_i x_i \right)^2 \sigma_m^2} \\ &\text{subject to} \quad \sum_{i=1}^{n} x_i = 1,\ 0 \leq x_i \leq \gamma_i,\ i = 1, \ldots, n. \end{aligned}$$

Let $\boldsymbol{x}(q)$ and $Z(q)$ denote an optimal solution and the optimal value of problem (7) for a fixed q, respectively. Since the objective function in the above problem is a strictly convex function, the optimal solution is uniquely determined. Then, we shall prove the following theorem:

Theorem 1 *Let $(\boldsymbol{x}(q^*), q^*)$ be an optimal solution $(\boldsymbol{x}(q), q)$ of problem (7) satisfying $Z(q) = \mu_{\tilde{G}}(q)$. Suppose that an optimal value q^* of problem (6) satisfies $R^*(1) < q^* < R^*(0)$. Then, $(\boldsymbol{x}(q), q)$ is equivalent to an optimal solution $(\boldsymbol{x}^*, q^*)$ of problem (6) iff and only if $Z(q) = \mu_{\tilde{G}}(R(q))$ holds.*

Proof: If $(\boldsymbol{x}(q), q)$ satisfies $Z(q) = \mu_{\tilde{G}}(R(q))$, then it holds from the definition that $(\boldsymbol{x}(q), q) = (\boldsymbol{x}(q^*), q^*)$ and $Z(q^*) = \mu_{\tilde{G}}(R(q^*))$. Suppose that $(\boldsymbol{x}(q^*), q^*)$ is not optimal to problem (6). This means that there exists some $\hat{\boldsymbol{x}}$ and $\hat{q}$ satisfying

$$\hat{q} < q^*$$

and

$$\sum_{i=1}^{n} \left\{ \bar{\alpha}_i + \beta_i \bar{r}_m + \hat{q}\eta_i \right\} \hat{x}_i - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 \hat{x}_i^2 + \left(\sum_{i=1}^{n} \beta_i \hat{x}_i \right)^2 \sigma_m^2} \geq \mu_{\tilde{G}}^*(R(\hat{q})).$$

Then, the following inequality holds:

$$
\begin{aligned}
Z(q^*) &= \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + q^* \eta_i\} x_i(q^*) - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(q^*)^2 + \left(\sum_{i=1}^{n} \beta_i x_i(q^*) \right)^2 \sigma_m^2} \\
&\geq \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + q^* \eta_i\} x_i(\hat{q}) - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(\hat{q})^2 + \left(\sum_{i=1}^{n} \beta_i x_i(\hat{q}) \right)^2 \sigma_m^2} \\
&> \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + \hat{q} \eta_i\} x_i(\hat{q}) - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(\hat{q})^2 + \left(\sum_{i=1}^{n} \beta_i x_i(\hat{q}) \right)^2 \sigma_m^2} \\
&\geq \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + \hat{q} \eta_i\} \hat{x}_i(q) - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 \hat{x}_i(q)^2 + \left(\sum_{i=1}^{n} \beta_i \hat{x}_i(q) \right)^2 \sigma_m^2} \\
&= \mu_{\tilde{G}}(R(\hat{q})) > \mu_{\tilde{G}}(R(q^*)).
\end{aligned}
$$

This contradicts the assumption $Z(q^*) = \mu_{\tilde{G}}(R(q^*))$.

Assume that $Z(q) = \mu_{\tilde{G}}(R(q))$ does not hold when $(\boldsymbol{x}(q), q)$ is equivalent to an optimal solution $(\boldsymbol{x}^*, q^*)$ of problem (6). Then since $(\boldsymbol{x}(q), q)$ is an admissible solution of (7), it holds

$$
\sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + q \eta_i\} x_i(q) - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(q)^2 + \left(\sum_{i=1}^{n} \beta_i x_i(q) \right)^2 \sigma_m^2} \geq \mu_{\tilde{G}}(R(q)).
$$

Considering the assumption that $Z(q) \neq \mu_{\tilde{G}}(R(q))$,

$$
Z(q) = \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + q \eta_i\} x_i(q) - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(q)^2 + \left(\sum_{i=1}^{n} \beta_i x_i(q) \right)^2 \sigma_m^2} > \mu_{\tilde{G}}(R(q)).
$$

Then there exists an $\check{q}$ smaller than the q satisfying

$$
\sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + \check{q} \eta_i\} x_i(q) - K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(q)^2 + \left(\sum_{i=1}^{n} \beta_i x_i(q) \right)^2 \sigma_m^2} = \mu_{\tilde{G}}(R(\check{q})).
$$

This means that $(\boldsymbol{x}(q), \check{q})$ is also an admissible solution of problem (6), and it contradicts the assumption that $(\boldsymbol{x}(q), \check{q})$ is optimal to problem (6). The proof is complete. □

Theorem 2 *$Z(q)$ is a monotone decreasing function of q.*

Proof: Suppose that $0 < q_1 < q_2 < 1$. Then it holds that

$$\begin{aligned}
Z(q_2) &= \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + q_2 \eta_i\} x_i(q_2) + K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(q_2)^2 + \left(\sum_{i=1}^{n} \beta_i x_i(q_2)\right)^2 \sigma_m^2} \\
&\geq \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + q_2 \eta_i\} x_i(q_1) + K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(q_1)^2 + \left(\sum_{i=1}^{n} \beta_i x_i(q_1)\right)^2 \sigma_m^2} \\
&> \sum_{i=1}^{n} \{\bar{\alpha}_i + \beta_i \bar{r}_m + q_1 \eta_i\} x_i(q_1) + K_\theta \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i(q_1)^2 + \left(\sum_{i=1}^{n} \beta_i x_i(q_1)\right)^2 \sigma_m^2} \\
&= Z(q_1).
\end{aligned}$$

The proof is complete. □

Using the above theorems, we construct an algorithm for solving problem (6).

Algorithm for solving fuzzy random portfolio selection problems

Step 1 Set $q := R^*(1)(= 0)$ and solve problem (7). If $Z(q) \geq \mu^*_{\tilde{G}}(R(q))$, then set $\boldsymbol{x}^* := \boldsymbol{x}(q)$ and terminate. Otherwise, set $\underline{q} \leftarrow 0$ and go to step 2.

Step 2 Set $q := R^*(0)(= t_R)$ and solve problem (7). If $Z(q) \geq \mu^*_{\tilde{G}}(R(q))$, then terminate (the given fuzzy goal is too tight and it is necessary to loose the fuzzy goal). Otherwise, set $\overline{q} \leftarrow t_R$ and go to step 3.

Step 3 If $\overline{q} - \underline{q} < \varepsilon$ (a sufficiently small positive constant), then terminate. Otherwise, go to step 4.

Step 4 Set $q_c := (\overline{q} + \underline{q})/2$ and solve problem (7). If $Z(q_c) = \mu^*_{\tilde{G}}(R(q_c))$, then set $\boldsymbol{x}^* =: \boldsymbol{x}(q_c)$ and terminate. Otherwise, go to step 5.

Step 5 If $Z(q_c) > \mu^*_{\tilde{G}}(R(q_c))$, then set $q_c := (q_c + \underline{q})/2$ and solve (7). Return to step 3.

Step 6 If $Z(q_c) < \mu^*_{\tilde{G}}(R(q_c))$, then set $q_c := (q_c + \overline{q})/2$ and solve (7). Return to step 3.

5 Model using a necessity measure In the previous section, we have considered a model using a possibility measure, which is useful in making a decision with an optimistic notion. This section devotes to investigating a model using a necessity measure and deals with the following problem:

$$\begin{array}{ll}
\text{maximize} & h \\
\text{subject to} & \Pr\left(N_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(\tilde{G}) \geq h\right) \geq \theta, \sum_{i=1}^{n} x_i = 1 \\
& 0 \leq x_i \leq \gamma_i,\ i = 1, \ldots, n
\end{array} \tag{8}$$

where $N_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}$ denotes a necessity measure and

$$N_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(\tilde{G}) = \inf_{y} \max\left\{1 - \mu_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(y), \mu_{\tilde{G}}(y)\right\}. \tag{9}$$

Then, $N_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(\tilde{G}) \geq h$ implies

$$\begin{aligned}
&\inf_y \max\{1-\mu_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(y),\ \mu_{\tilde{G}}(y)\} \\
&\Longleftrightarrow \forall y : 1-\mu_{\tilde{\boldsymbol{C}}(\omega)\boldsymbol{x}}(y) < h \Rightarrow \mu_{\tilde{G}}(y) \geq h \\
&\Longleftrightarrow \forall y : \sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_m(\omega)-L^*(1-h)\xi_i\}\,x_i < y < \sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_i(\omega)+R^*(1-h)\eta_i\}\,x_i \\
&\qquad \Rightarrow y \geq \mu^*_{\tilde{G}}(h) \\
&\Longleftrightarrow \sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i\gamma_i(\omega)+R^*(1-h)\eta_i\}\,x_i \geq \mu^*_{\tilde{G}}(h).
\end{aligned}$$

Consequently, problem (8) is rewritten as

$$\text{(10)}\quad \begin{array}{ll}
\text{maximize} & h \\
\text{subject to} & Pr\Big(\mu^*_{\tilde{G}}(h) \leq \sum_{i=1}^{n}\{\alpha_i(\omega)+\beta_i r_m(\omega)-L^*(1-h)\xi_i\}\,x_i\Big) \geq \theta,\ \sum_{i=1}^{n} x_i = 1 \\
& 0 \leq x_i \leq \gamma_i,\ i=1,\ldots,n.
\end{array}$$

Since

$$\frac{\displaystyle\sum_{i=1}^{n}\{\beta_i(r_m(\omega)-\bar{r}_m)+\alpha_i(\omega)-\bar{\alpha}_i\}\,x_i}{\sqrt{\displaystyle\sum_{i=1}^{n}\sigma_i^2x_i^2+\left(\sum_{i=1}^{n}\beta_i x_i\right)^2\sigma_m^2}}$$

is a random variable having the standard normal distribution $N(0,\ 1)$, problem (8) is equivalent to the following deterministic problem:

$$\text{(11)}\quad \begin{array}{ll}
\text{maximize} & h \\
\text{subject to} & \displaystyle\sum_{i=1}^{n}\{\bar{\alpha}_i+\beta_i\bar{r}_m-L^*(1-h)\xi_i\}\,x_i - K_\theta\sqrt{\sum_{i=1}^{n}\sigma_i^2x_i^2+\left(\sum_{i=1}^{n}\beta_i x_i\right)^2\sigma_m^2} \geq \mu^*_{\tilde{G}}(h) \\
& \displaystyle\sum_{i=1}^{n} x_i = 1,\ 0 \leq x_i \leq \gamma_i,\ i=1,\ldots,n.
\end{array}$$

Apparently, problem (11) is solved by the solution algorithm described in the previous section by simply replacing $R^*(h)$ with $-L^*(1-h)$.

6 Conclusion This paper has addressed a portfolio selection problem where the rate of return on investment in single index model is represented with a fuzzy random variable. An decision making model has been constructed on the basis of stochastic programming and possibilistic programming. It has been shown that the problem based on the constructed model, which includes both randomness and fuzziness, is transformed into the deterministic equivalent problem. By using the fact that the deterministic problem can be regarded as a convex programming problem including a parameter, a solution method for obtaining an optimal solution of the problem has been proposed. In addition, a model using a necessity measure has been considered, and it has been shown that the problem based on the model using a necessity measure can be also solved by the similar method for the model using a possibility measure.

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