SOLID VARIETIES, TRANSITION SEMIGROUPS AND UNARY CLONES

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Abstract. There is a well-known connection between hyperidentities of an algebra and identities satisfied by the clone of the algebra. The clone of an algebra is a heterogeneous algebra, and the correspondence between hyperidentities and clone identities is rather complicated to work with. It is also of interest to study this correspondence in a restricted setting, that of hyperidentities of unary algebras (of arbitrary unary type) and identities in the unary clone of unary term operations. This unary clone is just a monoid, usually called the transition monoid of the unary algebra. This correspondence is an important one in automata theory, since any finite unary algebra \( A \) can be regarded as an automaton: the set \( A \) is regarded as a set of states, with one state chosen as an initial state and a subset of \( A \) chosen as the set of final states, and the operations of the algebra regarded as inputs. Then terms and identities over the unary algebra \( A \) correspond to monoid words from the free monoid generated by the operations of \( A \). In this paper we study the correspondence between hyperidentities and clone identities in this special case of unary algebras and transition monoids. We also look at generalizations of this approach to algebras of type \((n, n, \ldots)\), and the corresponding \(n\)-clones, which correspond to tree automata of type \((n, n, \ldots)\).

1 Preliminaries There is a well-known connection, described by Taylor in [Tay;81], between hyperidentities of an algebra and identities satisfied by the clone of the algebra. The clone of an algebra is a heterogeneous algebra, and the correspondence between hyperidentities and clone identities is rather complicated to work with. In this paper we study this correspondence in a restricted setting, that of hyperidentities of unary algebras (of arbitrary unary type) and identities in the unary clone of unary term operations. This unary clone is just a monoid, usually called the transition monoid of the unary algebra. Since unary algebras correspond to finite automata, this correspondence is an important one in automata theory. In 1965 Schützenberger ([Sch;65]) found an interesting connection between star-free languages and aperiodic monoids. The precise formulation of the correspondence between certain sets of formal languages and sets of finite semigroups is due to Eilenberg (see [Eil;74]) and uses the concept of a pseudovariety. The connection between finite deterministic automata and regular languages is given by Kleene’s famous theorem ([Kle;50]). In this paper we consider the third side, the connection between finite deterministic automata and their transition monoids. This connection allows for instance to apply the decomposition methods of finite semigroups using a wreath product in which the factors are, alternately, finite groups and finite aperiodic semigroups in automata theory ([Kro-R;65]). If \( s \approx t \) is an identity in a finite unary algebra, then the word \( t^A \) is recognized by the automaton corresponding to the unary algebra iff the word \( s^A \) is recognized.

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is even an identity in the transition monoid, and as we will prove, a hyperidentity in the algebra $A$, then all words which arise from $s^A$ by replacing the input letters by arbitrary words are recognized iff the same holds for $t^A$. Therefore one may expect more applications of our results in Automata Theory. We also look at generalizations of this approach to algebras of type $(n, n, \ldots)$, and the corresponding $n$-clones, which correspond to tree automata of type $(n, n, \ldots)$.

We begin by defining a type $\tau$ of algebras to be unary if all the operation symbols of the type have arity 1. We will assume throughout that $\tau_0$ is such a unary type (possibly with an infinite number of operation symbols). We let $(f_i)_{i \in I}$ be an indexed set of unary operation symbols, corresponding to the type $\tau_0$. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of individual variables, and for each $n \geq 1$ let $X_n = \{x_1, \ldots, x_n\}$. We shall denote by $W_{\tau_0}(X_n)$ the set of all $n$-ary terms of type $\tau_0$, and by $W_{\tau_0}(X) = \cup_{n \geq 1} W_{\tau_0}(X_n)$ the set of all (finitary) terms of type $\tau_0$. This set is the universe of the algebra $F_{\tau_0}(X) := (W_{\tau_0}(X); (f_i)_{i \in I})$, with operations defined by $f_i(t) := f_i(t)$, for $t \in W_{\tau_0}(X)$; the algebra $F_{\tau_0}(X)$ is well known to be the absolutely free algebra of type $\tau_0$ on the set $X$. In the special case $X_1 = \{x_1\}$, the elements of $W_{\tau_0}(\{x_1\})$ are called unary terms. An identity $s \approx t$ of terms of unary type, (or more formally the pair of terms $(s, t)$), contains at most two different variables. A unary identity containing only one variable is called a regular identity. A unary algebra is a pair $A = (A; (f_i^A)_{i \in I})$ consisting of a set $A$ and a set of operations defined on $A$ which are all unary. Thus a unary algebra is an algebra of type $\tau_0$ for some unary type $\tau_0$. Let $A$ be a unary algebra. Then every unary term $t \in W_{\tau_0}(\{x_1\})$ induces a unary term operation $t^A$ on $A$, which is inductively defined by the following steps:

\begin{enumerate}
  \item The variable $x_1 \in X_1$ induces the identity mapping $id_A : A \to A$, that is, $x_1^A := id_A$.
  \item If $t^A$ is the unary term operation induced by the unary term $t'$ and if $t = f_i(t')$ is a compound term, then $t^A := f_i^A \circ t^A$, where $\circ$ denotes the composition of unary operations defined on $A$.
\end{enumerate}

We will use $T^{(1)}(A)$ for the set of all unary term operations of the unary algebra $A$. This set is the universe of a monoid generated by $\{f_i^A \mid i \in I\} \cup \{id_A\}$, which is called the transformation or transition monoid of $A$ or the unary clone of $A$. We shall also use the name $T^{(1)}(A)$ for this monoid, so

$T^{(1)}(A) = (\{f_i^A \mid i \in I\} \cup \{id_A\}; \circ, id_A)$.

The corresponding semigroup $S(A)$ generated by $\{f_i^A \mid i \in I\}$ is called the transition semigroup of $A$. Unary algebras are sometimes called $X$-algebras (see e.g. [Pet-C-B:02]).

2 Hypersubstitutions and Substitutions To consider identities in unary clones or transition semigroups, we need to build up a language of type $(2,0)$ or $(2)$, respectively, consisting of monoid or semigroup words. Let $\{F_i \mid i \in I\}$ be a new set of variables, also indexed by $I$. Then we denote by $F_S(\{F_i \mid i \in I\})$ the free semigroup generated by $\{F_i \mid i \in I\}$, and similarly we denote by $F_M(\{F_i \mid i \in I\})$ the free monoid generated by $\{F_i \mid i \in I\} \cup \{\lambda\}$, where $\lambda$ is the empty word. We shall denote the binary operation in both the free semigroup and monoid by $\circ$. The basis of our work will be the following definition of a mapping $\varphi$, which will effect the translation between unary terms of our type $\tau_0$ and the monoid words on our new meta-level alphabet $\{F_i \mid i \in I\}$. Specifically, we want to define a mapping $\varphi$ between the set $W_{\tau_0}(\{x_1\})$ of unary terms of type $\tau_0$ and the monoid generated by $\{F_i \mid i \in I\} \cup \{\lambda\}$. 

Definition 2.1 The mapping \( \varphi: W_\tau(\{x_1\}) \rightarrow \langle \{F_i \mid i \in I\} \cup \{\lambda\} \rangle \) is inductively defined as follows:

(i) \( \varphi(x_1) := \lambda, \)

(ii) \( \varphi(f_i(t)) := F_i \circ \varphi(t). \)

On both the set \( W_\tau(\{x_1\}) \) of unary terms and the monoid \( F_M(\{F_i \mid i \in I\}) \) we are using the operation of superposition of terms. The superposition of unary terms is defined inductively by \( x_1(t) := t \), and if \( t' = f_i(t'') \), then \( t'(t) := f_i(t''(t)) \). It is sometimes convenient to make this explicit with a new operation symbol \( S_1^i \) for the superposition of unary terms; in this case, we have \( S_1^i(x_1, t) := t \) and \( S_1^i(f_i(t''), t) := f_i(S_1^i(t'', t)) \). This approach will be used again in Section 6. This operation \( S_1^i \) is a binary operation, and is clearly associative; the clone axioms for the unary clone \( T^{(1)}(A) \) make this algebra a monoid. The binary composition \( \circ \) in \( F_M(\{F_i \mid i \in I\}) \) is also an instance of the general superposition operation \( S_1^1 \).

In the remainder of this section we examine the connection between certain special identities of unary algebras and identities in their unary clones. Hyperidentities are identities on the complexity (the number of occurrences of operation symbols) of \( \varphi \).

Proposition 2.2 For any two terms \( t', t'' \in W_\tau(\{x_1\}) \) we have

\[
\varphi(t'(t'')) = \varphi(t') \circ \varphi(t'').
\]

Proof. We give a proof by induction on the complexity of the term \( t' \). If \( t' = x_1 \) then \( \varphi(x_1(t'')) = \varphi(t'') = \lambda \circ \varphi(t'') = \varphi(x_1) \circ \varphi(t'') \). If \( t' = f_i(t) \) and if we assume inductively that \( \varphi(t(t'')) = \varphi(t) \circ \varphi(t'') \), then we have \( \varphi(f_i(t)(t'')) = \varphi(f_i(t(t''))) = F_i \circ \varphi(t(t'')) = (F_i \circ \varphi(t)) \circ \varphi(t'') = \varphi(f_i(t)) \circ \varphi(t''). \)

Proposition 2.3 The mapping \( \varphi: W_\tau(\{x_1\}) \rightarrow \langle \{F_i \mid i \in I\} \cup \{\lambda\} \rangle \) is bijective.

Proof. By definition, \( \varphi \) maps terms in \( W_\tau(\{x_1\}) \) to words in \( F_M(\{F_i \mid i \in I\}) \), and it is clearly surjective. Assuming that \( \varphi(t_1) = \varphi(t_2) \), we prove the injectivity of \( \varphi \) by induction on the complexity (the number of occurrences of operation symbols) of \( t_1 \). If \( t_1 \) is the variable \( x_1 \in X_1 \), then \( \varphi(x_1) = \lambda = \varphi(x_1) \). Since \( x_1 \) is the only unary term of type \( \tau_0 \) which is mapped to \( \lambda \), we then have \( t_1 = t_2 = x_1 \). Now suppose that \( \varphi(t_1) = \varphi(t_2) \Rightarrow t_1 = t_2 \), and consider the term \( t_1' = f_i(t_1) \). We may assume that \( t_1' \) is also not a variable, since otherwise \( \varphi(t_1') = \lambda = \varphi(t_1') \) and \( t_1' = x_1 \). Hence there is a unary operation symbol \( f_j \) and a unary term \( t_2' \) such that \( f_j(t_2') = f_i(t_1') \). Then we have \( \varphi(t_2') = \varphi(f_j(t_2')) = F_j \circ \varphi(t_2') \), while \( \varphi(t_1') = \varphi(f_i(t_1)) = F_i \circ \varphi(t_1) \). Thus

\[
\varphi(t_1') = \varphi(t_2') \Rightarrow F_i \circ \varphi(t_1) = F_j \circ \varphi(t_2) \Rightarrow F_i = F_j \text{ and } \varphi(t_1) = \varphi(t_2) \Rightarrow f_i = f_j \text{ and } t_1 = t_2 \Rightarrow t_1' = t_2'.
\]

using the fact that \( F_M(\{F_i \mid i \in I\}) \) is free.
Definition 2.4 A hypersubstitution of unary type is a mapping
\[ \sigma : \{ f_i \mid i \in I \} \longrightarrow W_{\tau_0}(\{ x_1 \}). \]
Any hypersubstitution \( \sigma \) of unary type induces a mapping \( \hat{\sigma} \) defined on the set of all unary terms of the type, as follows.

Definition 2.5 Let \( \sigma \) be a hypersubstitution of unary type. Then \( \sigma \) induces a mapping \( \hat{\sigma} : W_{\tau_0}(\{ x_1 \}) \longrightarrow W_{\tau_0}(\{ x_1 \}) \), by setting
\[ \begin{align*}
(i) \quad & \hat{\sigma}[x_1] := x_1, \\
(ii) \quad & \hat{\sigma}[f_i(t)] := \sigma(f_i)(\hat{\sigma}[t]) \quad (= S_1^1(\sigma(f_i), \hat{\sigma}[t])).
\end{align*} \]
Let \( Hyp(\tau_0) \) be the set of all hypersubstitutions of the unary type \( \tau_0 \). Then a binary operation \( \circ_h \) on \( Hyp(\tau_0) \) can be defined, by setting \( \sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \hat{\sigma}_2 \). We also have an identity hypersubstitution \( \sigma_{id} \) defined by \( \sigma_{id}(f_i) = f_i(x_1) \). Altogether this gives a monoid \( (Hyp(\tau_0); \circ_h, \sigma_{id}) \). We shall show that hypersubstitutions of unary type are closely related to substitutions on the monoid \( F_M(\{ F_i \mid i \in I \}) \), that is, substitutions of \( \langle \{ F_i \mid i \in I \} \cup \{ \lambda \} \rangle \). For the analogous semigroup case, when we do not use the empty word \( \lambda \), we need the concept of a pre-hypersubstitution. A hypersubstitution \( \sigma \) of unary type is called a pre-hypersubstitution if for every \( i \in I \) the term \( \sigma(f_i) \) is not a variable. We denote by \( Pre(\tau_0) \) the set of all pre-hypersubstitutions of type \( \tau_0 \). It is easy to see that this set is closed under the composition \( \circ_h \), and that \( Pre(\tau_0) \) is a submonoid of \( Hyp(\tau_0) \).

Any mapping \( \eta : \{ F_i \mid i \in I \} \longrightarrow \langle \{ F_i \mid i \in I \} \cup \{ \lambda \} \rangle \) is called a substitution. The extension \( \overline{\eta} \) of the substitution \( \eta \) is an endomorphism of the free monoid \( F_M(\{ F_i \mid i \in I \} \cup \{ \lambda \}) \) and is uniquely determined by \( \eta \). Now we consider the set \( Subst_M(2) \) of all such substitutions \( \eta \). We can define a composition \( \circ \) on this set, by defining \( \eta_1 \circ \eta_2 := \overline{\eta}_1 \circ \overline{\eta}_2 \), and we have an identity for this operation which is the identity mapping \( id \) on \( F_M(\{ F_i \mid i \in I \} \cup \{ \lambda \}) \). Thus here too we obtain a monoid \( (Subst_M(2); \circ, id) \). Since the sets \( \{ f_i \mid i \in I \} \) of operation symbols of type \( \tau_0 \) and \( \{ F_i \mid i \in I \} \) of new variable symbols are both indexed by \( I \), there is a bijection \( \pi \) between them, with \( \pi(f_i) = F_i \) for \( i \in I \). Using \( \pi \) we define a mapping \( \overline{\pi} : W_{\tau_0}(\{ x_1 \}) \longrightarrow \langle \{ F_i \mid i \in I \} \cup \{ \lambda \} \rangle \), by setting
\[ \begin{align*}
(i) \quad & \overline{\pi}(x_1) = \lambda \\
(ii) \quad & \overline{\pi}(f_i(t)) = \pi(f_i) \circ \overline{\pi}(t).
\end{align*} \]
Using the inverse mapping \( \pi^{-1} \) of the bijection \( \pi \) we define a mapping \( \overline{\pi}^{-1} \) by
\[ \begin{align*}
(i) \quad & \overline{\pi}^{-1}(\lambda) = x_1, \\
(ii) \quad & \overline{\pi}^{-1}(w_1 \circ w_2) = \overline{\pi}^{-1}(w_1)(\overline{\pi}^{-1}(w_2)(x_1)).
\end{align*} \]
Then we have
Lemma 2.6 \( \overline{\pi} = \varphi \) and \( \overline{\pi}^{-1} = \varphi^{-1} \)

Proof. We give a proof by induction on the term complexity. It is clear that for the variable \( x_1 \) we have \( \overline{\pi}(x_1) = \lambda = \varphi(x_1) \). Assume inductively that \( t = f_i(t') \) and that \( \overline{\pi}(t') = \varphi(t') \). Then \( \overline{\pi}(f_i(t')) = \pi(f_i) \circ \overline{\pi}(t') = \pi(f_i) \circ \varphi(t') = \varphi(f_i(t')) \). For \( \overline{\pi}^{-1} \) we have \( \overline{\pi}^{-1}(\lambda) = x_1 = \varphi^{-1}(\lambda) \) and \( \overline{\pi}^{-1}(w_1 \circ w_2) = \overline{\pi}^{-1}(w_1)(\overline{\pi}^{-1}(w_2)(x_1)) = \varphi^{-1}(w_1)(\varphi^{-1}(w_2)(x)) = \varphi^{-1}(w_1 \circ w_2) \).
The diagram (D) below shows that for every hypersubstitution $\sigma$ of unary type, the mapping $\eta := \varphi \circ \sigma \circ \pi^{-1}$ is a substitution.

**Proposition 2.7** If $\eta$ is a substitution of the form $\varphi \circ \sigma \circ \pi^{-1}$, then the following are satisfied:

1. $\overline{\eta} = \varphi \circ \sigma \circ \varphi^{-1}$,
2. $\varphi \circ (\sigma_1 \circ_{h} \sigma_2) \circ \pi^{-1} = (\varphi \circ \sigma_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_2 \circ \pi^{-1})$.

**Proof.**

(i) For the base case we have $\overline{\eta}(F_i) = \eta(F_i) = (\varphi \circ \sigma \circ \pi^{-1})(F_i) = \varphi(\sigma(f_i)) = \varphi(\sigma(1_{F_i}(x_1))) = \varphi(\sigma(\varphi^{-1}(F_i))) = (\varphi \circ \sigma \circ \varphi^{-1})(F_i)$. Since the extension of $\eta$ is uniquely determined, we get (i). (ii) This follows from $\varphi \circ (\sigma_1 \circ_{h} \sigma_2) \circ \pi^{-1} = (\varphi \circ \sigma_1 \circ \sigma_2 \circ \pi^{-1}) = (\varphi \circ \sigma_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_2 \circ \pi^{-1})$.

Now we can establish our isomorphism.

**Theorem 2.8** The monoid $(\text{Hyp}(\tau_0); \circ_{h}, \sigma, id)$ is isomorphic to $(\text{Subst}_M(2); \circ, id)$.

**Proof.** Consider the mapping $\psi : \text{Hyp}(\tau_0) \rightarrow \text{Subst}_M(2)$ defined by $\psi(\sigma) := \varphi \circ \sigma \circ \pi^{-1}$. Clearly $\psi$ is well-defined. To see that $\psi$ is one-to-one, suppose that $\psi(\sigma_1) = \psi(\sigma_2)$. Then $\varphi \circ \sigma_1 \circ \pi^{-1} = \varphi \circ \sigma_2 \circ \pi^{-1}$ and since $\varphi$ and $\pi^{-1}$ are both bijections, we get $\sigma_1 = \sigma_2$. For surjectivity of $\psi$, let $\eta \in \text{Subst}_M(2)$. Then the mapping $\varphi^{-1} \circ \eta \circ \pi$ from $\{f_i \mid i \in I\}$ to $W_{\tau_0}(\{x_1\})$ is a hypersubstitution of unary type, and applying $\psi$ gives $\psi(\varphi^{-1} \circ \eta \circ \pi) = \varphi(\varphi^{-1} \circ \eta \circ \pi) = \varphi(\varphi^{-1} \circ \eta \circ \pi) = \varphi(\varphi^{-1} \circ \eta \circ \pi) = \varphi(\varphi^{-1} \circ \eta \circ \pi) = \varphi(\varphi^{-1} \circ \eta \circ \pi)$. Finally, $\psi$ is a homomorphism, since $\psi(\sigma_1 \circ_{h} \sigma_2) = \varphi(\sigma_1 \circ_{h} \sigma_2) \circ \pi^{-1} = (\varphi \circ \sigma_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_2 \circ \pi^{-1}) = (\varphi \circ \sigma_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_2 \circ \pi^{-1}) = (\varphi \circ \sigma_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_2 \circ \pi^{-1})$, by Proposition 2.7 and the definition of the operation $\circ$ on $\text{Subst}_M(2)$.

We noted above that the free monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$ is isomorphic to the unary clone $\text{Clone}^{(1)}(\tau_0)$ of the type $\tau_0$. If in diagram (D) above we replace the monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$ by $\text{Clone}^{(1)}(\tau_0)$ and the set $\{F_i \mid i \in I\}$ by $\{f_i \mid i \in I\}$, then instead of $\text{Subst}_M(2)$ we can consider the set of all clone substitutions. Then Theorem 2.8 can be interpreted as asserting the existence of a one-to-one mapping between sets of hypersubstitutions and...
clone substitutions. A more general version of this interconnection, using all the clone of terms for arbitrary types, can be found in [Den-G;96] (Corollary 3.5); see also [Den-W;00]. The results of this section can be modified to the semigroup case, excluding the variable $x_1$ and the empty word $\lambda$, by using pre-hypersubstitutions. These are unary hypersubstitutions $\sigma$ which map all the operation symbols $f_i$ to terms other than the variable $x_1$. In this case we consider the restriction $\varphi_S$ of the mapping $\varphi$ which maps $W_{\tau_0}(\{x_1\}) \setminus \{x_1\}$ to the free semigroup $F_S(\{F_i \mid i \in I\})$ generated by the variables $\{F_i \mid i \in I\}$. Then instead of monoid substitutions we consider semigroup substitutions $\eta_S : \{F_i \mid i \in I\} \longrightarrow (\{F_i \mid i \in I\}$ and the monoid $(\text{Subst}_S; \circ, \text{id})$. The mappings of $\pi_S$ and $\pi_S^{-1}$ are defined by the equations $\pi_S(f_i(t)) = \pi_S(f_i) \circ \pi_S(t)$ and $\pi_S^{-1}(w_1 \circ w_2) = \pi_S^{-1}(w_1) (\pi_S^{-1}(w_2)(x_1))$. Then by a proof similar to that of Theorem 2.8, we have the following analogous result.

**Theorem 2.9** The monoid $(\text{Pre}(\tau_0); \circ_h, \sigma_id)$ is isomorphic to $(\text{Subst}_S(2); \circ, \text{id})$.

We can use the isomorphism of Theorem 2.8 in the following way. Sets of substitutions with certain properties which are preserved by composition of substitutions will correspond to submonoids of hypersubstitutions. For instance, in Section 4 we will consider the following two monoids of hypersubstitutions:

$$
B := \{\sigma \mid \sigma \in Hyp(\tau_0) \text{ and } \varphi \circ \hat{\sigma} \circ \varphi^{-1} \text{ is bijective}\}
$$

and

$$
C := \{\sigma \mid \sigma \in Hyp(\tau_0) \text{ and } \varphi \circ \hat{\sigma} \circ \varphi^{-1} \text{ is surjective}\}.
$$

### 3 Identities and Hyperidentities

Let $s$ and $t$ be unary terms of type $\tau_0$, and let $A$ be a unary algebra of type $\tau_0$. The identity $s \approx t$ is satisfied in $A$ if the term operations induced by $s$ and $t$ are equal, that is, if $s^A = t^A$. An equivalent condition that we shall frequently use is that an identity $s \approx t$ is satisfied in $A$ if for every valuation mapping $v : X \to A$ we have $\overline{\tau}(s) = \overline{\tau}(t)$ where $\overline{\tau}$ is the uniquely determined extension of $v$. We shall denote by $IdA$ the set of all identities satisfied in the algebra $A$. If $V$ is a variety of unary type $\tau_0$, then $IdV$ denotes the set of all identities which are satisfied in all algebras of $V$. An identity $s \approx t$ is called a hyperidentity of a variety $V$ if for every $\sigma \in Hyp(\tau_0)$ the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is in $IdV$. The identity is called a pre-hyperidentity of $V$ if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is in $IdV$ for every pre-hypersubstitution $\sigma$. In the first case we write $V \models s \approx t$ and in the second case $V \models_{\text{pre-hyp}} s \approx t$. If $w_1$ and $w_2$ are monoid words from $F_M(\{F_i \mid i \in I\})$, then the pair $w_1 \approx w_2$ is an identity in the monoid $T^{(1)}(A)$ if $w_1^{T^{(1)}(A)} = w_2^{T^{(1)}(A)}$. Analogously, in the semigroup case where $w_1$ and $w_2$ are semigroup words from $F_S(\{F_i \mid i \in I\})$, then $w_1 \approx w_2$ is an identity in the semigroup $S(A)$ when $w_1^{S(A)} = w_2^{S(A)}$. We need one more mapping $h$ which assigns to each variable $F_i$ the fundamental operation $f_i^A$ of our algebra $A$. The mapping $h$ can be uniquely extended to a homomorphism $\overline{h} : \mathcal{F}_M(\{F_i \mid i \in I\}) \to (T^{(1)}(A), \circ, id_A)$ since $\mathcal{F}_M(\{F_i \mid i \in I\})$ is free. For $S(A)$ and $\mathcal{F}_S(\{F_i \mid i \in I\})$ we denote the corresponding mapping by $hS$. We use our isomorphism $\varphi$ from the previous section and the mapping $h$ to relate hyperidentities on an algebra $A$ to clone identities on the unary clones of the algebra $A$.

**Lemma 3.1** For every term $s \in W_{\tau_0}(\{x_1\})$ and for every algebra $A$ of type $\tau_0$, we have $s^A = h(\varphi(s))$; and for every term $t \in W_{\tau_0}(\{x_1\}) \setminus \{x_1\}$ we have $t^A = hS(\varphi_S(t))$.

**Proof.** We will give a proof by induction on the complexity of the term $s$. If $s = x_1$, then $x_1^A = id_A = h(\lambda) = h(\varphi(x_1))$. Now let $s = f_i(s_1)$ be a compound term, with $s_1^A =
Lemma 3.1 will allow us to compare identities \( s \approx t \) (of the right form) on an algebra \( A \) with their images \( \varphi(s) \approx \varphi(t) \) on the unary clone of \( A \). In order to apply \( \varphi \) to terms \( s \) and \( t \), and hence to identities, the terms must be terms on the alphabet \( X_1 \) only. For unary type, any term contains at most one variable, and hence any identity contains at most two variables. We cannot apply \( \varphi \) to identities such as \( f_1(f_2(x_1)) \approx f_2(f_1(x_2)) \) which contain two different variables, and hence we restrict ourselves to identities which contain only the single variable \( x_1 \). Such identities are regular, since they have the same variable on each side. Let \( s \approx t \in IdV \) be a regular identity of a variety \( V \), so that for every algebra \( A \in V \) we have \( s^A = t^A \). Then by Lemma 3.1 the elements \( \widehat{h}(\varphi(s)) \) and \( \widehat{h}(\varphi(t)) \) agree on \( T^{(1)}(A) \) for every \( A \). This means that a set \( \Sigma \) of identities satisfied in an algebra \( A \) of type \( \tau_0 \) corresponds to a set of equations between the generating elements \( \{ f_i^A \mid i \in I \} \) of the unary clone \( T^{(1)}(A) \). Such sets of equations are also called relations of the algebra. For more information on presentations of an algebra by generators and relations the reader is referred to [Bur-S;81] in the arbitrary case and to [Cli-P;67] for semigroups.

Now let \( A \) be an algebra of type \( \tau_0 \). On the free monoid \( F_M(\{ F_i \mid i \in I \}) \) we define the so-called Myhill congruence \( \mu_A \) of \( A \) ([Pet-C-B;02]) by

\[
(\pi_1, \pi_2) \in \mu_A \iff \varphi^{-1}(\pi_1) \approx \varphi^{-1}(\pi_2) \in IdA.
\]

For \( V \) a variety, we define the congruence \( \mu_V \) to be the intersection of all the congruences \( \mu_{A} \) for all \( A \in V \). Analogously we define \( \mu_A^* \) on \( F_S(\{ F_i \mid i \in I \}) \) by

\[
(\pi_1, \pi_2) \in \mu_A^* \iff \varphi_S^{-1}(\pi_1) \approx \varphi_S^{-1}(\pi_2) \in IdA.
\]

It is easy to see that \( \mu_A \) is a congruence and therefore we may form the quotient monoid \( F_M(\{ F_i \mid i \in I \})/\mu_A \) for a variety \( F_M(\{ F_i \mid i \in I \})/\mu_V \). In a corresponding way we may consider \( F_S(\{ F_i \mid i \in I \})/\mu_A^* \) and \( F_S(\{ F_i \mid i \in I \})/\mu_V^* \), respectively. Then we have

**Proposition 3.2** The monoid \( (T^{(1)}(A), \circ, \text{id}_A) \) is isomorphic to the quotient monoid \( F_M(\{ F_i \mid i \in I \})/\mu_A \), and in the semigroup case \( S(A) \) is isomorphic to \( F_S(\{ F_i \mid i \in I \})/\mu_A^* \).

**Proof.** We show first that \( \mu_A \) is the kernel of \( \widehat{h} \). We have

\[
(\pi_1, \pi_2) \in \mu_A \iff \varphi^{-1}(\pi_1) \approx \varphi^{-1}(\pi_2) \in IdA
\]

\[
\iff \varphi^{-1}(\pi_1)A = \varphi^{-1}(\pi_2)A
\]

\[
\iff \widehat{h}(\varphi^{-1}(\pi_1)) = \widehat{h}(\varphi^{-1}(\pi_2)) \quad \text{by Lemma 3.1 and the bijectiv}-
\]

\[
\iff \widehat{h}(\pi_1) = \widehat{h}(\pi_2)
\]

\[
\iff (\pi_1, \pi_2) \in \ker \widehat{h},
\]

ity of \( \varphi \). Since \( \widehat{h} \) is a surjective homomorphism from \( F_M(\{ F_i \mid i \in I \}) \) onto \( T^{(1)}(A) \), by the homomorphism theorem we have \( T^{(1)}(A) \cong F_M(\{ F_i \mid i \in I \})/\ker \widehat{h} \cong F_M(\{ F_i \mid i \in I \})/\mu_A \).

The proof for \( S(A) \) is similar. \( \blacksquare \)

If \( \eta : \{ F_i \mid i \in I \} \to F_M(\{ F_i \mid i \in I \}) \) is a substitution then the composition \( \widehat{h} \circ \eta : \{ F_i \mid i \in I \} \cup T^{(1)}(A) \) is a valuation of the variables \( F_i \) by elements of the clone \( T^{(1)}(A) \). The next theorem connects hyperidentities of the algebra or variety with identities of the unary clone.

**Theorem 3.3** A regular identity \( s \approx t \) is a hyperidentity in an algebra \( A \) of type \( \tau_0 \) iff the equation \( \varphi(s) \approx \varphi(t) \) is an identity in the monoid \( T^{(1)}(A) \). That is,
\[ A \models s \approx t \iff T^{(1)}(A) \models \varphi(s) \approx \varphi(t). \]

**Proof.** We first assume that \( s \approx t \) is a hyperidentity of \( A \). Then for every \( \sigma \in H_{hyp}(\tau_0) \) we have \( \hat{\sigma}[s]^A = \hat{\sigma}[t]^A \). Let \( \nu : \{ F_i \mid i \in I \} \to T^{(1)}(A) \) be an arbitrary valuation mapping. Since \( \hat{\eta} \) is surjective, there exists a substitution \( \eta \), with \( \nu = \hat{\eta} \circ \eta \), using the axiom of choice. Then we have \( \tau(\varphi(s)) = (\hat{\eta} \circ \tau)(\varphi(s)) = \tau \circ (\varphi \circ \hat{\eta} \circ \varphi^{-1})(\varphi(s)) = (\tau \circ \varphi \circ \hat{\eta})(s) = (\hat{\sigma}_\eta[s])^A = (\hat{\sigma}_\eta[t])^A = \tau(\varphi(t)) \), and this means that \( \varphi(s) \approx \varphi(t) \in IdT^{(1)}(A) \). Conversely, let \( \varphi(s) \approx \varphi(t) \in IdT^{(1)}(A) \). Then \( s, t \in W_{\tau_0}(\{ x_1 \} \) and for every valuation \( \nu \) we have \( \tau(\varphi(s)) = \tau(\varphi(t)) \). Let \( \sigma \) be a hypersubstitution. Then, using Lemma 3.1 and the fact that \( \hat{\eta} \circ \varphi \circ \hat{\sigma} \circ \varphi^{-1} \) is the extension of a valuation mapping, we obtain \( \hat{\sigma}[s]^A = \hat{\eta}(\varphi(\hat{\sigma}[s])) = \hat{\eta}(\varphi(\hat{\sigma}(\varphi^{-1}(\varphi(s))))) = \hat{\eta}(\varphi(\hat{\sigma}(\varphi^{-1}(\varphi(t))))) = \hat{\sigma}[t]^A \). This shows that \( \hat{\sigma}[s]^A = \hat{\sigma}[t]^A \), and \( s \approx t \) is a hyperidentity in \( A \). □

A normal regular identity, of type \( \tau_0 \), is a regular identity \( s \approx t \) in which either both \( s \) and \( t \) are equal to the variable \( x_1 \) or neither are. In the following analogue of Theorem 3.3, normal pre-hyperidentities of an algebra correspond to identities in the transition semigroups of the algebra.

**Theorem 3.4** Let \( s \approx t \) be a non-trivial regular normal identity of a unary algebra \( A \). Then \( s \approx t \) is a pre-hyperidentity of \( A \) iff the equation \( \varphi_S(s) \approx \varphi_S(t) \) is an identity in the transition semigroup \( \text{S}(A) \).

**Proof.** The proof is similar to that of Theorem 3.3. □

A variety \( V \) of type \( \tau_0 \) is called solid if every identity in \( V \) is satisfied as a hyperidentity, and is called presolid if every identity in \( V \) is satisfied as a pre-hyperidentity. The presolid varieties of type \( \tau_0 \) form a complete sublattice of the lattice of all varieties of type \( \tau_0 \), and the solid varieties form a complete sublattice of the lattice of all presolid varieties. Solidity and presolidity are special cases of the general concept of \( M \)-solidity, for any submonoid \( M \) of hypersubstitutions; see [Den-W, 00] for more information on \( M \)-solid varieties. Let \( V \) be a variety defined by a set \( \Sigma \) of regular equations of type \( \tau_0 \), so that \( V = \text{Mod} \Sigma \) is the class of all algebras of type \( \tau_0 \) which satisfy every equation from \( \Sigma \) as an identity. It is well-known that \( V \) is solid or presolid iff every equation from \( \Sigma \) is satisfied as a hyperidentity or pre-hyperidentity, respectively, of \( V \).

**Corollary 3.5** Let \( A \) be an algebra of unary type \( \tau_0 \) with the property that the variety \( V(A) \) which is generated by \( A \) is regular. Then \( V(A) \) is solid iff the monoid \( T^{(1)}(A) \) is free with respect to itself (Hall-free), meaning that every mapping from \( \{ f_i^A \mid i \in I \} \) to \( T^{(1)}(A) \) can be extended to an endomorphism of \( T^{(1)}(A) \).

**Proof.** Using the equivalence from Theorem 3.3, we will show that \( T^{(1)}(A) \) is free iff for every identity \( s \approx t \in IdA \) the equation \( \varphi(s) \approx \varphi(t) \) is an identity in \( T^{(1)}(A) \). Suppose first that \( T^{(1)}(A) \) is free with respect to itself, freely generated by the independent set \( \{ f_i^A \mid i \in I \} \), and let \( s \approx t \in IdA \). Then by Lemma 3.1, the elements \( \tau(\varphi(s)) \) and \( \tau(\varphi(t)) \) are equal in \( T^{(1)}(A) \). Let \( \nu : \{ F_i \mid i \in I \} \to T^{(1)}(A) \) be an arbitrary valuation mapping. We define a mapping \( \alpha_\nu : \{ f_i^A \mid i \in I \} \to T^{(1)}(A) \) by \( \alpha_\nu(f_i^A) := \nu(F_i) \) for every \( i \in I \). The mapping \( \alpha_\nu \) is well-defined since \( \{ f_i^A \mid i \in I \} \) is an independent set and so, for \( i, j \in I \) we have \( f_i^A = f_j^A \Rightarrow i = j \Rightarrow F_i = F_j \Rightarrow \nu(F_i) = \nu(F_j) \Rightarrow \alpha_\nu(f_i^A) = \alpha_\nu(f_j^A) \).
Since $\mathcal{F}_M(\{F_i \mid i \in I\})$ is free in the class of all monoids, the mapping $v = \alpha_v \circ h$ can be uniquely extended to $\overline{\alpha_v} = \overline{\alpha_v} \circ \overline{h}$. Then
\[
\overline{\alpha_v}(\varphi(s)) = \overline{\alpha_v}(\varphi(t)) \Rightarrow \overline{\alpha_v}(\overline{h}(\varphi(s))) = \overline{\alpha_v}(\overline{h}(\varphi(t))) \Rightarrow \overline{\alpha_v}(\varphi(s)) = \overline{\alpha_v}(\varphi(t)) \Rightarrow \varphi(s) \approx \varphi(t) \in IdT^{(1)}(A).
\]
Therefore $s \approx t$ is a hyperidentity in $A$, and we have shown that $V(A)$ is solid.

For the converse direction, we assume that $V(A)$ satisfies the solidity condition. To show that $T^{(1)}$ is free, we let $\alpha : \{f_i^A \mid i \in I\} \to T^{(1)}(A)$ be any mapping, and show that it can be extended to an endomorphism. Our extension will be the mapping $\overline{\alpha} : T^{(1)}(A) \to T^{(1)}(A)$ defined by $\overline{\alpha}(u^A) := \overline{\alpha \circ h}(\varphi(u))$, for each $u^A \in T^{(1)}(A)$. Note that $\overline{\alpha \circ h}$ is the uniquely defined extension of the valuation $\alpha \circ h : \{F_i \mid i \in I\} \to T^{(1)}(A)$. To check that $\overline{\alpha}$ is well-defined, we let $u^A = t^A$. Then $u \approx t \in IdA$ and by the solidity assumption we have $\varphi(u) \approx \varphi(t) \in IdT^{(1)}(A)$. Then $\alpha \circ h(\varphi(u)) = \alpha \circ h(\varphi(t))$ since $\alpha \circ h$ is a valuation, and by definition we get $\overline{\alpha}(u^A) = \overline{\alpha \circ h}(\varphi(u)) = \overline{\alpha \circ h}(\varphi(t)) = \overline{\alpha \circ h}(\varphi(t)) = \overline{\alpha \circ h}(\varphi(t)) = (\alpha \circ h)(I) = \alpha(h(I)) = \alpha(f^A).

If $V$ is a regular variety of algebras of type $\tau_0$ and if $\mathcal{F}_V(X)$ is the free algebra generated by the countably infinite alphabet $X$, then $T^{(1)}(\mathcal{F}_V(X))$ is called the transition monoid of the variety $V$. Corollary 3.5 says that $V$ is solid iff this transition monoid $T^{(1)}(\mathcal{F}_V(X))$ is free with respect to itself. If $V = Mod\Sigma$ then $V$ is solid iff $T^{(1)}(\mathcal{F}_V(X))$ is free with respect to itself and has the free presentation $(\{f_i \mid i \in I\}; \varphi(\Sigma))$. (Mar.66). We remark that Corollary 3.5 is a special case of the equivalence of the solidity of a variety of arbitrary type $\tau$ and the freeness of the clone of that variety, if we regard the clone as a multibased (heterogeneous) algebra. In a similar way, Theorem 3.4 can be used to show that for every set $\Sigma$ of normal equations and regular equations of type $\tau_0$, the variety $V = Mod\Sigma$ is presolid iff the semigroup presented by $(\{f_i \mid i \in I\}, \overline{\alpha}(\varphi_S(\Sigma)))$ is free with respect to itself. Theorem 3.3 can be generalized in the following way. The isomorphism $\psi : Hyp(\tau_0) \to Subst_M(2)$ used in the proof of Theorem 2.8 maps submonoids of $Hyp(\tau_0)$ to submonoids of $Subst_M(2)$. Let $G \subseteq Hyp(\tau_0)$ be a submonoid and let $\psi(G) \subseteq Subst_M(2)$ be the corresponding monoid of substitutions.

**Definition 3.6** An equation $w_1 \approx w_2$ between monoid words from $\mathcal{F}_M(\{F_i \mid i \in I\})$ is called a $\psi(G)$-identity in a semigroup $S$ if for all $\eta \in \psi(G)$ the equations $\overline{h}(\overline{\eta}(w_1)) = \overline{h}(\overline{\eta}(w_2))$ are satisfied.

Then analogously to Theorem 3.3 we get the following Theorem.

**Theorem 3.7** Let $G$ be a monoid of hypersubstitutions of type $\tau_0$ and let $A$ be an algebra of type $\tau_0$. Then the regular identity $s \approx t$ is a $G$-hyperidentity in $A$ iff $\varphi(s) \approx \varphi(t)$ is a $\psi(G)$-identity in $T^{(1)}(A)$ \ . In this case we write $T^{(1)}(A) \models_{\psi(G) - id} \varphi(s) \approx \varphi(t)$.

For the semigroup case, for terms $s, t \neq x_1$ we get a similar result:

**Theorem 3.8** Let $G$ be a monoid of hypersubstitutions of type $\tau_0$ and let $A$ be an algebra of type $\tau_0$. Then the regular and normal identity $s \approx t$ is a $G$-hyperidentity in $V$ iff $\varphi_S(s) \approx \varphi_S(t)$ is a $\psi(G)$-identity in $S(A)$ \ . In this case we write $S(A) \models_{\psi(G) - id} \varphi_S(s) \approx \varphi_S(t)$.
4 σ-Closed and τ-Closed Varieties of Unary Type

There are some interesting interconnections between varieties \( V \) of unary type \( \tau_0 \) and the classes \( \{T^{(1)}(A) \mid A \in V\} \) of transition monoids or \( \{S(A) \mid A \in V\} \) of transition semigroups formed from algebras in \( V \). One such connection uses the concept of \( \sigma \)-closed varieties of type \( \tau_0 \) introduced in [Pet-C-B:02].

**Definition 4.1** Let \( V \) be a variety of unary type \( \tau_0 \). Then \( V \) is called \( \sigma \)-closed if whenever \( A \in V \) and \( T^{(1)}(B) \cong T^{(1)}(A) \), then also \( B \in V \). The variety \( V \) is called \( \sigma^* \)-closed if whenever \( A \in V \) and \( S(B) \cong S(A) \) then also \( B \in V \).

We recall from the end of Section 2 the definition of the submonoid \( O \) of hypersubstitutions:

\[
O := \{ \sigma \mid \sigma \in hyp(\tau_0) \text{ and } \varphi \circ \sigma \circ \varphi^{-1} \text{ is surjective} \}.
\]

A unary variety \( V \) is called \( O \)-solid if for all identities \( s \approx t \) in \( IdV \) and all \( \sigma \in O \) we have \( \delta[s] \approx \delta[t] \in Id V \). We shall show that \( \sigma \)-closed varieties and \( O \)-solid varieties are closely related to each other. To formulate and prove this interrelation we introduce some new notation and concepts. For any algebra \( A \) (of arbitrary type) we denote by \( ConA \) the lattice of congruences on \( A \). By the second isomorphism theorem, if two congruences \( \theta \) and \( \rho \) on \( A \) satisfy \( \theta \subseteq \rho \), it follows that there is a homomorphism from \( A/\theta \) onto \( A/\rho \); but the converse implication is not always true.

**Definition 4.2** Let \( A \) be an algebra of arbitrary type. A congruence \( \theta \in ConA \) is said to be weakly invariant if for every \( \rho \in ConA \) the following condition is satisfied: if there exists a homomorphism from \( A/\theta \) onto \( A/\rho \) then \( \theta \subseteq \rho \).

**Definition 4.3** A set \( C \) of congruences of an algebra \( A \) of arbitrary type \( \tau \) is said to be isomorphically closed if whenever \( \theta \in C \) and \( A/\theta \cong A/\rho \) it follows that \( \rho \in C \).

The following results were proved for semigroups in [Pet-C-B:02], and can easily be generalized to algebras of arbitrary type \( \tau \).

**Theorem 4.4** Let \( A \) be an algebra of arbitrary type \( \tau \), let \( V \) be a variety of type \( \tau \) and let \( F_V(X) \) be the free algebra with respect to \( V \), freely generated by \( X \). Then (i) A congruence \( \theta \) on \( A \) is weakly invariant iff the principal filter generated by \( \theta \) in \( ConA \) is isomorphically closed.

(ii) Every weakly invariant congruence on \( A \) is invariant under all surjective endomorphisms of \( A \).

(iii) The set \( Con_{w\tau}A \) of all weakly invariant congruences of \( A \) forms a complete lattice.

(iv) Every fully invariant congruence on \( F_V(X) \) is weakly invariant.

**Lemma 4.5** ([Pet-C-B:02]) Let \( V \) be a unary variety of type \( \tau_0 \). If \( V \) is \( \sigma \)-closed, then \( \mu_V \in ConF_M(\{F_i \mid i \in I\}) \) is weakly invariant.

Now we are able to prove:

**Theorem 4.6** A regular variety \( V \) of type \( \tau_0 \) is \( \sigma \)-closed iff it is \( O \)-solid.

**Proof.** Assume that \( V \) is \( O \)-solid. It follows from the theory of \( M \)-hyperidentities and \( M \)-solid varieties that there is a set \( \Sigma \) of regular equations of type \( \tau_0 \) such that \( V \) is the \( O \)-hypermodel class \( V = H_O Mod\Sigma \) defined by \( \Sigma \), meaning that \( V \) is the class of all algebras of type \( \tau_0 \) satisfying all the equations from \( \Sigma \) as \( O \)-hyperidentities. Thus \( A \in V \)
iff \( \mathcal{A} \) \( \mathcal{O} \)-hypersatisfies \( \Sigma \). We will use the notation \( \varphi(\Sigma) \) to denote the set of identities obtained by applying \( \varphi \) to each identity in \( \Sigma \); this makes sense because the identities in \( \Sigma \) are all regular. Now let \( \mathcal{A} \) be any algebra in \( V \). Then \( \mathcal{A} \models \Sigma \) if and only if \( T^{(1)}(A) \models \varphi(\Sigma) \). If \( T^{(1)}(A) \cong T^{(1)}(B) \), then \( T^{(1)}(B) \models \varphi(\Sigma) \); this gives \( B \models \varphi(\Sigma) \) and thus \( B \in V = H_{\mathcal{O}}\text{Mod}\Sigma \). This shows that \( V \) is \( \sigma \)-closed. Conversely, we suppose that \( V \) is a regular variety of type \( \tau_0 \) which is \( \sigma \)-closed. Then by Lemma 4.5 the Myhill congruence \( \mu_V \in \text{Con} \mathcal{F}_M(\{(F_i|i \in I)\}) \) is weakly invariant. By Theorem 4.4 \( \mu_V \) is invariant under all surjective endomorphisms of \( \mathcal{F}_M(\{(F_i|i \in I)\}) \). Therefore for every pair \((w_1, w_2) \in \mu_V \) and every surjective endomorphism \( \varphi : \mathcal{F}_M(\{(F_i|i \in I)\}) \rightarrow \mathcal{F}_M(\{(F_i|i \in I)\}) \) we have \((\varphi(w_1), \varphi(w_2)) \in \mu_V \). Then by Theorem 2.8 there is a hypersubstitution \( \sigma \) such that \(((\varphi \circ \sigma \circ \varphi^{-1})(w_1), (\varphi \circ \sigma \circ \varphi^{-1})(w_2)) \in \mu_V \). By definition of \( \mu_V \) for every algebra \( \mathcal{A} \in V \) and hypersubstitution \( \sigma \) such that \( \varphi \circ \sigma \circ \varphi^{-1} \) is surjective we get that from \( \varphi^{-1}(w_1) \approx \varphi^{-1}(w_2) \in Id \mathcal{A} \) there follows \( \varphi^{-1} \circ (\varphi \circ \sigma \circ \varphi^{-1})(w_1) \approx \varphi^{-1} \circ (\varphi \circ \sigma \circ \varphi^{-1})(w_2) \in Id \mathcal{A} \) and thus \( \hat{\sigma}^{-1}(w_1) \approx \hat{\sigma}^{-1}(w_2) \in Id \mathcal{A} \). This means that \( \varphi^{-1}(w_1) \approx \varphi^{-1}(w_2) \) is an \( \mathcal{O} \)-hyperidentity in \( V \). Since every regular identity \( s \approx t \in Id V \) can be written in the form \( \varphi^{-1}(w_1) \approx \varphi^{-1}(w_2) \), this completes the proof.

In a similar way it can be shown that a regular and normal variety \( V \) of type \( \tau_0 \) is \( \sigma^* \)-closed iff \( V \) is \( \sigma \)-closed. We generalize Definition 4.1 in the following way:

**Definition 4.7** A variety \( V \) of type \( \tau_0 \) is said to be \( s \)-closed if for every algebra \( \mathcal{B} \) of type \( \tau_0 \), whenever \( \mathcal{A} \in V \) and \( T^{(1)}(\mathcal{B}) \) is isomorphic to a submonoid of \( T^{(1)}(\mathcal{A}) \), it follows that \( \mathcal{B} \in V \). The variety \( V \) is said to be \( v \)-closed if for every algebra \( \mathcal{B} \) of type \( \tau_0 \), whenever \( \mathcal{A} \in V \) and \( Id T^{(1)}(\mathcal{B}) \supseteq Id T^{(1)}(\mathcal{A}) \) (that is, if \( T^{(1)}(\mathcal{B}) \in V(T^{(1)}(\mathcal{A})) \), then it follows that \( \mathcal{B} \in V \). (Analogous definitions can be made for \( v^* \)-closed and \( s^* \)-closed varieties, using the transition semigroups \( S(\mathcal{A}) \) and \( S(\mathcal{B}) \).)

**Proposition 4.8** If a variety \( V \) of type \( \tau_0 \) is \( v \)-closed, then it is \( \sigma \)-closed and \( s \)-closed.

**Proof.** Let \( V \) be \( v \)-closed, and let \( \mathcal{A} \in V \) and \( T^{(1)}(\mathcal{B}) \cong T^{(1)}(\mathcal{A}) \). Then \( Id T^{(1)}(\mathcal{B}) = Id T^{(1)}(\mathcal{A}) \), and so \( \mathcal{B} \in V \) since \( V \) is \( v \)-closed. Therefore \( V \) is \( \sigma \)-closed. If \( T^{(1)}(\mathcal{B}) \) is isomorphic to a submonoid of \( T^{(1)}(\mathcal{A}) \), then \( Id T^{(1)}(\mathcal{B}) \supseteq Id T^{(1)}(\mathcal{A}) \) and so \( \mathcal{B} \in V \) since \( V \) is \( v \)-closed. This shows that \( V \) is \( s \)-closed.

In a corresponding way it can be shown that every \( v^* \)-closed variety of type \( \tau_0 \) is \( \sigma^* \)-closed and \( s^* \)-closed.

**Theorem 4.9** A regular variety \( V \) of type \( \tau_0 \) is \( v \)-closed iff it is solid.

**Proof.** Let \( V \) be a solid regular variety. Then there exists a set \( \Sigma \) of regular equations with \( V = H_{\mathcal{O}}\text{Mod}\Sigma \), such that for every algebra \( \mathcal{A} \in V \) we have \( \mathcal{A} \models \Sigma \). Then by Theorem 3.3 for every \( \mathcal{A} \in V \) we have \( T^{(1)}(\mathcal{A}) \models \varphi(\Sigma) \). If \( \mathcal{B} \) is an algebra with \( Id T^{(1)}(\mathcal{B}) \supseteq Id T^{(1)}(\mathcal{A}) \) then \( T^{(1)}(\mathcal{B}) \models \varphi(\Sigma) \) and by Theorem 3.3 we have \( \mathcal{B} \models \Sigma \); thus \( \mathcal{B} \in V \) is \( \mathcal{O} \)-closed. Now let \( V \) be \( v \)-closed, and let \( \mathcal{A} \in V \) with \( \mathcal{A} = (A; (f_i^{A})_{i \in I}) \). We consider the derived algebra \( \sigma(\mathcal{A}) = (A; (\sigma(f_i^{A})_{i \in I})) \) for a hypersubstitution \( \sigma \). Since the operations \( \sigma(f_i^{A}) \) are term operations of \( \mathcal{A} \) we have \( \{\langle \sigma(f_i^{A}) | i \in I \} \cup \{id_A\} = T^{(1)}(\sigma(\mathcal{A})) \subseteq T^{(1)}(\mathcal{A}) = \{\langle f_i^{A} | i \in I \} \cup \{id_A\} \). Therefore \( Id T^{(1)}(\sigma(\mathcal{A})) \supseteq Id T^{(1)}(\mathcal{A}) \). Since \( V \) is \( v \)-closed, we have \( \sigma(\mathcal{A}) \in V \). This shows that any derived algebra formed from an algebra in \( V \) is also in \( V \), and this is known to be equivalent to solidity of \( V \) (see [Den-W:00]).
In a corresponding way it can be shown that a regular and normal variety of type $\tau_0$ is $v^*$-closed iff it is pre-solid. Now we prove that $v$-closure and $s$-closure are equivalent concepts for varieties of type $\tau_0$.

**Theorem 4.10** Let $V$ be a regular variety of type $\tau_0$. Then $V$ is $v$-closed iff it is $s$-closed.

**Proof.** One direction is given by Proposition 4.8. Conversely, suppose that $V$ is $s$-closed, and let $\mathcal{A} \in V$. Then as in the proof of Theorem 4.9 we have $T^{(1)}(\sigma(\mathcal{A})) \subseteq T^{(1)}(\mathcal{A})$ and then $\sigma(\mathcal{A}) \in V$. This shows that $V$ is solid, and hence by Theorem 4.9 it is also $v$-closed. □

This means that for a regular variety of type $\tau_0$, the concepts of $v$-closure, $s$-closure and solidity are equivalent. An analogous result holds for $v^*$-closure, $s^*$-closure and presolidity of normal regular varieties. In Section 2 we introduced the concept of a $B$-solid variety. Since $B$ is a submonoid of $\mathcal{O}$, every $\mathcal{O}$-solid variety of type $\tau_0$ is also $B$-solid. Let $V$ be a $B$-solid regular variety of type $\tau_0$. Then there exists a set $\Sigma$ of regular equations of type $\tau_0$ such that $V = H_B Mod \Sigma$ and if $C \in V$, then $C \models \Sigma$. Therefore $T^{(1)}(\mathcal{C}) \models \varphi(\Sigma)$.

If $T^{(1)}(\mathcal{A}) \cong T^{(1)}(\mathcal{C})$ then $T^{(1)}(\mathcal{A}) \models \varphi(\Sigma)$ and $\mathcal{A} \models \Sigma$ by Theorem 3.7, and it follows that $\mathcal{A} \in V$ and $V$ is $\sigma$-closed. By Theorem 4.6 $V$ is $\mathcal{O}$-solid. This proves the following result.

**Theorem 4.11** A regular variety $V$ of type $\tau_0$ is $B$-solid iff it is $\mathcal{O}$-solid. □

5 Example In this section we illustrate the application of Theorems 4.9 and 4.10 with an example. We consider the unary type $\tau_0 = (1, 1)$, so that we have two unary operation symbols $f$ and $g$. Let $V$ be the model class of the following equations:

$$f(f(x)) \approx x, \quad g(g(x)) \approx x, \quad f(g(f(x))) \approx g(x), \quad g(f(g(x))) \approx f(x).$$

We want to show that $V$ is solid. Since the four identities defining $V$ are regular, the variety $V$ is regular, and we can use the fact that such varieties are solid iff they are $s$-closed. Let $\mathcal{A} \in V$, so $\mathcal{A} = (A; f^A, g^A)$ is an algebra of type $(1, 1)$ satisfying the four identities above. Then the transition monoid $T^{(1)}(\mathcal{A})$ has the presentation $T^{(1)}(\mathcal{A}) := \langle \{ f^A, g^A \}, \{(f^A)^2 = id_A, (g^A)^2 = id_A, f^A \circ g^A \circ f^A = g^A, g^A \circ f^A \circ g^A = f^A \} \rangle$. $T^{(1)}(\mathcal{A})$ consists of the four elements $id_A, f^A, g^A$, and $f^A g^A$. The Cayley-table of $T^{(1)}(\mathcal{A})$ has the form

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>$id_A$</th>
<th>$f^A$</th>
<th>$g^A$</th>
<th>$f^A g^A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$id_A$</td>
<td>$id_A$</td>
<td>$f^A$</td>
<td>$g^A$</td>
<td>$f^A g^A$</td>
</tr>
<tr>
<td>$f^A$</td>
<td>$f^A$</td>
<td>$id_A$</td>
<td>$f^A g^A$</td>
<td>$g^A$</td>
</tr>
<tr>
<td>$g^A$</td>
<td>$g^A$</td>
<td>$f^A g^A$</td>
<td>$id_A$</td>
<td>$f^A$</td>
</tr>
<tr>
<td>$f^A g^A$</td>
<td>$f^A g^A$</td>
<td>$g^A$</td>
<td>$f^A$</td>
<td>$id_A$</td>
</tr>
</tbody>
</table>

$T^{(1)}(\mathcal{A})$ has exactly 4 proper submonoids, which have the following sets as universes:

$$\{id_A\}, \{id_A, f^A\}, \{id_A, g^A\}, \{id_A, f^A g^A\}.$$ 

Since these submonoids are isomorphic to trivial monoids or two-element groups, by commutativity the identities $x^2 \approx x, xyx \approx y, yxy \approx x$ are satisfied. Let $C$ be an arbitrary submonoid of $T^{(1)}(\mathcal{A})$. If $B$ is an algebra of type $(1, 1)$ such that $T^{(1)}(B) \cong C$, then $T^{(1)}(B)$ satisfies the identities $x^2 \approx x, xyx \approx y, yxy \approx x$, and this means $B \in V$. Therefore, $V$ is $s$–closed and using Theorem 4.9 and Theorem 4.10 $V$ is solid.
6 Extension to n-Clones and Type (n,n,...) In this section we show that some results of the previous sections also hold for n-clones of algebras whose operations are all n-ary, for some \( n \geq 1 \). We will call a type of algebras n-ary if all its operations are n-ary, and an algebra of this type will be called an n-ary algebra. Throughout this section we assume that \( \tau \) is such an n-ary type, for some fixed \( n \geq 1 \), and that the operation symbols of type \( \tau \) are \( \{ f_i \} \in I \), indexed by some set \( I \). The set \( W_\tau(X_n) \) of all n-ary terms of type \( \tau \), on the alphabet \( X_n = \{ x_1, x_2, \ldots, x_n \} \), forms the universe of an algebra of type \( (n+1,0,\ldots,0) \), which has an n+1-ary operation \( S_n^{n+1}(X_n) \) called superposition and n nullary operations or special elements \( e_1, \ldots, e_n \), and which satisfies the three clone axioms (see [Den-W:00]):

\[
\begin{align*}
\text{(C1)} & \quad S_n^n(F_i, S_n^n(F_{i_1}, F_{i_2}, \ldots, F_{i_n})), \\
\text{(C2)} & \quad S_n^n(e_i, F_i, \ldots, F_n) = F_i, \\
\text{(C3)} & \quad S_n^n(F_i, e_1, \ldots, e_n) = F_i
\end{align*}
\]

for \( 1 \leq i \leq n \). The terms occurring in these axioms are terms of an language of type \( \tau = (n+1,0,\ldots,0) \) built up by operation symbols \( S_n^n, e_1, \ldots, e_n \) and an alphabet \( \{ F_i \mid i \in I \} \) with a bijection \( \pi \) between \( \{ F_i \mid i \in I \} \) and \( \{ f_i \mid i \in I \} \). Then an n-clone is defined as any algebra of type \( \tau = (n+1,0,\ldots,0) \) satisfying (C1), (C2), (C3). The special n-clone whose universe is the set \( W_\tau(X_n) \) of all n-ary terms is called the n-clone of type \( \tau \), and is denoted by \( \text{Clone}_n(\tau) \). (For convenience, the n+1-ary operation of this concrete n-clone will also be denoted by \( S_n^n \)). Notice that in the case \( n = 1 \) the clone axioms just say that the superposition operation \( S_1^1 \) is associative and that \( e_1 \) is a two-sided identity for \( S_1^1 \); that is, that the 1-clone is a monoid. Then we can define a map \( \varphi : W_\tau(X_n) \to \{ F_i \mid i \in I \} \cup \{ e_1, \ldots, e_n \} \) inductively by \( \varphi(x_j) = e_j \), for \( 1 \leq j \leq n \),

\[
\begin{align*}
\varphi(f_i(t_1, \ldots, t_n)) &= S_n^n(F_i, \varphi(t_1), \ldots, \varphi(t_n)), \\
\varphi(s_1) &= S_n^n(\varphi(t), \varphi(s_1), \ldots, \varphi(s_n)).
\end{align*}
\]

Inductively, let \( t = f_i(t_1, \ldots, t_n) \) for some \( i \in I \) and some n-ary terms \( t_1, \ldots, t_n \). Then

\[
\begin{align*}
\varphi(S_n^n(t, s_1, \ldots, s_n)) &= \varphi(f_i(t_1, s_1, \ldots, s_n), \ldots, t_n(s_1, \ldots, s_n)), \\
&= S_n^n(F_i, \varphi(s_1), \ldots, \varphi(s_n)), \\
&= S_n^n(F_i, \varphi(S_n^n(t_1, s_1, \ldots, s_n), \ldots, \varphi(s_1), \ldots, \varphi(s_n))), \\
&= S_n^n(F_i, \varphi(t), \varphi(s_1), \ldots, \varphi(s_n)).
\end{align*}
\]

Since \( \varphi \) is easily shown to be a bijection (the analogue of Proposition 2.3), we have an isomorphism between \( \text{Clone}_n(\tau) \) and the free n-clone \( \mathcal{F}_n \) on our new language. Next we define \( \text{Subst}_n \) to be the set of all substitutions \( \eta \) from the generating set \( \{ F_i \mid i \in I \} \) to the free n-clone \( \mathcal{F}_n \). We define a binary composition operation \( \circ \) on \( \text{Subst}_n \) by \( (\eta_1 \circ \eta_2)(F_i) = \mathcal{P}(\eta_2(F_i)) \), where \( \mathcal{P} \) is the obvious extension of \( \eta_1 \). Then \( \text{Subst}_n \) forms a monoid under
this operation, with the identity substitution as identity element. Then we can prove that
the monoid \( Hyp(\tau) \) of all hypersubstitutions of type \( \tau \) is isomorphic to the monoid \( Subst_n \),
using the isomorphism \( \psi : \sigma \mapsto \varphi \circ \sigma \circ \pi^{-1} \), as in the proof of Theorem 2.8. Now let \( A \) be
any algebra of our \( n \)-ary type \( \tau \). It is well known that every \( n \)-ary term \( t \) of type \( \tau \) induces
an \( n \)-ary term operation \( t^A \) on \( A \), by \( t^A = \text{the } j \)-th projection operation \( e_j \) on \( A \), for \( t = x_j \),
for \( 1 \leq j \leq n \),
\[ t^A = f^A(t^A_1, \ldots, t^A_n), \]
for \( t = f_i(t_1, \ldots, t_n) \). The collection \( \{ t^A \mid t \in W_X(X_n) \} \) forms the
universe of an \( n \)-clone \( T^{(n)}(A) \), called the term \( n \)-clone of \( A \). Then we can prove inductively,
as in Lemma 3.1, that for any \( n \)-ary term \( t \), we have \( t^A = \overline{h}(\varphi(t)) \). Now we want to apply
this result to identities of the appropriate form, to get an analogue of Theorem 3.3. For
unary type, the property we needed was that only the variable \( x_1 \) was used on either side
of the identity, which is equivalent to regularity for unary. For \( n \geq 2 \), we need that the
identity is \( n \)-ary, meaning that it uses only the variables \( x_1, \ldots, x_n \) so that \( \varphi \) can be applied;
note that this is not equivalent to regular for \( n \geq 2 \). We will call a variety \( V \)-ary if \( V = Mod_\Sigma \) for some set \( \Sigma \) of \( n \)-ary identities. Note that the property on \( n \)-ary identities is not
an equational theory, since a variety \( V = Mod_\Sigma \) for a set \( \Sigma \) of \( n \)-ary identities still satisfies
identities which use more than just the variables \( x_1, \ldots, x_n \).

**Theorem 6.2** Let \( V \) be an \( n \)-ary algebra of type \( \tau \). Then an \( n \)-ary identity \( s \approx t \) is a
hyperidentity of \( V \) iff \( \varphi(s) \approx \varphi(t) \) is an identity of the term \( n \)-clone \( T^{(n)}(A) \).

In the unary case most of the results from Section 2 had two versions, one for the transition
monoid which is the unary clone, and one for the corresponding transition semigroup. For
\( n \geq 2 \), it is also possible to consider this additional structure. We define \( S^{(n)}(A) \) to be the
set of term operations on \( A \) induced by non-variable terms from \( W^*_X(X_n) \). This set is closed
under the \( n+1 \)-ary operation \( S_n^0 \), and we shall call the algebra \( (S^{(n)}(A); S_n^0) \) the \( n \)-transition
algebra or the pre-\( n \)-clone of \( A \). It is straightforward then to define the restriction \( \varphi_S \) of the
isomorphism \( \varphi \) to the set \( S^{(n)}(A) \), and to prove analogous results in this case. We conclude
with an example. We take \( n = 2 \), and type \( \tau = (2) \) with a single binary operation \( f \). We let
\( \Sigma \) be the set consisting of the following identities of type \( (2) \)
\[ f(x, x) \approx x, \quad f(x, f(y, x)) \approx f(f(x, y), x) \approx x, \quad f(f(x, y), y) \approx f(x, f(y, y)), \]
\[ f(x, f(x, y)) \approx f(f(x, x), y). \]
Now let \( V = Mod_\Sigma \), the variety with the set \( \Sigma \) as a basis. Notice that the identities in \( \Sigma \) are all \( 2 \)-ary, and hence \( V \) is a \( 2 \)-ary variety; and that \( V \) is not a variety of semigroups, since associativity
is not satisfied here, and indeed is not a \( 2 \)-ary identity. It is easy to calculate that for any
algebra \( A \in V \), we have \( T^{(2)}(A) = \{ x_1^A, x_2^A, f^A(x_1, x_2), f^A(x_2, x_1) \} \). This 2-clone has only
two 2-subclones, the trivial (projection) clone \( \{ x_1^A, x_2^A \} \) and the 2-clone itself. It is not
difficult to see that any algebra \( B \) of type \( (2) \) with a 2-clone isomorphic to a subclone of
\( T^{(2)}(A) \) belongs to the variety \( V \). Therefore \( V \) is \( s \)-closed. It is also straightforward to check
that \( V \) is a solid variety of type \( (2) \); any hypersubstitution of this type is \( \sim_V \) equivalent to
one of \( \sigma_{x_1}, \sigma_{x_2}, \sigma_{f(x_1,x_2)} \) or \( \sigma_{f(x_2,x_1)} \), and application of any of these to the basis identities
in \( \Sigma \) yields an identity of \( V \).

**References**

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