SMARANDACHE DISJOINT IN $BCK/D$-ALGEBRAS

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Abstract. In this paper we include several new families of Smarandache-type $P$-algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

Let $(X, *)$ be a binary system/algebra. Then $(X, *)$ is a Smarandache-type $P$-algebra if it contains a subalgebra $(Y, *)$, where $Y$ is non-trivial, i.e., $|Y| \geq 2$, or $Y$ contains at least two distinct elements, and $(Y, *)$ is itself of type $P$. Thus, we have Smarandache-type semigroups (the type $P$-algebra is a semigroup), Smarandache-type groups (the type $P$-algebra is a group), Smarandache-type abelian groups (the type $P$-algebra is an abelian group). Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [2]). Smarandache-type groups are of course a larger class than Kandasamy’s Smarandache semigroups since they may include non-associative algebras as well.

In this paper we include several new families of Smarandache-type $P$-algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

A $d$-algebra ([4]) $(X, *, 0)$ is an algebra satisfying the following axioms: (i) $x * x = 0$ for all $x \in X$; (ii) $0 * x = 0$ for all $x \in X$; (iii) $x * y = y * x = 0$ if and only if $x = y$.

If $X = [0, \infty) = \{x \in R | x \geq 0\}$, where $R$ is the collection of real numbers, and if $x * y = \max\{0, x - y\}$, then $(X, *, 0)$ is a $d$-algebra. $d$-algebras are quite common and occur in many situations as the example above indicates.

Instead of asking whether a $d$-algebra can be a semigroup using the same operation, we can instead ask the much wider question: Can a $d$-algebra be a Smarandache-type semigroup?

Theorem 1. If $(X, *, 0)$ is a $d$-algebra, then it cannot be a Smarandache-type semigroup.

Proof. Suppose that it is in fact a Smarandache-type semigroup. Let $|Y| \geq 2$, where $(Y, *)$ is a subalgebra of $(X, *)$, which is also a semigroup. Thus, if $y \in Y$, then $y * y = 0 \in Y$ as well. Therefore $(y * y) * y = 0 * y = 0 = y * (y * y) = y * 0$. But $y * 0 = 0 * y = 0$ by condition (iii) for $d$-algebras implies $y = 0$, so that in fact $Y = \{0\}$ and $|Y| = 1$, a contradiction.

Corollary 2. If $(X, *, 0)$ is a $d$-algebra, then it cannot be a Smarandache-type group.

We can say the following however:

Theorem 3. If $(X, *)$ is a semigroup, then it cannot be a Smarandache-type $d$-algebra.

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Proof. Suppose that \((Y, *, 0)\) is a non-trivial sub-\(d\)-algebra of \((X, *)\). Then if \(y \in Y\), we have \(y \ast y = 0\) and \(0 \in Y\) and \((y \ast y) \ast y = 0 \ast y = 0 = y \ast (y \ast y) = y \ast 0 = 0\), so that \(y = 0\) and \(|Y| = 1\), \(Y = \{0\}\). Hence, the condition \(|Y| \geq 2\) is impossible, i.e., \((X, \ast)\) is not a Smarandache-type \(d\)-algebra.

Given a \(d\)-algebra and abelian group, we can construct an algebra which is both a Smarandache-type \(d\)-algebra and a Smarandache-type abelian group. Let \((Y, \ast, 0)\) be a \(d\)-algebra and \((Z, +)\) be the additive abelian group of natural numbers. Let \((X, \Diamond)\) be defined for \(X = Y \times Z = \{(a, m) \mid a \in Y, m \in Z\}\) as follows: \((a, m) \Diamond (b, n) := (a \ast b, m + n)\). Then \((a, 0) \Diamond (b, 0) = (a \ast b, 0)\), and thus \((Y \times \{0\}, \Diamond, (0, 0))\) is a subalgebra of \((X, \Diamond)\) which is isomorphic to the \(d\)-algebra \((Y, \ast, 0)\). On the other hand, \((\{0\} \times Z, \Diamond)\) has a product \((0, m) \Diamond (0, n) = (0 \ast 0, m + n) = (0, m + n)\) so that \((\{0\} \times Z, \Diamond)\) is an example of an algebra which is a Smarandache-type abelian group. Hence \((X, \ast)\) is both a Smarandache-type \(d\)-algebra and Smarandache-type abelian group.

Given algebra types \((X, \ast)\) (type-\(P_1\)) and \((X, \circ)\) (type-\(P_2\)), we shall consider them to be Smarandache disjoint if the following two conditions hold:

(A) If \((X, \ast)\) is a type-\(P_1\)-algebra with \(|X| > 1\) then it cannot be a Smarandache-type-\(P_2\)-algebra \((X, \circ)\);
(B) If \((X, \circ)\) is a type-\(P_2\)-algebra with \(|X| > 1\) then it cannot be a Smarandache-type-\(P_1\)-algebra \((X, \ast)\).

This condition does not exclude the existence of algebras \((X, \circ)\) which are both Smarandache-type-\(P_1\)-algebras and Smarandache-type-\(P_2\)-algebras. In fact we have already produced an example of such an algebra where \(P_1 \equiv \text{‘semigroup’}\) and \(P_2 \equiv \text{‘d-algebra’}\).

If \((X, \ast, 0)\) is a \(d\)-algebra which also satisfy the following conditions:

(iv) \((x \ast (x \ast y)) \ast y = 0\) for all \(x, y \in X\);
(v) \(((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0\) for all \(x, y, z \in X\),

then it is a BCK-algebra (see [1, 3]). Since BCK-algebras are \(d\)-algebras it follows that:

**Theorem 4.** Semigroups and \(d\)-algebras are Smarandache disjoint.

**Corollary 5.** Semigroups and BCK-algebras are Smarandache disjoint.

**Corollary 6.** Groups and \(d\)-algebras are Smarandache disjoint.

Of course, since groups are semigroups we have:

**Corollary 7.** Groups and semigroups are not Smarandache disjoint.

Consider the collection of left semigroups, i.e., semigroups \((X, \ast)\) with an associative product \(x \ast y = x\). If \((Y, \ast)\) is a subgroup of \((X, \ast)\), then \(x \ast y = x\) means that \(y\) is the multiplicative identity of \((Y, \ast)\) and since \(y \in Y\) is arbitrary, it follows that \(|Y| = 1\) as well. Hence we show that:

**Theorem 8.** If \((X, \ast)\) is a left semigroup, then \((X, \ast)\) cannot be a Smarandache-type group.

If \((X, \ast, e)\) is a group and if \((Y, \ast)\) is a left semigroup, then being closed, we have for \(x, y \in Y, x \ast y = x\), whence from the group structure of \((X, \ast)\) we find \(y = e\), and since \(y \in Y\)
is arbitrary, $|Y| = 1$ as well. Thus $(X, \ast)$ cannot be a Smarandache-type left semigroup and hence we have shown that:

**Theorem 9.** Left semigroups and groups are Smarandache disjoint.

**Corollary 10.** If $(X, \ast)$ with $x \ast y = y$ is a right semigroup, then it follows that right semigroups and groups are Smarandache disjoint.

The notion of Smarandache disjointness illustrated here appears to be novel and of interest as well.

**Question.** Give an example of a special class of $d$-algebras which is Smarandache disjoint from the class of $BCK$-algebras.

**References**


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