PSEUDO-MTL ALGEBRAS AND PSEUDO-\(R_0\) ALGEBRAS

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Received July 8, 2003; revised June 27, 2004

Abstract. The relations between pseudo-MTL algebras and pseudo-\(R_0\) algebras are discussed. The main results are as follows: pseudo-IMTL algebras are equivalent to weak pseudo-\(R_0\) algebras; pseudo-NM algebras are equivalent to pseudo-\(R_0\) algebras.

1. Introduction

The notion of MTL algebras was introduced by Esteva and Godo in [3] as a generalization of BL algebras [6]. In [1,2,4], Georgescu et.al proposed the notion of pseudo-BL algebras as a noncommutative extension of BL algebras. Afterwards, Georgescu and Popescu [5] proposed the notion of pseudo-MTL algebras (called weak pseudo-BL algebras) as a noncommutative extension of MTL algebras. In [7], we generalized Georgescu’s ideas to \(R_0\) algebras and proposed the concept of pseudo-\(R_0\) algebras. In this paper, we discuss the relations between pseudo-MTL algebras and pseudo-\(R_0\) algebras. We prove that pseudo-IMTL algebras are equivalent to weak pseudo-\(R_0\) algebras, and that pseudo-NM algebras are equivalent to pseudo-\(R_0\) algebras.

Now let us recall the definition of MTL algebras (see [3]).

An MTL algebra is a structure \(L = (L, \lor, \land, \rightarrow, 0, 1)\) such that

(i) \((L, \lor, \land, 0, 1)\) is a bounded lattice,

(ii) \((L, \rightarrow, 1)\) is an abelian monoid, i.e. \(\rightarrow\) is commutative and associative and \(x \rightarrow 1 = 1\)

(iii) the following conditions hold for all \(x, y, z \in L\):

(1) \(x \rightarrow y \leq z\) if and only if \(x \leq y \rightarrow z\) (residuation),

(2) \((x \rightarrow y) \lor (y \rightarrow x) = 1\) (prelinearity).

An MTL algebra \(L\) is called an IMTL algebra, if the following condition holds:

(3) \((x \rightarrow 0) \rightarrow 0 = x\).

An IMTL algebra \(L\) is called a NM algebra, if the following condition holds:

(4) \((x \circ y \rightarrow 0) \lor (x \land y \rightarrow x \circ y) = 1\).

2. Pseudo-MTL algebras

Definition 2.1. (see [5]) A pseudo-MTL algebra is a structure \(L = (L, \lor, \land, \circ, \rightarrow, \neg, 0, 1)\) of type \(2, 2, 2, 2, 0, 0\), which satisfies the following axioms:

(C1) \((L, \lor, \land, 0, 1)\) is a bounded lattice,

(C2) \((L, \circ, 1)\) is a monoid, i.e. \(\circ\) is associative and \(x \circ 1 = 1 \circ x = x\),

(C3) \(x \circ y \leq z\) if and only if \(x \leq y \rightarrow z\) if and only if \(y \leq x \rightarrow z\),

(C4) \((x \rightarrow y) \lor (y \rightarrow x) = (x \neg y) \lor (y \neg x) = 1\).

The following example shows that pseudo-MTL algebras exist.

Example 2.2. Let \(L = \{0, a, b, c, 1\}\) and satisfy \(0 \leq a \leq b \leq c \leq 1\). We define \(x \land y = \min\{x, y\}\), \(x \lor y = \max\{x, y\}\), and define \(\circ, \rightarrow, \neg\) as follows:

\[
\begin{align*}
&x \land y = \min\{x, y\}, \\
&x \lor y = \max\{x, y\}, \\
&x \circ y = \min\{x, y\}, \\
&x \rightarrow y = \max\{0, x, y\}, \\
&x \neg y = \min\{0, x, y\}.
\end{align*}
\]

2000 Mathematics Subject Classification. 06D99, 03G25.
Key words and phrases. MTL algebra, pseudo-MTL algebra, pseudo-\(R_0\) algebra.
It is easily checked that \((L, \land, \lor, \circ, \leadsto, \sim, 0, 1)\) is a pseudo-MTL algebra.

**Lemma 2.3.** (see [5]) Let \(L\) be a pseudo-MTL algebra. The following properties hold:

1. \(x \leadsto x = x \sim x = 1\),
2. \(1 \leadsto x = 1 \sim x = x\),
3. \(x \circ y \leq x \land y, y \circ x \leq x \land y\),
4. \(x \leadsto (y \sim z) = y \leadsto (x \sim z)\),
5. \(x \circ (x \sim y) \leq x \land y, (x \sim y) \circ x \leq x \land y\),
6. If \(x \leq y\), then \(x \circ z \leq y \circ z, z \circ x \leq z \circ y\),
7. \(x \leq y\) if and only if \(x \leadsto y = 1\) if and only if \(x \sim y = 1\),
8. \(y \sim z \leq (x \leadsto y) \rightarrow (x \sim z), y \sim z \leq (x \sim y) \rightarrow (x \sim z)\),
9. \(y \sim z \leq (z \sim x) \rightarrow (y \leadsto x), y \sim z \leq (z \sim x) \rightarrow (y \leadsto x)\),
10. \((x \circ y) \sim z = y \leadsto (x \sim z), (x \circ y) \sim z = x \leadsto (y \sim z)\),
11. \((x \circ (y \lor z)) = (x \circ y) \lor (x \circ z), (y \lor z) \circ x = (y \circ x) \lor (z \circ x)\),
12. \((x \circ (y \land z)) = (x \circ y) \land (x \circ z), (y \land z) \circ x = (y \circ x) \land (z \circ x)\).

**Remark 2.4.** The identity \(x \land y = x \circ (x \sim y) = (x \sim y) \circ x\) does not hold in pseudo-MTL algebras. In fact, in Example 2.2, take \(x = b, y = a\), we have \(x \land y = b \land a = a\), but \((x \sim y) \circ x = (b \rightarrow a) \circ b = a \circ b = b\).

Let \(L\) be a pseudo-MTL algebra. Define

\[\sim x = x \rightarrow 0, \sim x = x \sim x = 0.\]

Obviously, \(\sim\) and \(\sim\) are unary operations on \(L\).

**Definition 2.5.** A pseudo-MTL algebra is called a pseudo-IMTL algebra, if it satisfies \((C5) x \sim \sim x = \sim x\).

**Definition 2.6.** A pseudo-IMTL algebra is called a pseudo-NM algebra, if it satisfies \((C6) (x \circ y \rightarrow 0) \lor (x \land y \rightarrow x \circ y) = 1, (x \circ y \sim 0) \lor (x \land y \sim x \circ y) = 1\).

**Lemma 2.7.** Let \(L\) be a pseudo-IMTL algebra. The following properties hold:

1. \(x \leadsto y \sim y \rightarrow 0, x \rightarrow y = \sim y \rightarrow \sim x\),
2. \(x \leq y\) if and only if \(\sim y \leq \sim x\) if and only if \(\sim y \leq \sim x\),
3. \(\sim (x \lor y) = \sim x \sim y, \sim (x \land y) = \sim x \land \sim y\),
4. \(\sim (x \land y) = \sim x \lor \sim y, \sim (x \lor y) = \sim x \lor \sim y\),
5. \(x \circ y = \sim y \rightarrow x, x \circ y = \sim y \rightarrow x\),
6. \(x \sim 0 = \sim (x \lor 0), x \rightarrow y = \sim (x \lor 0)\),
7. \(x \leadsto (y \lor z) = (x \leadsto y) \lor (x \leadsto z), x \rightarrow (y \land z) = (x \rightarrow y) \lor (x \rightarrow z)\).

**Proof.** (1) By Lemma 2.3 and \((C5)\), we have \(x \leadsto y \leq (y \sim 0) \rightarrow (x \sim 0) = y \rightarrow y \rightarrow x \leq (\sim x \sim 0) \land (\sim y \sim 0) = \sim x \land \sim y = x \sim y\), and so \(x \sim y = \sim y \rightarrow x\).

Similarly, \(x \rightarrow y = \sim y \rightarrow x\).

(2) From Lemma 2.3 and (1), it follows that \(x \leq y\) if and only if \(x \sim y = 1\) if and only if \(y \rightarrow x = 1\) and only if \(y \rightarrow y = 1\) if and only if \(y \rightarrow x = 1\). Similarly, \(x \leq y\) if and only if \(y \rightarrow y = 1\).

(3) Since \(x \leq y \rightarrow y\), we have \(\sim (x \lor y) \leq \sim x, \sim (x \land y) \leq \sim x\). Let \(t \leq x, t \leq y\), by (2) and \((C5)\), we have \(x \leadsto t \leq \sim t, y \leadsto t \leq \sim t, t \leq \sim x\), hence \(x \land y \leq \sim t\). Using (2) and \((C5)\) again, we have \(t \leadsto \sim t \leq (x \land y)\), thus \(\sim (x \lor y) = \sim x \land \sim y\). Similarly, \(\sim (x \lor y) = \sim x \land \sim y\).

(4) The proof is similar to (3).
Similarly, if and only if $y$.

(6) From (5) and (C5), it follows that $x \sim x \sim x$ if and only if $\neg(y \sim y) \sim \neg t = t$. Thus $x \sim y = (y \sim x)$. (7) By (6), (3), (4) and Lemma 2.3, we have $x \sim y = (y \sim z)$.

Remark 2.8. Lemma 2.7(5) shows that operator $\sim$ can be defined by operators $\neg, \sim$, in pseudo-$MTL$ algebras.

3. Main results Pseudo-$R_0$ algebras were introduced by the authors in [7] as a noncommutative extension of $R_0$ algebras [8]. In this section, we discuss the relations between pseudo-$MTL$ algebras and the pseudo-$R_0$ algebras.

Definition 3.1. (see [7]) A pseudo-$R_0$ algebra is a structure

$L = (L, \lor, \land, \rightarrow, \sim, \neg, 0, 1)$

of type $(2,2,2,1,1,0,0,0)$, which satisfies the following axioms, for all $x, y, z \in L$:

(i) $(L, \lor, \land, \rightarrow, 0, 1)$ is a bounded lattice,

(ii) The following conditions hold:

(PR1) $x \rightarrow y = y \rightarrow x$, $\neg x \rightarrow \neg y = y \rightarrow x$,

(PR2) $1 \rightarrow x = 1 \rightarrow x = x$,

(PR3) $(y \sim z) \lor ((x \sim x) \sim (x \sim z)) = (x \sim y) \sim (x \sim z)$,

(PR4) $y \sim (y \sim z)$ if and only if $x \rightarrow y = 1$ if and only if $x \sim y = 1$, $x \rightarrow y = z$ if and only if $y \leq x \sim z$,

(PR5) $x \rightarrow (y \sim z) = (x \rightarrow y) \sim (x \sim z)$,

(PR6) $x \sim (y \sim z) = (x \sim y) \sim (x \sim z)$,

(PR7) $\neg x = x \rightarrow 0, \neg x = x \rightarrow 0$.

If a bounded lattice $L$ satisfies (PR1)-(PR5) and (PR7), then $L$ is called a weak pseudo-$R_0$ algebra.

Lemma 3.2 (see [7]). Let $L$ be a pseudo-$R_0$ algebra. The following hold:

(i) $x = \sim x = \sim x$,

(ii) $x \rightarrow x = x \sim x = 1$,

(iii) $y \leq y$ if and only if $x \rightarrow y = 1$ if and only if $x \sim y = 1$,

(iv) $x \rightarrow y = z$ if and only if $y \leq x \sim z$,

(v) $x \rightarrow (y \sim y) = \sim x \land y$, $x \sim (y \sim y) = x \land y$,

(vi) $x \land y = \sim x \land y$, $x \land y = x \sim y$.

The following theorem is a characterization of a pseudo-$R_0$ algebra.

Theorem 3.3. Let $(L, \lor, \land, 0, 1)$ be a bounded lattice. $\neg$ and $\sim$ are two unary

operations on $L$, and $\rightarrow, \sim$ are two binary operations on $L$. Then $L$ is a pseudo-$R_0$ algebra if and only if $L$ satisfies (PR1), (PR4), (PR5), (PR6) and (PR8), where

(PR8) $x \leq y$ if and only if $x \rightarrow y = 1$ if and only if $x \rightarrow y = 1$.

Proof. Let $L$ be a bounded lattice and satisfy (PR1), (PR4)-(PR6) and (PR8). Since $x \leq x$, (PR8) we have $x \rightarrow x \rightarrow x = 1$. By (PR4), $x \rightarrow (1 \rightarrow x) = 1 \rightarrow (x \rightarrow x) = 1 \sim 1 = 1$, so by (PR8) $x \leq 1 \rightarrow x$. Conversely, from (PR4) it follows that $1 \sim (1 \rightarrow x) = (1 \sim x) \rightarrow (1 \sim x) = 1$. By (PR8) we have $(1 \rightarrow x) \rightarrow x = 1$ and $(1 \rightarrow x) \leq x$. Therefore $1 \sim x = x$. Similarly, $1 \rightarrow x = x$. Thus (PR2) holds. From (PR1) and (PR2), it
follows that \( x \sim 0 = \sim 0 \rightarrow x = 1 \rightarrow \sim x = x, x \rightarrow 0 = \sim 0 \sim \neg x = 1 \sim \neg x = \neg x. \) Thus (PR7) holds. Finally, from (PR5) we have if \( x \leq y, \) then \( z \sim x \leq z \sim y, z \rightarrow x \leq z \rightarrow y. \) On the other hand, from (PR4) and (PR8), it follows that \( x \leq y \sim z \) if and only if \( y \leq x \rightarrow z. \) Now let \( t \leq y \rightarrow z, \) then \( y \leq t \sim z \) and \( x \rightarrow y \leq x \rightarrow (t \sim z) = t \sim (x \rightarrow z). \) Hence \( t \leq (x \rightarrow y) \rightarrow (x \rightarrow z). \) This implies \( y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z). \) Therefore \( (x \rightarrow z) \lor ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (x \sim y) \rightarrow (x \sim z). \) Similarly, \( (y \sim z) \lor ((x \sim y) \sim (x \sim z)) = (x \sim y) \sim (x \sim z). \) Thus (PR3) holds. Consequently, \( L \) is a pseudo-\( R_0 \) algebra. The converse is obvious. 

**Corollary 3.4.** Let \( (L, \lor, \land, 0, 1) \) be a bounded lattice. \( \lor \) and \( \land \) are two unary operations on \( L, \) and \( - \) are two binary operations on \( L. \) Then \( (L, \lor, \land, \rightarrow, \sim, 0, 1) \) is a weak pseudo-\( R_0 \) algebra if and only if \( L \) satisfies (PR1), (PR4), (PR5) and (PR8).

From Lemmas 2.3, 2.7 and Corollary 3.4, we have the following theorem:

**Theorem 3.5.** Each pseudo-\( IMTL \) algebra is a weak pseudo-\( R_0 \) algebra.

**Corollary 3.6.** Each pseudo-\( NM \) algebra is a pseudo-\( R_0 \) algebra.

**Proof.** Let \( L \) be a pseudo-\( NM \) algebra. By Theorem 3.5, \( L \) is a weak pseudo-\( R_0 \) algebra. Now we only prove that \( L \) satisfies (PR6). By Lemma 2.7(6) we have \( x \sim y = \sim (y \sim x) \). Hence \( (x \rightarrow y) \lor ((x \rightarrow y) \sim (x \rightarrow y)) = (x \sim y) \lor ((x \sim y) \sim (x \sim y)) = (x \sim y) \lor ((x \sim y) \sim (x \sim y)) = (x \sim y) \lor ((x \sim y) \sim (x \sim y)) = 1. \) Similarly, \( (x \sim y) \lor ((x \sim y) \sim (x \sim y)) = (x \sim y) \lor ((x \sim y) \sim (x \sim y)) = 1. \) Thus (PR6) holds. By Theorem 3.3, \( L \) is a pseudo-\( R_0 \) algebra.

**Corollary 3.7.** Each \( IMTL \) algebra is a weak \( R_0 \) algebra. Further, each \( NM \) algebra is a \( R_0 \) algebra.

**Theorem 3.8.** Let \( L = (L, \lor, \land, \rightarrow, \sim, 0, 1) \) be a weak pseudo-\( R_0 \) algebra. For all \( x, y \in L, \) we define \( x \circ y = \sim (y \sim x). \)

Then \( L = (L, \land, \lor, \circ, \sim, 0, 1) \) is a pseudo-\( IMTL \) algebra.

**Proof.** It suffices to check the conditions (C1)-(C5) hold. From Definition 3.1 and Lemma 3.2, it follows that (C1) and (C5) hold.

(C2) By the definition of \( \circ, \) we have \( x \circ 1 = \sim (1 \sim x) = \sim x = x, 1 \circ x = \sim (x \sim 1) = \sim (x \sim 0) = \sim x = x. \) And \( x \circ y \circ z = \sim (z \sim (y \sim x)) = \sim (z \sim (y \sim x)) = \sim (z \sim (y \sim x)) = \sim (z \sim (y \sim x)) = \sim (y \sim x) = x \circ (y \sim z). \) This shows that \( \circ \) is associative. Hence (C2) holds.

(C3) If \( x \circ y \leq z, \) by Lemma 3.2(3) we have \( (x \circ y) \sim z = 1, \) i.e., \( (x \sim y) \sim z \sim x = 1. \) Using (PR1) we have \( \sim (y \sim x) \sim y \sim x = \sim y \sim x = y \sim x, \) and so \( z \sim y \sim x = z \sim y \sim x. \) This implies that \( y \leq x \sim z. \) Conversely, if \( y \leq x \sim z, \) then \( y \leq x \sim z \sim x, \) thus \( y \sim (z \sim x) = 1. \) By (PR1) and (PR4) we have \( x \circ y \sim z = \sim (y \sim x) \sim z \sim y \sim x = y \sim x \sim y = 1. \) Hence \( x \circ y \leq z. \) On the other hand, \( y \leq x \sim z \) if and only if \( y \rightarrow (x \sim z) = 1 \) and only if \( x \sim (y \rightarrow z) = y \rightarrow (x \sim z) = 1 \) if and only if \( x \leq y \rightarrow z. \) Thus (C3) holds.

(C4) Since \( x \leq x \lor y, \) then \( x = (x \lor y) = \sim (x \lor y) \sim x = x. \) By (PR3) we have \( \sim (y \sim x) \sim (y \sim x) = (y \sim x) \sim (y \sim x) = 1. \) Therefore \( x \lor y \sim y \leq x \rightarrow y. \) Similarly, \( x \lor y \leq x \rightarrow y \rightarrow x. \) Thus \( (x \lor y) \sim (y \sim x) \leq x \rightarrow y \rightarrow x. \) Since \( 1 = (x \lor y) \sim (y \sim x) = (x \lor y) \sim (y \sim x) = 1. \) Similarly, \( (x \sim y) \sim (y \sim x) = 1. \) By Definition 2.5, \( L \) is a pseudo-\( IMTL \) algebra. □
Corollary 3.9. Let $L = (L, ∨, ∧, →, ∼, −, 0, 1)$ be a pseudo-$R_0$ algebra. For all $x, y \in L$, we define

$$x \circ y = (y \sim \sim x).$$

Then $L = (L, ∧, ∨, ⊗, →, ¬, ∼, 0, 1)$ is a pseudo-$NM$ algebra.

**Proof.** From Theorem 3.8, it follows that $L$ is a pseudo-$IMTL$ algebra. Now we only prove that $L$ satisfies (C6). Firstly, we prove

$$x \circ y = (x \rightarrow ¬y).$$

Indeed, from Theorem 3.8, it follows that $x \circ y \leq t$ if and only if $x \leq y \rightarrow t = t \sim ¬y$ if and only if $¬t \leq x \rightarrow ¬y$ if and only if $(x \rightarrow ¬y) \leq ¬t = t$. Consequently, $x \circ y = (x \rightarrow ¬y).

(C6) Since $x \circ y = (x \rightarrow ¬y)$, we have $(x \circ y) \rightarrow 0 = x \rightarrow ¬y$. Then $((x \circ y) \rightarrow 0) \lor ((x \land y) \rightarrow (x \circ y)) = (x \rightarrow ¬y) \lor ((x \circ y) \sim ¬(x \land y)) = (x \rightarrow ¬y) \lor ((x \land ¬y) \sim ¬(x \lor ¬y))$. By (PR6) we have $(x \rightarrow ¬y) \lor ((x \land ¬y) \sim ¬(x \lor ¬y)) = 1$. Thus $((x \circ y) \rightarrow 0) \lor ((x \land y) \rightarrow (x \circ y)) = 1$. On the other hand, since $x \circ y = (y \sim \sim x)$, then $(x \circ y) \sim 0 = y \sim \sim x$. Hence $((x \circ y) \sim 0) \lor ((x \land y) \sim (x \circ y)) = (y \sim \sim x) \lor (x \lor ¬y) \sim ¬(x \lor ¬y)) = (y \sim \sim x) \lor (y \sim \sim x) \rightarrow ¬(x \lor ¬y)) = 1$. By Definition 2.6, $L$ is a pseudo-$NM$ algebra.

**Corollary 3.10.** Each weak $R_0$ algebra is an $IMTL$ algebra. Further, each $R_0$ algebra is a NM algebra.

**Remark 3.11.** Theorems 3.5 and 3.8 show that pseudo-$IMTL$ algebras are equivalent to weak pseudo-$R_0$ algebras. Corollaries 3.6 and 3.9 show that pseudo-$NM$ algebras are equivalent to pseudo-$R_0$ algebras.

**Acknowledgement.** We thank G. Georgescu, A.Iorgulescu and A.Popescu for sending us their papers.

**References**


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