THE FDS-PROPERTY AND SPACES IN WHICH COMPACT SETS ARE CLOSED

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Abstract. A space in which every infinite set contains an infinite subset with only a finite number of accumulation points is said to have the finite derived set property. We study this property in the class of spaces in which compact sets are closed – the KC-spaces – and apply our results to show that among hereditarily Lindelöf spaces, minimal KC-spaces are compact. This result generalizes a theorem of [2] and gives a partial answer to a question of R. Larson.

1 Which spaces have the FDS-property? A space X is said to have the finite derived set property (hereafter abbreviated as the FDS-property) if each infinite subset A ⊆ X contains an infinite subset with only finitely many accumulation points (in X). This concept was introduced in [8] in order to study properties of the lattice of T₁-topologies on a set X. In a subsequent paper [2], we studied the class of KC-spaces, that is to say the class of spaces in which all compact subsets are closed; such spaces are clearly T₁ and every Hausdorff space is KC. The KC-spaces have also been called T₃₃-spaces (for instance in [6]). A problem ascribed to R. Larson in [6] is whether a space is maximal with respect to being compact if and only if it is minimal with respect to being KC. In [2], it was shown that in the class of KC-spaces, each countable space has the FDS-property and this result was used to prove that every countable minimal KC-space is compact, thus giving a (very) partial answer to the above-mentioned question of Larson. A KC-space (X, τ) is said to be Katětov-KC if there is a minimal KC-topology σ ⊆ τ. In [6], Fleissner showed that not every KC-space is Katětov-KC, but no characterization of Katětov-KC spaces is known.

In the first section of this paper we continue our study of those spaces which have the FDS-property, while in Section 2, we apply our results to show that in some fairly wide classes of KC-spaces, including all hereditarily Lindelöf spaces, minimal KC implies compact. We also prove that certain classes of KC-spaces are Katětov-KC. All spaces considered here are (at least) T₁ and all undefined notation and terminology can be found in [5], but note that the symbol ⊂ is used exclusively to denote proper containment. The following result is obvious:

Remark 1.1 If X is a KC-space, then no infinite subspace of X can have the cofinite topology.

Lemma 1.2 Each infinite subspace of a KC-space contains an infinite discrete subspace.

Proof: Let (X, τ) be a KC-space and suppose A ⊆ X is infinite; since A does not have the cofinite topology, there is some open set U₀ in X such that A ∩ U₀ ≠ ∅ and A \ U₀ is

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infinite; choose $x_0 \in A \cap U_0$. Having chosen open sets $U_0, \ldots, U_{n-1}$ and points $x_0, \ldots, x_{n-1}$ in such a way that for all $m \in \{0, \ldots, n-1\}$,

i) $x_m \in A \cap U_m$;

ii) $x_k \notin U_m$ if $k \neq m$; and

iii) $A_{n-1} = A \setminus \bigcup \{U_m : 0 \leq m \leq n-1\}$ is infinite.

Then, by Remark 1.1, $A_{n-1}$ does not have the cofinite topology and so we can find $U_n \in \tau$ such that $x_m \notin U_n$ for each $m < n$, $U_n \cap A_{n-1} \neq \emptyset$ and $A_{n-1} \setminus U_n$ is infinite; we then choose $x_n \in U_n \cap A_{n-1}$. This completes our inductive construction; let $D = \{x_n : n \in \omega\}$. Clearly $D$ is infinite and is discrete since $U_n \cap D = \{x_n\}$ for all $n \in \omega$.

A family $\mathcal{A} = \{A_\alpha : \alpha \in \kappa\} \subseteq [\omega]^\omega$ (the set of of infinite subsets of $\omega$) has the strong finite intersection property if the intersection of any finite subfamily of $\mathcal{A}$ is infinite. If $A, B \in [\omega]^\omega$ and $A \setminus B$ is finite, then we write $A \subseteq^* B$. A set $B$ is a pseudointersection of the family $\mathcal{A}$ if $B \subseteq^* A_\alpha$ for all $\alpha \in \kappa$. Recall that $p$ is the smallest cardinal such that there exists a family of infinite subsets $\{A_\alpha : \alpha < p\}$ of $\omega$ with the strong finite intersection property but no infinite pseudointersection. It is known that $\omega_1 \leq p \leq c$.

**Theorem 1.3** A KC-space with the property that $\chi(p, X) < p$ for each $p \in X$ has the FDS-property.

**Proof:** Let $(X, \tau)$ be a KC-space such that $\chi(p, X) < p$ for all $p \in X$ and let $A \subseteq X$ be infinite. By Lemma 1.2, we can find a countably infinite discrete subspace $D \subseteq A$. If $D$ is closed in $X$ then we are done. If not, then let $x$ be an accumulation point of $D$ and let $\{U_\alpha : \alpha < \lambda\}$ be a local base of open sets at $x$, where $\lambda = \chi(x, X) < p$. Then the family $\{U_\alpha \cap D : \alpha < \lambda\}$ is a family of subsets of $D$ with the strong finite intersection property. By the definition of $p$, there is an infinite subset $S \subseteq D$ such that $S \setminus U_\alpha$ is finite for each $\alpha < \lambda$. Clearly then, the countably infinite set $S$ converges to $x$ and so $S \cup \{x\}$ is compact and hence closed in $X$. Therefore $|\text{cl}(S) \setminus S| = 1$.

**Remark 1.4** A similar argument can be used to show that if $X$ is a KC-space such that $\chi(x, X) < p$ for each $x \in X$, then every separable subspace of $X$ is Hausdorff.

If $(X, \tau)$ is a Hausdorff space, then for each $p \in X$, set $\psi_c(p, X) = \min\{|\mathcal{U}| : \mathcal{U} \subset \tau, p \in \mathcal{U} \text{ for all } U \in \mathcal{U} \text{ and } \bigcap\{U : U \in \mathcal{U}\} = \{p\}\}$; as in [7], we can then define the closed pseudocharacter of $X$, $\psi_c(X) = \sup\{\psi_c(p, X) : p \in X\}$.

If in the above theorem, the space $(X, \tau)$ is Hausdorff, then in order to conclude that $x$ is the only possible accumulation point of $S$, it suffices that there exist $\lambda < p$ such that $\bigcap\{U_\alpha : \alpha < \lambda\} = \{x\}$. Hence we have proved:

**Theorem 1.5** Each Hausdorff space $X$ such that $\psi_c(x, X) < p$ for any $x \in X$ has the FDS-property.

Bounding the size of $X$ also allows us to prove that a space has the FDS-property in case it is either Tychonoff or compact and KC.

**Theorem 1.6** Every Tychonoff space of cardinality less than $c$ has the FDS-property.

**Proof:** Suppose to the contrary that $X$ is a Tychonoff space of cardinality less than $c$ in which there is an infinite subset $A \subseteq X$ (which we may suppose to be discrete) such that every infinite subset of $A$ has infinitely many accumulation points. Clearly $Y = \text{cl}_X(A)$ is pseudocompact and hence if $f : Y \to \mathbb{R}$ is continuous, then $f(Y)$ is a compact subset of $\mathbb{R}$ of cardinality less than $c$; thus $f[Y]$ must be countable. Let $f^\beta : \beta Y \to \mathbb{R}$ be the
extension of $f$ to $\beta Y$; clearly $f^\beta[\beta Y] = f[Y]$ and so every real-valued continuous image of $\beta Y$ is countable. Thus by a result of Shapirovski, (see [9]), $\beta Y$, and hence $Y$, is scattered. But then, if $x$ is an isolated point of $Y \setminus A$, it follows by the regularity of $Y$ that there is a closed neighbourhood $U$ of $x$ in $Y$ contained in $A \cup \{x\}$. Since $U$ is compact, there is a sequence in $A$ converging to $x$, contradicting our hypothesis regarding the set $A$.

Before our next result, we note that a countable compact $KC$-space with only a finite number of accumulation points is either finite or a topological union of convergent sequences. Thus a countably compact $KC$-space has the $FDS$-property if and only if it is sequentially compact.

**Theorem 1.7** A compact $KC$-space of cardinality less than $c$ has the $FDS$-property (and hence is sequentially compact).

**Proof:** Suppose that $(X, \tau)$ is a compact $KC$-space and $|X| < c$. We assume to the contrary that $X$ does not have the $FDS$-property and so there is some countably infinite subset $A \subseteq X$ such that every infinite subset of $A$ has infinitely many accumulation points in $X$. By Lemma 1.2, without loss of generality, we can assume that $A$ is discrete and that $\text{cl}_X(A) = X$. Let $x \in X \setminus A$; if every neighbourhood $U$ of $x$ is such that $A \setminus U$ is finite, then $A \cup \{x\}$ is compact, hence closed in $X$ and so $x$ is the unique accumulation point of $A$, a contradiction. Thus we can choose an open neighbourhood $V$ of $x$, such that $A \setminus V$ is infinite. The closed subspace $X \setminus V$ of $X$ is compact and $A$ is countable, and hence Lindelöf, and so $A \cup (X \setminus V)$ is Lindelöf. This in its turn implies that $A \cup (X \setminus V)$ is not countably compact, for otherwise it would be compact, but being a proper dense subspace of $X$, it is not closed in $X$, a contradiction. Thus there is a discrete set $D \subseteq A \cup (X \setminus V)$ which is closed in $A \cup (X \setminus V)$. However, since $X \setminus V$ is compact, only a finite number of points of $D$ lie in $X \setminus V$ and hence $D \cap V$ is infinite and all its accumulation points (by our hypothesis, an infinite number in $X$) must lie in $V$; that is to say, $\text{cl}_X(D \cap V) \subseteq V$. Thus we have constructed two infinite sets $D \cap V$ and $A \setminus V$ whose closures are disjoint in $X$. Since $D \cap V \subseteq A$, each of these sets has the property that every infinite subset has an infinite number of accumulation points and the above argument can be repeated using $D \cap V$ and $A \setminus V$ in place of $A$. A standard binary tree argument can now be used to show that $|X| \geq c$.

**Remark 1.8** Since any topology stronger than a topology with the $FDS$-property has the $FDS$-property, it follows from Theorem 1.7 that in the search for a $KC$-topology (respectively Hausdorff topology) with no weaker compact $KC$-topology, assuming $\omega_1 < c$, it suffices to find a $KC$-topology (respectively, Hausdorff topology) on a set of size $\omega_1$ without the $FDS$-property.

Note that if $\omega_1 = c$ then there is a countably compact subspace of $\beta \omega$ of cardinality $\omega_1$ which does not have the $FDS$-property. On the other hand, if $\omega_1 < p$, and $\tau$ is any Hausdorff topology on $\omega_1$, then $(\omega_1, \tau)$ can be condensed onto a Hausdorff topology of weight $\omega_1$, which, by Theorem 1.5, has the $FDS$-property, implying in its turn that $\tau$ has the $FDS$-property.

**Question 1.9** If $p = \omega_1 < c$, does there exist a $KC$ (or even a Hausdorff) topology on $\omega_1$, without the $FDS$-property?

Again using a tree argument, it is easy to show, under $\omega_1 < c$, that a countably compact Hausdorff topology on $\omega_1$ without the $FDS$-property cannot be Urysohn or even weakly regular (each non-empty open set contains the closure of a non-empty open set).
Theorem 1.10 A hereditarily Lindelöf KC-space has the FDS-property.

Proof: Suppose to the contrary that \((X, \tau)\) is a hereditarily Lindelöf KC-space which does not have the FDS-property. Then there is some infinite subset \(D \subseteq X\), which by Lemma 1.2 we may assume to be discrete, such that every infinite subset of \(D\) has infinitely many accumulation points. As a consequence, for all infinite \(C \subseteq D\), and \(x \in \text{cl}_\tau(C) \setminus C\), there is an open neighbourhood \(U_x\) of \(x\) such that \(C \setminus U_x\) is infinite, for otherwise, \(C \cup \{x\}\) is compact, hence closed in \(X\), contradicting our assumption regarding \(D\). Clearly we can assume also that \(D\) is countable. We will construct recursively a strictly increasing nested family of open sets in a subspace of \(X\) of length \(\omega_1\), contradicting the fact that \(X\) is hereditarily Lindelöf.

To this end, let \(D = D_0\) and choose \(x_0 \in \text{cl}(D) \setminus D\) and an open neighbourhood \(U_0\) of \(x_0\) such that \(D_1 = D_0 \setminus U_0\) is infinite.

Suppose that for some ordinal \(\alpha \in \omega_1\) we have chosen points \(\{x_\beta : \beta \in \alpha\}\), infinite subsets \(\{D_\beta : \beta \in \alpha\}\) of \(D\) and open sets \(\{U_\beta : \beta \in \alpha\}\), such that

i) \(x_\beta \in U_\beta\) for all \(\beta \in \alpha\),

ii) \(x_\beta \in (\text{cl}(D_\beta) \setminus D_\beta) \setminus U_\gamma\) for all \(\gamma < \beta < \alpha\),

iii) \(D_\beta \subseteq^* D_\alpha \setminus U_\gamma\) for all \(\gamma < \beta < \alpha\), and

iv) \(D_\beta \setminus U_\beta\) is infinite for all \(\beta \in \alpha\),

we proceed to choose \(x_\alpha\) and \(U_\alpha\) as follows:

By iii) and iv), \(D_\beta \cap D_\alpha\) is infinite for each \(\gamma \in \beta \in \alpha\) and, since \(|\alpha| = \omega < p\), there is some infinite set \(D_\alpha \subseteq^* D_\beta\) for all \(\beta \in \alpha\). Again by iii), we have \(D_\beta \subseteq^* D_\alpha \subseteq^* D_\alpha \setminus U_\gamma\) for all \(\gamma \in \beta \in \alpha\) whence it follows that \(D_\alpha \subseteq^* D_\alpha \setminus U_\gamma\) for all \(\gamma \in \alpha\) and hence all accumulation points of \(D_\alpha\) lie outside \(U_\gamma\) for each \(\gamma \in \alpha\). Choose \(x_\alpha \in \text{cl}(D_\alpha) \setminus D_\alpha\) and an open neighbourhood \(U_\alpha\) of \(x_\alpha\) such that \(D_\alpha \setminus U_\alpha\) is infinite. It is clear that \(\{x_\beta : \beta \leq \alpha\}\), \(\{D_\beta : \beta \leq \alpha\}\) and \(\{U_\beta : \beta \leq \alpha\}\) satisfy i)-iv) above.

Let \(L = \{x_\alpha : \alpha \in \omega_1\}\); by construction, each \(x_\alpha \in L\) has an open neighbourhood \(U_\alpha \cap L\) contained in \(\{x_\beta : \beta \leq \alpha\}\); that is to say, for each \(\alpha \in \omega_1\), \(\{x_\beta : \beta \in \alpha\}\) is open in \(L\) and the result follows.

We note that the above result can be somewhat improved since the recursive construction can be continued as far as any cardinal \(\mu < p\). Thus we have actually proved:

Corollary 1.11 If \(X\) is a KC-space with \(hL(X) < p\), then \(X\) has the FDS-property.

Since consistently \(p = \aleph_0\) and \(\aleph_0\) is regular we have:

Corollary 1.12 It is consistent that every KC-space \(X\) with \(hL(X) < \aleph_0\) has the FDS-property.

A space \(X\) is said to be weakly discretely generated if whenever \(A \subset X\) is not closed, then there is some discrete subset \(D \subseteq A\) such that \(\text{cl}(D) \setminus A \neq \emptyset\). It was shown in Proposition 3.1 of [4] that every compact Hausdorff space is weakly discretely generated and a similar proof applying Lemma 2.3 of [1] can be used for compact KC-spaces.

Of course, a space with a countable network is hereditarily Lindelöf and so it is worth noting that a Hausdorff space with \(\sigma\)-discrete network need not have the FDS-property. The Katětov extension, \(\kappa \omega\) of \(\omega\) (see [5, 3.12.6]) is strongly \(\sigma\)-discrete, hence has a \(\sigma\)-discrete network, but lacks the FDS-property. A modification of the topology of the Stone-\v{C}ech compactification \(\beta X\) of van Douwen’s countable maximal space \(X\) (see [3]) obtained by declaring \(X\) and each of its supersets to be open is an \(H\)-closed space with a \(\sigma\)-discrete network which is neither weakly discretely generated nor has the FDS-property.
Properties of minimal KC-spaces. Our first lemma in this section generalizes Theorem 10 of [2] and gives a partial answer to the question of R. Larson mentioned in the first paragraph of Section 1.

Lemma 2.1 A hereditarily Lindelöf, minimal KC-space is compact.

Proof: Suppose that \((X, \sigma)\) is a hereditarily Lindelöf minimal KC-space; by Theorem 1.10, \(X\) has the FDS-property. If \((X, \sigma)\) is not compact then since it is Lindelöf, it is not countably compact and hence there is some countably infinite closed discrete subspace \(D = \{d_n : n \in \omega\} \subseteq X\). Fix \(p \in X\) and a free ultrafilter \(G \in \beta \omega \setminus \omega\) and define a new topology \(\mu\) on \(X\) as follows:

(i) If \(p \notin U\), then \(U \in \mu\) if and only if \(U \in \sigma\),

and

(ii) If \(p \in U\), then \(U \in \mu\) if and only if \(U \in \sigma\) and \(\{n \in \omega : d_n \in U\} \in G\).

Clearly \((X, \mu)\) is a \(T_1\)-space, \(\mu \subseteq \sigma\) and for each \(B \subseteq X\), \(\text{cl}_{\mu}(B) \subseteq \text{cl}_\sigma(B) \cup \{p\}\); since \((X, \sigma)\) has the FDS-property, it follows that \((X, \mu)\) does as well. We proceed to show that \((X, \mu)\) is a KC-space. To this end, suppose to the contrary that \(A\) is a non-closed, compact subset of \((X, \mu)\). Obviously \(p \in \text{cl}_\mu(A)\) and there are two cases to consider:

(a) If \(p \notin A\), then \(\mu|A = \sigma|A\) and so \(A\) is compact and hence closed in \((X, \sigma)\). Thus \(U = X \setminus A\) is open and \(p \in U\). If \(\{n \in \omega : d_n \in A\} \notin G\), then \(\{n \in \omega : d_n \in D \setminus A\} \in G\) and for each \(d \in D \setminus A\), \(d \in U\) and so \(p \in U \in \mu\) contradicting the fact that \(p \in \text{cl}_\mu(A)\). Thus \(\{n \in \omega : d_n \in A\} \notin G\) and hence there is some infinite set \(S \subseteq A \cap D\) such that \(\{n \in \omega : d_n \in S\} \notin G\) and \(S\) is then an infinite closed discrete subset of \(A\) in \((X, \mu)\), implying that \((A, \mu|A)\) is not compact, again a contradiction.

(b) If \(p \in A\), then \(\text{cl}_\mu(A) = \text{cl}_\sigma(A)\), implying that \(A\) is not closed in \((X, \sigma)\). Thus \(A\) is not compact and since \(A\) is Lindelöf, it is not countably compact in \((X, \sigma)\). Thus there is a countably infinite, discrete subset \(C \subseteq A\) which is closed in \((A, \sigma|A)\). However, \(C\) is not closed in \((A, \mu|A)\) and so \(\text{cl}_\mu(C) \cap A = C \cup \{p\}\). This implies that \(\{n \in \omega : d_n \in \text{cl}_\mu(C)\} \in G\).

If \(P = \{n \in \omega : d_n \in C\}\) is infinite, then there is some infinite subset \(S \subseteq P\) such that \(S \notin G\) and hence \(\{d_n : n \in S\}\) is a closed, discrete subspace of \((A, \mu|A)\), contradicting the compactness of this space. If, on the other hand, \(P\) is finite, then since \((X, \mu)\) has the FDS-property, there is an infinite subset \(B \subseteq C\) with only a finite number of accumulation points in \((X, \mu)\). Thus \(\{n \in \omega : d_n \in \text{cl}_\mu(B)\} \notin G\) which implies that \(B\) is closed and discrete in \((A, \mu|A)\), implying in its turn that \(A\) is not compact in \((X, \mu)\).

In fact Lemma 2.1 can be improved.

Theorem 2.2 A hereditarily Lindelöf minimal KC-space is compact and sequential.

Proof: Suppose that \((X, \tau)\) is a hereditarily Lindelöf minimal KC-space; the previous lemma shows that \(X\) is compact and we proceed to show that it is sequential. To this end, suppose that \(A \subseteq X\) is not closed and hence not compact. Since \(X\) is hereditarily Lindelöf, \(A\) is not countably compact and hence we can find a countable discrete subset \(D = \{x_n : n \in \omega\} \subseteq A\) which is closed in \(A\); that is to say, all of the accumulation points of \(D\) lie outside of \(A\). By Theorem 1.10, \(X\) has the FDS-property, and so there is some countably infinite set \(E \subseteq D\) with only a finite number of accumulation points in \(X\), all of which lie in \(\text{cl}(A) \setminus A\). Thus \(\text{cl}(E)\) is a countable, compact KC-space and it follows from Corollary 3 of [2], \(\text{cl}(E)\) is sequential; thus there is a sequence in \(E\) converging out of \(E\) and hence out of \(A\).
Clearly, we have shown in the previous theorem that a compact, hereditarily Lindelöf $KC$-space is sequential but need not be first countable as the one-point compactification of the rationals illustrates. It is interesting to note that there are $H$-closed, hereditarily Lindelöf, Hausdorff spaces which are not sequential. Let $\mu$ denote the usual metric topology on $[0,1]$ and consider the topology $\tau$ on $[0,1]$ generated by the family of sets of the form
\[ \{ U \setminus D : U \in \mu \text{ and } D \text{ is closed and discrete in } [0,1] \setminus Q \}, \]
where $Q$ denotes the set of rational numbers.

The completely Hausdorff space $(0,1,\tau)$ has a countable network, and is $H$-closed (since its semiregularization is the compact space $(0,1,\mu)$), but is not sequential, because $[0,1] \setminus Q$ is sequentially closed but not closed.

In a first countable non-Hausdorff space there always exists a sequence convergent to two distinct points and hence it is clear that a first countable $KC$-space is Hausdorff. Hence in Theorem 2.2 we have actually proved:

**Corollary 2.3** A second countable minimal $KC$-space is compact Hausdorff.

By way of contrast, second countable, non-compact, minimal Hausdorff spaces are known and the one-point compactification of the rationals is a non-Hausdorff, Fréchet-Urysohn, minimal $KC$-space. The question then arises as to whether a first countable minimal $KC$-space is compact. In fact, we can prove a stronger result:

**Theorem 2.4** A sequential minimal $KC$-space is compact.

**Proof:** Let $(X,\tau)$ be a non-compact space satisfying the hypothesis of the theorem. Fix $a \in X$ and define a new topology $\sigma$ on $X$ as follows:
\[ \sigma = \{ U \in \tau : a \notin U \} \cup \{ U \in \tau : a \in U \text{ and } X \setminus U \text{ is compact} \}. \]

Clearly $(X,\sigma)$ is a compact $T_1$-space and $\sigma \subset \tau$. Thus to complete the proof, it suffices to show that $(X,\sigma)$ is a $KC$-space. To this end, suppose that $S \subseteq X$ is a compact subset of $(X,\sigma)$. It is clear that $\text{cl}_\sigma(S) \subseteq \text{cl}_\tau(S) \cup \{ a \}$ and that if $a \notin S$, then $\sigma|S = \sigma|S$. There are then two possibilities:

(i) If $a \notin S$, then by the preceding remarks, $S$ is compact, and hence closed, in $(X,\tau)$ and so $X \setminus S$ is an open $\sigma$-neighbourhood of $a$. Thus $a \notin \text{cl}_\sigma(S)$ and so $\text{cl}_\sigma(S) = \text{cl}_\tau(S) = S$.

(ii) If $a \in S$ then $\text{cl}_\sigma(S) = \text{cl}_\tau(S)$ and so if $S$ is not closed in $(X,\sigma)$, then it is not closed in $(X,\tau)$ either. Thus there is some $x \in \text{cl}_\tau(S) \setminus S$ and a sequence $\{ x_n \}_{n \in \omega}$ in $S$ convergent to $x$. Since $a \neq x$, we may assume that $x_n \neq a$ for all $n \in \omega$. Then $K = \{ x_n : n \in \omega \} \cup \{ x \}$ is compact in $(X,\tau)$, hence closed in $(X,\tau)$ and since $a \notin K$, it is closed in $(X,\sigma)$. Thus $K \cap S = \{ x_n : n \in \omega \}$ is a closed subset of the compact space $(S,\sigma|S)$ and thus is compact. Since $a \notin K \cap S$, we have $\sigma|(K \cap S) = \tau|(K \cap S)$ so $K \cap S$ is compact in $(X,\tau)$ and hence closed in $(X,\tau)$. However, $x \in \text{cl}_\tau(K \cap S) \setminus (K \cap S)$ which is a contradiction.

The next result is an immediate consequence of Theorem 2.4 and the comments preceding Corollary 2.3.

**Corollary 2.5** A first countable $KC$-space is minimal $KC$ if and only if it is compact Hausdorff.

**Corollary 2.6** Every sequential $KC$-space is Katětov-$KC$.

**Corollary 2.7** Each countable Hausdorff space is Katětov-$KC$. 
**Theorem 2.8** A Hausdorff $k$-space is minimal KC if and only if it is compact.

**Proof:** The sufficiency is clear. For the necessity, let $(X, \tau)$ be a non-compact space which satisfies the hypothesis of the theorem. Define $\sigma$ as in Theorem 2.4. Again, we claim that $(X, \sigma)$ is a KC-space. If $S$ is a compact subset of $(X, \sigma)$ and $a \notin S$, then the proof proceeds as in (i) of Theorem 2.4. If on the other hand, $a \in S$, then $cl_\sigma(S) = cl_\tau(S)$ and so if $S$ is not closed in $(X, \sigma)$, then it is not closed in $(X, \tau)$ either. Since $(X, \tau)$ is a $k$-space, there is some compact set $C$ in $(X, \tau)$ such that $C \cap S$ is not closed in $C$. Furthermore, if the chosen compact set $C$ has the property that $a \in C$, then since $(X, \tau)$ is Hausdorff, given $x \in cl_\tau(C \cap S) \setminus (C \cap S)$, we can find disjoint open neighbourhoods $U, V$ of $x$ and $a$ respectively. Then $C \setminus V$ is a compact subset of $(X, \tau)$ with the property that $S \cap (C \setminus V)$ is not closed in $C \setminus V$. Hence we have shown that it is possible to choose $C$ so that $a \notin C$. Then $cl_\tau(C \cap S) \subseteq C$ is a closed, hence compact subset of $(X, \tau)$ which does not contain $a$ and hence is also closed in $(X, \sigma)$. Thus $T = S \cap cl_\tau(C \cap S)$ is a $\sigma$-closed subset of $S$ and hence is compact in $(X, \sigma)$. However, since $a \notin T$, it follows that $\tau|T = \sigma|T$ and hence $T$ is compact in $(X, \tau)$, a contradiction, since $x \in cl_\tau(T) \setminus T$.

In the proof of the above theorem, we have constructed a compact KC-topology $\sigma$ on $X$ with $\sigma \subset \tau$. Thus we have also proved:

**Corollary 2.9** A Hausdorff $k$-space is Katětov-KC.

Larson’s original question remains open but appears to be a difficult problem. However, considering the results obtained above, a number of interesting and possibly more tractable questions remain; below we mention a few of them.

**Question 2.10** Can a non-compact minimal Hausdorff space be minimal KC? Alternatively, is every Hausdorff minimal KC-space compact? Is every minimal Hausdorff space a $k$-space?

**Question 2.11** Is a closed subspace of a minimal KC-space, minimal KC?

Note that a positive answer to the last question implies that a minimal KC-space is countably compact. Furthermore, if $X$ is Hausdorff and every closed subspace is minimal KC, then every closed subspace is $H$-closed and then by a result of Stone (see [5, 3.12.5]), $X$ is compact. Since a hereditarily Lindelöf Hausdorff space has countable pseudocharacter, we are led to ask:

**Question 2.12** Does a Lindelöf KC-space with countable pseudocharacter have the FDS-property?

**Question 2.13.** Can every KC-space (or each $T_2$-space) be embedded in a compact KC-space? Is the Wallman compactification of a KC-space KC?

However, there is no way of embedding a KC-space in some power of a compact KC-space since the square of a non-Hausdorff compact KC-space is never KC (the diagonal is compact but not closed). Indeed, it is easy to show that if $X$ is a KC-space then for each $\kappa \geq 2$, $X^\kappa$ is KC if and only if each compact subspace of $X$ is Hausdorff.

**Question 2.14.** Does there exist a compact KC-space in which every open set is dense?

If the answer to Question 2.13 is affirmative, then so is the answer to Question 2.14, for if $X$ is the co-countable topology on an uncountable set, then in any $T_1$-compactification of $X$, all open sets are dense.
References


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