ON THE ALUTHGE TRANSFORMATIONS OF $\infty$-HYPONORMAL OPERATORS

SHIZUO MIYAJIMA and ISAO SAITO

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Abstract. A bounded linear operator $T$ is called $\infty$-hyponormal if $T$ is $p$-hyponormal for every $p > 0$. In this paper $\infty$-hyponormality of the Aluthge transformations of $\infty$-hyponormal operators is investigated. It is shown that the Aluthge transformation of an $\infty$-hyponormal operator is not necessarily $\infty$-hyponormal. It is also shown that the (generalized) Aluthge transformation of an $\infty$-hyponormal operator $T$ is $\infty$-hyponormal provided $|T| |T^*| = |T^*||T|$. Moreover we give an example of an $\infty$-hyponormal operator $T$ whose Aluthge transformation $\bar{T}$ is $\infty$-hyponormal but $|T||T^*| \neq |T^*||T|$.

1 Introduction A bounded linear operator $T$ on a Hilbert space is called $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$ ($p > 0$). (The notion of $p$-hyponormal operators for $p \in \mathbb{N}$ was introduced first by Fujii and Nakatsu[6]. A 1-hyponormal operator is nothing but a hyponormal operator.) Concerning $p$-hyponormal operators, many interesting results have been obtained (e.g., [1], [4], [5], [7]–[10], [14]). The unilateral shift is a simple example being $p$-hyponormal for every $p > 0$. In the hope of getting fruitful results, the authors [11], [12] investigated operators that are $p$-hyponormal for every $p > 0$ and they called them $\infty$-hyponormal. Let us recall the definition of $\infty$-hyponormal operators.

Definition. A bounded linear operator $T$ on a Hilbert space is called $\infty$-hyponormal if it is $p$-hyponormal for every $p > 0$. By the Löwner-Heinz theorem, $T$ is $\infty$-hyponormal if and only if $(T^*T)^n \geq (TT^*)^n$ for every $n \in \mathbb{N}$, or $|T|^n \geq |T^*|^n$ for every $n \in \mathbb{N}$.

Concerning $\infty$-hyponormal operators, the authors [11], [12] proved that the outer boundary of the spectrum of a pure $\infty$-hyponormal operator is the circle with the radius $||T||$, and showed the existence of non-trivial invariant subspaces for any $\infty$-hyponormal operator. (Recall that an operator $T$ is said to be pure if $T$ has no reducing subspace on which it is normal.) For other facts (e.g., algebraic properties of the set of $\infty$-hyponormal operators), see [11].

On the other hand, Aluthge[1] introduced the operator $\bar{T} = |T|^p U|T|^p$ for a $p$-hyponormal operator $T$, where $T = U|T|$ is the polar decomposition of $T$. Further Furuta[7] introduced the operator $\bar{T}_{s,t} = |T|^s U|T|^t$ for a $p$-hyponormal operator $T$ and $s, t > 0$. The operators $\bar{T} = |T|^p U|T|^p$ and $\bar{T}_{s,t} = |T|^s U|T|^t$ are called the Aluthge transformation and the generalized Aluthge transformation of $T$, respectively. These operators have nice properties as follows.

Theorem A[1]. For a $p$-hyponormal $(0 < p \leq 1)$ operator with the polar decomposition $T = U|T|$, Aluthge transformation $\bar{T} = |T|^p U|T|^p$ of $T$ is $\min\left\{p + \frac{1}{2}, 1\right\}$-hyponormal.

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Theorem B[8], [9], [14]. Let $T$ be a $p$-hyponormal ($0 < p \leq 1$) operator with the polar decomposition $T = U|T|$. Then the following assertions hold.

1. If $s, t > 0$, $\max\{s, t\} \geq p$ hold, then $\tilde{T}_{s,t} = |T|^s U |T|^t$ is $p_{\min\{s,t\}}$-hyponormal.

2. If $s, t > 0$, $\max\{s, t\} \leq p$ hold, then $\tilde{T}_{s,t} = |T|^s U |T|^t$ is hyponormal.

Roughly speaking, these theorems say that the Aluthge transformations or generalized Aluthge transformations “improve” the degree of hyponormality, although restricted in the range of $0 < p \leq 1$. So, it is natural to ask whether the Aluthge transformations of $\infty$-hyponormal operators preserve the $\infty$-hyponormality or not.

In section 2, we answer the question in the negative by giving an example of an $\infty$-hyponormal operator for which the Aluthge transformation is not $\infty$-hyponormal.

However, for many $\infty$-hyponormal operators (e.g., the unilateral shift, a unilateral weighted shift operator with increasing weight sequence, a bilateral weighted shift operator with increasing weight sequence and a quasinormal operator), their Aluthge transformations are also $\infty$-hyponormal. The common properties of these $\infty$-hyponormal operators $T$ are that $T$ is hyponormal and $|T||T^*| = |T^*||T|$. (In [2], [3] and [6], operators satisfying $|T||T^*| = |T^*||T|$ were investigated and many interesting results were obtained. Especially Fujii and Nakatsu[6] noted that a hyponormal operator $T$ satisfying $|T||T^*| = |T^*||T|$ is $\infty$-hyponormal.)

In section 3, we first show that the Aluthge transformation $\tilde{T} = |T|^s U |T|^t$ and more generally the generalized Aluthge transformation $\tilde{T}_{s,t} = |T|^s U |T|^t$ of an $\infty$-hyponormal operator $T$ are also $\infty$-hyponormal provided $|T||T^*| = |T^*||T|$. Next we show that the equality $|T||T^*| = |T^*||T|$ is not a necessary condition for the Aluthge transformation $\tilde{T}$ of an $\infty$-hyponormal operator $T$ to be $\infty$-hyponormal.

2 Example of an $\infty$-hyponormal operator whose Aluthge transformation is not $\infty$-hyponormal

We show that the Aluthge transformations of $\infty$-hyponormal operators are not necessarily $\infty$-hyponormal by giving an example. To do so, we need the following lemma, where $2 \times 2$ matrices are considered as bounded linear operators on the Hilbert space $C^2$.

Lemma 2.1 Let $A$ and $B$ are positive $2 \times 2$ matrices and let $p_1, p_2 (p_1 \leq p_2)$ and $q_1, q_2 (q_1 \leq q_2)$ be eigenvalues of $A$ and $B$, respectively.

Then $A^n \geq B^n$ holds for all $n \in \mathbb{N}$ if and only if the following assertion (i) or (ii) holds.

(i) $q_1 \leq q_2 \leq p_1 \leq p_2$

(ii) $q_1 \leq p_1 < q_2 \leq p_2$ and $\mathcal{N}(B - q_1) = \mathcal{N}(A - p_1)$ hold, where $\mathcal{N}(T)$ is the null space of $T$.

To prove this lemma, we use the following theorem[13]. For a proof, see [11], [13].

Theorem C [13]. Let $A, B$ be positive operators on a Hilbert space with spectral resolutions $A = \int \lambda dP(\lambda), B = \int \lambda dQ(\lambda)$, respectively. Then $A^n \geq B^n$ holds for every $n \in \mathbb{N}$ if and only if $P(\lambda) \leq Q(\lambda)$ holds for every $\lambda \geq 0$.

Now we give a proof of Lemma 2.1.

Proof of Lemma 2.1. Suppose that $A^n \geq B^n$ holds for every $n \in \mathbb{N}$. First we show that $q_1 \leq p_1$ and $q_2 \leq p_2$. Suppose that $q_1 > p_1$ or $q_2 > p_2$ holds. Then $Q(p_1) = 0 < P(p_1)$ or $Q(p_2) < I = P(p_2)$, where $A = \int \lambda dP(\lambda), B = \int \lambda dQ(\lambda)$ are spectral resolutions, respectively. ($Q(\lambda) < P(\lambda)$ means $Q(\lambda) \leq P(\lambda)$ and $Q(\lambda) \neq P(\lambda)$.) This contradicts Theorem C. Hence $q_1 \leq p_1$ and $q_2 \leq p_2$ hold.
If \( q_2 \leq p_1 \), then (i) \( q_1 \leq q_2 \leq p_1 \) holds. If \( q_2 > p_1 \), then \( q_1 \leq p_1 < q_2 \leq p_2 \) and so \( \mathcal{N}(B - q_1) = \mathcal{R}(Q(q_1)) = \mathcal{R}(Q(p_1)) \supseteq \mathcal{R}(P(p_1)) = \mathcal{N}(A - p_1) \) by Theorem C. (\( \mathcal{R}(T) \) denotes the range of \( T \)) Since \( \mathcal{N}(B - q_1) \) and \( \mathcal{N}(A - p_1) \) are not equal to \((0)\) nor \( C^2 \), we obtain \( \mathcal{N}(B - q_1) = \mathcal{N}(A - p_1) \). Hence (ii) holds.

Conversely, suppose that (i) holds. Then \( P(\lambda) = 0 \leq Q(\lambda) \) \((0 \leq \lambda < p_1)\), \( P(\lambda) \leq I = Q(\lambda) \) \((q_2 \leq \lambda)\). Hence \( P(\lambda) \leq Q(\lambda) \) for every \( \lambda \geq 0 \). By Theorem C, \( A^n \geq B^n \) holds for every \( n \in \mathbb{N} \).

Next suppose that (ii) holds. Since \( \mathcal{R}(Q(q_1)) = \mathcal{N}(B - q_1) = \mathcal{N}(A - p_1) = \mathcal{R}(P(p_1)) \) holds, we obtain \( P(\lambda) = 0 \leq Q(\lambda) \) \((0 \leq \lambda < p_1)\), \( P(\lambda) = Q(\lambda) \) \((p_1 \leq \lambda < q_2)\), and \( P(\lambda) \leq I = Q(\lambda) \) \((q_2 \leq \lambda)\). Hence \( P(\lambda) \leq Q(\lambda) \) for every \( \lambda \geq 0 \). By Theorem C, \( A^n \geq B^n \) holds for every \( n \in \mathbb{N} \).

Remark. If the assertion (i) holds in the above Lemma 2.1, then \( A \geq p_1 I \geq q_2 I \geq B \) holds. From this inequality, we can easily obtain the inequality \( A^n \geq p_1^n I \geq q_2^n I \geq B^n \) for every \( n \in \mathbb{N} \) without using Theorem C.

Now we give an example of an \( \infty \)-hyponormal operator for which the Aluthge transformation is not \( \infty \)-hyponormal.

**Example 2.2** Let

\[
A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},
\]

and set

\[
T = \begin{pmatrix} 0 & \begin{pmatrix} A_1 & 0 \\ \end{pmatrix} \\ \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} A_2 & 0 \\ \end{pmatrix} \\ \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} A_2 & 0 \\ \end{pmatrix} \\ \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} \\ \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} & \begin{pmatrix} \end{pmatrix} \end{pmatrix}
\]

on \( \bigoplus_{n=1}^{\infty} H_n \), where \( H_n = C^2 \) for \( n \in \mathbb{N} \).

First we show that \( T \) is \( \infty \)-hyponormal. A calculation yields

\[
|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_2 \\ \end{pmatrix}
\]

\[
|T^*| = (TT^*)^{\frac{1}{2}} = \begin{pmatrix} 0 & A_1 \\ A_2 & A_2 \\ \end{pmatrix}
\]

The eigenvalues of \( A_1 \) and \( A_2 \) are 0, 2, and 2, respectively. This implies \( A_1^n \leq A_2^n \) for every \( n \in \mathbb{N} \) because of Lemma 2.1 (i). Hence \( |T|^n \geq |T^*|^n \) for any \( n \in \mathbb{N} \), and so \( T \) is \( \infty \)-hyponormal.
Next we show that the Aluthge transformation $\tilde{T}$ is not $\infty$-hyponormal. The polar decomposition of $T$ is $T = V|T|$, where

$$V = \begin{pmatrix} 0 & I & 0 & I & 0 & \ldots \\ I & 0 & I & 0 & \ldots \\ & & & & & \ddots \\ \end{pmatrix}$$

By a calculation, we obtain

$$\tilde{T}^*\tilde{T} = (|T|^2 V|T|^2)^* (|T|^2 V|T|^2) = \begin{pmatrix} A_1^\frac{1}{2} A_2 A_1^\frac{1}{2} & A_2 \\ A_2 & A_2 \\ & \ddots \end{pmatrix},$$

$$\tilde{T}\tilde{T}^* = (|T|^2 V|T|^2)(|T|^2 V|T|^2)^* = \begin{pmatrix} 0 & A_2^\frac{1}{2} A_1 A_2^\frac{1}{2} \\ A_2^\frac{1}{2} A_1 A_2^\frac{1}{2} & A_2 \\ & \ddots \end{pmatrix}.$$ 

Hence $\tilde{T}$ is $\infty$-hyponormal if and only if $A_2^n \geq (A_2^\frac{1}{2} A_1 A_2^\frac{1}{2})^n$ holds for any $n \in \mathbb{N}$. A calculation yields

$$A_2^\frac{1}{2} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix},$$

$$A_2^\frac{1}{2} A_1 A_2^\frac{1}{2} = \begin{pmatrix} 3 + 2\sqrt{2} & 1 \\ 1 & 3 - 2\sqrt{2} \end{pmatrix},$$

since

$$A_2^\frac{1}{2} = \frac{1}{2} \begin{pmatrix} 2 + \sqrt{2} & 2 - \sqrt{2} \\ 2 - \sqrt{2} & 2 + \sqrt{2} \end{pmatrix}.$$ 

The eigenvalues of $A_2^\frac{1}{2}$ and $A_1^\frac{1}{2} A_2 A_1^\frac{1}{2}$ are 4, 16 and 0, 6, respectively. Since

$$\mathcal{N}(A_2^2 - 4) = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} : t \in \mathbb{C} \right\} \neq \mathcal{N}(A_1^\frac{1}{2} A_2 A_1^\frac{1}{2}) = \left\{ t \begin{pmatrix} -1 \\ 3 + 2\sqrt{2} \end{pmatrix} : t \in \mathbb{C} \right\},$$

there exists an $n \in \mathbb{N}$ such that $A_2^n \not\geq (A_2^\frac{1}{2} A_1 A_2^\frac{1}{2})^n$ by Lemma 2.1 (ii). Hence $\tilde{T}$ is not $\infty$-hyponormal.

3 A sufficient condition for the generalized Aluthge transformations of $\infty$-hyponormal operators to be also $\infty$-hyponormal. We give a sufficient condition for the generalized Aluthge transformations of $\infty$-hyponormal operators to be also $\infty$-hyponormal.

**Theorem 3.1** Let $T$ be an $\infty$-hyponormal operator with the polar decomposition $T = U|T|$. If $|T||T^*| = |T||T^*|$ holds, then $\tilde{T}_{s,t} = |T|^s U|T|^t$ is $\infty$-hyponormal for every $s, t \geq 0$. 
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Proof. Let $T = U|T|$ be the polar decomposition of $T$, and let $\tilde{T}_{s,t} = |T|^s U|T|^t$ be the generalized Aluthge transformation of $T$. If $\tilde{T}_{s,t} = |T|^s U|T|^t$ is $\infty$-hyponormal for any $s, t > 0$, then $|\tilde{T}_{s,t}| \geq |\tilde{T}_{s,t}|^p$, $|\tilde{T}_{s,t}| \geq |\tilde{T}_{s,t}|^p$ and $|\tilde{T}_{s,t}| \geq |\tilde{T}_{s,t}|^p$ for $k \in \mathbb{N}$ and $p \geq 0$. By taking the limit as $k \to \infty$, we see that $\tilde{T}_{s,t} = |T|^s U|T|^t$ is also $\infty$-hyponormal for $s = 0$ or $t = 0$. Hence we may assume that $s, t > 0$.

It is known that $U^+ U$ and $UU^+$ are the orthogonal projection onto $\mathcal{R}(|T|) = \mathcal{R}(|T|^p) = N(|T|^p)^{\perp}$ and $\mathcal{R}(|T|) = N(T^*)^{\perp}$, respectively, and so $|T|^p U^+ U = |T|^p$, $T^* |UU^+ = |T|^p$ and $U^+ U|T|^p = |T|$, $UU^+ |T|^p = |T|^p$ hold for any $p > 0$. (\mathcal{R}(T) denotes the closure of the range of $T$.) Moreover it is also known that $|T|^p = U|T|^p U^+$ and $U^* |T|^p U = |T|^p$ hold for any $p > 0$. Hence

$$|\tilde{T}_{s,t}|^2 = \tilde{T}_{s,t}^* \tilde{T}_{s,t} = (|T|^s U|T|^t)(|T|^s U|T|^t) = (U^+ U)|T|^s U^+ |T|^t U |T|^t U^* = U^* (|T|^t U^* |T|^t) U$$

and

$$|\tilde{T}_{s,t}|^2 = \tilde{T}_{s,t}^* \tilde{T}_{s,t} = (|T|^s U|T|^t)(|T|^s U|T|^t) = |T|^s (|U|^s |T|^t U^* |T|^t) |T|^t = |T|^s |T^*|^2 |T|^t$$

hold. From the commutativity of $|T|$ and $|T^*|$, we obtain

$$|\tilde{T}_{s,t}|^{2n} = U^* |T^*|^n t |T|^t U |T|^t U^* |T^*|^n t U = U^* (|T^*|^n t U U^* |T^*|^n t U) |T|^t U^* |T^*|^n t U$$

and

$$|\tilde{T}_{s,t}|^{2n} = |T|^n |T|^t |T|^n |T|^t$$

for any $n \in \mathbb{N}$. Hence $\tilde{T}_{s,t}$ is $\infty$-hyponormal if and only if

$$|T|^n |T^*|^n t |T|^t U |T|^t U^* |T^*|^n t U$$

holds for any $n \in \mathbb{N}$. From the $\infty$-hyponormality of $T$, $|T|^p \geq |T|^p$ holds, and so $|T|^p \geq U|T|^p U^* |T|^p$ holds for any $p > 0$. So we obtain $U^* |T|^p U \geq |T|^p$ for any $p > 0$. Hence

$$|T|^n |T^*|^n t |T|^t U |T|^t U^* |T^*|^n t U \geq |T|^n |T|^t |T|^n |T|^t \geq |T|^n |T|^n |T|^n |T|^n$$

holds for any $n \in \mathbb{N}$. Hence $\tilde{T}_{s,t}$ is $\infty$-hyponormal. $\square$

Next we give an example of an $\infty$-hyponormal operator $T$ whose Aluthge transformation $\tilde{T}$ is also $\infty$-hyponormal but $|T||T^*| \neq |T^*||T|$. In [11, p. 365] the authors constructed an $\infty$-hyponormal operator $T$ satisfying $|T||T^*| \neq |T^*||T|$. To examine the $\infty$-hyponormality of the Aluthge transformation of $T$, we need to reconstruct $T$ into some operator matrix which is unitarily equivalent to $T$. The operator $T$ in the following Example 3.2 is such an operator matrix.

Example 3.2 Let $\{\lambda_n\}_{n=0}^\infty$ be a bounded increasing sequence of positive numbers with $\lambda_0 = \lambda_1$ and $\lambda_k < \lambda_{k+1}$ for every $k \geq 1$, and let

$$U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_{n+1} \end{pmatrix}$$
for $n \geq 0$ and set

$$U = \begin{pmatrix} \ddots & & & & & & & & \cdots \\ & 0 & & & & & & \cdots \\ & & U_0 & 0 & 0 & & & \cdots \\ & & 0 & U_0 & 0 & & \cdots \\ & & 0 & & U_0 & 0 & \cdots \\ & & \cdots & & \cdots & & \cdots & \end{pmatrix},$$

$$A = \begin{pmatrix} \ddots & & & & & & & & \cdots \\ & A_0 & & & & & & \cdots \\ & A_0 & A_0 & & & & & \cdots \\ & & & A_0 & & & & \cdots \\ & & & & A_1 & & & \cdots \\ & & & & & A_2 & & \cdots \\ & & & & & & \cdots & \end{pmatrix}.$$ 

Matrices $U, A$ are considered as bounded operators on $\bigoplus_{n=\infty}^{\infty} H_n$, where $H_n = \mathbb{C}^2$ for every $n \in \mathbb{N}$.

Consider the operator $T = UA$. Then $T = UA$ is the polar decomposition of $T$ since $U$ is unitary and hence $T^*T = AU^*UA = A^2$. Note that $|T| = A, |T^*| = (TT^*)^\frac{1}{2} = (UA^2U^*)^\frac{1}{2} = UA\;U^*$ hold. By a calculation, we obtain

$$|T^*| = UA\;U^* = \begin{pmatrix} \ddots & & & & & & & \cdots \\ & B_0 & & & & & & \cdots \\ & & B_0 & & & & & \cdots \\ & & & B_0 & & & & \cdots \\ & & & & B_0 & & & \cdots \\ & & & & & B_1 & & \cdots \\ & & & & & & \cdots & \end{pmatrix},$$

where

$$B_n = U_0A_nU_0^* = \frac{1}{2} \begin{pmatrix} \lambda_n + \lambda_{n+1} & \lambda_n - \lambda_{n+1} \\ \lambda_n - \lambda_{n+1} & \lambda_n + \lambda_{n+1} \end{pmatrix},$$

(Note that $B_0 = A_0$.)

First we show that $T$ is $\infty$-hyponormal. A simple calculation yields

$$A_n^k = \begin{pmatrix} \lambda_n^k & 0 \\ 0 & \lambda_{n+1}^k \end{pmatrix},$$

$$B_n^k = U_0A_n^kU_0^* = \frac{1}{2} \begin{pmatrix} \lambda_n^k + \lambda_{n+1}^k & \lambda_n^k - \lambda_{n+1}^k \\ \lambda_n^k - \lambda_{n+1}^k & \lambda_n^k + \lambda_{n+1}^k \end{pmatrix}.$$
and so
\[ A^k_{n+1} - B^k_n = \frac{1}{2} \left( \lambda^k_{n+1} - \lambda^k_n, \lambda^k_{n+1} - \lambda^k_n, 2 \lambda^k_{n+2} - \lambda^k_{n+1} - \lambda^k_n \right) \geq 0 \]
for \( n, k \geq 0 \). The last assertion follows from \( \text{tr}(A^k_{n+1} - B^k_n) = (\lambda^k_{n+2} - \lambda^k_n) \geq 0 \) and \( \det(A^k_{n+1} - B^k_n) = \frac{1}{2} (\lambda^k_{n+2} - \lambda^k_{n+1})(\lambda^k_{n+1} - \lambda^k_n) \geq 0 \), where \( \text{tr}(A^k_{n+1} - B^k_n) \) and \( \det(A^k_{n+1} - B^k_n) \) are the trace of \( (A^k_{n+1} - B^k_n) \) and the determinant of \( (A^k_{n+1} - B^k_n) \), respectively. Therefore \( |T|^k \geq |T^*|^k \) holds for any \( k \geq 0 \), and hence \( T \) is \( \infty \)-hyponormal.

Next we show that \( |T||T^*| \neq |T^*||T| \). By a calculation, we obtain
\[
A_{n+1}B_n = \frac{1}{2} \left( \lambda_{n+1}(\lambda_n + \lambda_{n+1}) \lambda_{n+1}(\lambda_n - \lambda_{n+1}) \lambda_{n+2}(\lambda_n + \lambda_{n+1}) \lambda_{n+2}(\lambda_n - \lambda_{n+1}) \right),
\]
\[
B_nA_{n+1} = \frac{1}{2} \left( \lambda_{n+1}(\lambda_n + \lambda_{n+1}) \lambda_{n+2}(\lambda_n - \lambda_{n+1}) \lambda_{n+1}(\lambda_n + \lambda_{n+1}) \lambda_{n+2}(\lambda_n - \lambda_{n+1}) \right).
\]
From the assumption, \( \lambda_{n+1} \neq \lambda_{n+2}, \lambda_{n+1} - \lambda_n > 0 \) and so \( B_nA_{n+1} \neq A_{n+1}B_n \) hold for every \( n \geq 1 \). Hence \( |T||T^*| \neq |T^*||T| \) holds.

Finally we show that the Aluthge transformation \( \tilde{T} \) of \( T \) is also \( \infty \)-hyponormal. By a calculation, we obtain the following equalities for \( \tilde{T} = |T||T^*|T^* = A^\dagger U A^\dagger \)
\[
\tilde{T}T^* = A^\dagger U A^\dagger T^* A^\dagger = \begin{pmatrix}
\cdot & A_0^2 & \cdot \\
A_0^2 & A_0^2 & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}
\]
\[
\tilde{T}^*\tilde{T} = A^\dagger T^* A^\dagger = \begin{pmatrix}
\cdot & A_0^2 & \cdot \\
A_0^2 & A_0^2 & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}
\]
where \( C_n = A_n^\dagger U_n^\dagger A_{n+1} U_n^\dagger A_{n+1} \), \( D_n = A_n^\dagger B_n A_n^\dagger \) for \( n \geq 0 \) under the convention that \( B_{-1} := B_0 = A_0 \). Hence \( \tilde{T} \) is \( \infty \)-hyponormal if and only if \( C_n^k \geq D_n^k \) for every \( n, k \geq 0 \). Let \( p_1(n), p_2(n) \) \( (p_1(n) \leq p_2(n)) \) be eigenvalues of \( 2C_n \), and let \( q_1(n), q_2(n) \) \( (q_1(n) \leq q_2(n)) \) be eigenvalues of \( 2D_n \). If \( q_2(n) \leq p_1(n) \) holds for any \( n \in \mathbb{N} \), then \( (2D_n)^k \leq (2C_n)^k \) and so \( D_n^k \leq C_n^k \) for every \( k \geq 0 \) by Lemma 2.1, and hence \( \tilde{T} \) is \( \infty \)-hyponormal. Therefore it suffices to show that \( q_2(n) \leq p_1(n) \) holds for every \( n \geq 0 \).

By a calculation,
\[
C_n = \frac{1}{2} \begin{pmatrix}
\lambda_n(\lambda_{n+1} + \lambda_{n+2}) \sqrt{\lambda_n} \lambda_{n+1}(\lambda_{n+1} - \lambda_{n+2}) \sqrt{\lambda_n} \\
\sqrt{\lambda_n} \lambda_{n+1}(\lambda_{n+1} - \lambda_{n+2}) \lambda_{n+1}(\lambda_{n+1} + \lambda_{n+2})
\end{pmatrix}.
\]
\[ D_0 = A_0^T A_0 = A_0^2, \]
\[ D_m = \frac{1}{2} \left( \begin{array}{cc}
\lambda_m (\lambda_{m+1} + \lambda_m) & \sqrt{\lambda_m \lambda_{m+1} (\lambda_m - \lambda_m)} \\
\lambda_m (\lambda_{m+1} + \lambda_m) & \lambda_{m+1} (\lambda_m - \lambda_m)
\end{array} \right) \]

for \( n \geq 0, m \geq 1 \). Solving the characteristic equation of matrices \( 2C_0, 2D_0 \), we obtain \( q_1^{(0)} = q_2^{(0)} = p_1^{(0)} = 2\lambda_1^2 \leq p_2^{(0)} = 2\lambda_1 \lambda_2 \). Next we fix an arbitrary \( n \) with \( n \geq 1 \). Then

\[
\text{tr}(2C_n) = (\lambda_n + \lambda_{n+1})(\lambda_{n+1} + \lambda_{n+2}), \quad \det(2C_n) = 4\lambda_n \lambda_{n+1}^2 \lambda_{n+2},
\]
\[
\text{tr}(2D_n) = (\lambda_n + \lambda_{n+1})(\lambda_n - \lambda_{n+1}), \quad \det(2D_n) = 4\lambda_n^2 \lambda_{n+1}^2.
\]

and

\[
p_1^{(n)} = \frac{\text{tr}(2C_n) - \sqrt{\text{tr}(2C_n)^2 - 4\det(2C_n)}}{2},
\]
\[
q_2^{(n)} = \frac{\text{tr}(2D_n) + \sqrt{\text{tr}(2D_n)^2 - 4\det(2D_n)}}{2}.
\]

From these expressions and the fact that \( \text{tr}(2C_n) \geq \text{tr}(2D_n) \), we can see that \( q_2^{(n)} \leq p_1^{(n)} \) is equivalent to

\[
\left( \sqrt{\text{tr}(2D_n)^2 - 4\det(2D_n)} + \sqrt{\text{tr}(2C_n)^2 - 4\det(2C_n)} \right)^2 \leq (\text{tr}(2C_n) - \text{tr}(2D_n))^2.
\]

So \( q_2^{(n)} \leq p_1^{(n)} \) holds if and only if

\[
2\sqrt{\text{tr}(2C_n)^2 - 4\det(2C_n)}\sqrt{\text{tr}(2D_n)^2 - 4\det(2D_n)} 
\leq (\text{tr}(2C_n) - \text{tr}(2D_n))^2 - (\text{tr}(2C_n)^2 - 4\det(2C_n)) - (\text{tr}(2D_n)^2 - 4\det(2D_n)).
\]

Hereafter, we denote the left-hand side and the right-hand side of (1) by (lhs) and (rhs), respectively. Moreover we introduce new variables \( a, x, y, z \) and setting

\[
\lambda_{n-1} = a, \quad \lambda_n = a + x, \quad \lambda_{n+1} = a + x + y, \quad \lambda_{n+2} = a + x + y + z.
\]

Since \( \{\lambda_n\}_{n=0}^\infty \) is an increasing sequence of positive numbers, \( a, x, y, z \geq 0 \). By using new variables, we can rewrite (rhs) as

\[
\text{(rhs)} = 16a^2 xy + 32a^2 y + 16xy^2 + 8a^2 y^2 + 36axy^2 + 28ay^2 + 8ay^3 + 12ay^3 + 8a^2x^2 + 16a^2 z^2 + 8a^2 z^2 + 16a^2 z^2 + 40axy + 162^3 y + 12a^2 z^2 + 14y^2 z,
\]

which shows that \( \text{rhs} \geq 0 \). Therefore \( q_2^{(n)} \leq p_1^{(n)} \) holds if and only if (lhs) \( \leq (\text{rhs})^2 \). By using new variables, we can write

\[
(\text{rhs})^2 - (\text{lhs})^2 = 192a^4 x^3 y + 768a^3 x^3 y^2 + 1152a^3 x^3 y^2 + 768a^3 x^3 y^2 + 192a^3 x^3 y^2 + 256a^4 xy^3 + 1472a^3 x^3 y^3 + 2880a^2 x^3 y^3 + 2368a^2 x^3 y^3 + 704x^3 y^3 + 640a^3 x^2 y^4 + 2240a^2 x^2 y^4 + 256a^2 x^2 y^4 + 960a^2 y^4 + 512a^2 x^2 y^4 + 1088a^2 x^2 y^4 + 576a^2 x^2 y^4 + 128a^4 x^3 y^5 + 128a^2 x^3 y^5 + 256a^4 x^3 y^5 + 256a^4 x^3 y^5 + 1024a^3 x^3 y^5 + 1536a^3 x^3 y^5 + 1024a^3 x^3 y^5 + 640a^4 x^2 y^6 + 3072a^3 x^2 y^6 + 5376a^2 x^2 y^6 + 4096a^4 x^2 y^6 + 1152a^4 x^3 y^7 + 256a^4 x^3 y^7 + 2304a^3 x^3 y^7 + 5696a^2 x^3 y^7 + 1856a^4 x^2 y^8 + 184a^4 x^2 y^8 + 5904a^3 x^3 y^8 + 2880a^2 x^3 y^8 + 128a^3 x^3 y^8 + 448a^4 x^2 y^9 + 320a^4 x^2 y^9 + 256a^4 x^2 y^9 + 1024a^3 x^2 y^9 + 1536a^2 x^2 y^9 + 1024a^3 x^2 y^9 + 256a^2 x^2 y^9 + 128a^3 x^2 y^9 + 2868a^3 x^2 y^9 + 2304a^3 x^2 y^9 + 704a^4 x^3 y^9 + 320a^4 x^3 y^9 + 128a^4 x^3 y^9 + 160a^4 x^3 y^9 + 640a^3 x^3 y^9 + 128a^2 x^2 y^9 + 320a x^2 y^9 + 192a x^2 y^9.
\]

Hence the inequality (lhs) \( \leq (\text{rhs})^2 \) holds and hence \( q_2^{(n)} \leq p_1^{(n)} \) holds for every \( n \geq 0 \) and so \( \hat{T} \) is \( \infty \)-hyponormal.
From the above Example 3.2, we see that the equality $|T||T^*| = |T^*||T|$ is not necessary for the Aluthge transformation $\tilde{T}$ of an \(\infty\)-hyponormal operator $T$ to be \(\infty\)-hyponormal. Hence there remains the problem of finding a necessary and sufficient condition for the Aluthge transformation of an \(\infty\)-hyponormal operator to be also \(\infty\)-hyponormal.

References


Shizuo MIYAJIMA
Department of Mathematics
Faculty of Science
Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku
Tokyo, 162 Japan
e-mail: miyajima@rs.kagu.sut.ac.jp

Isao SAITO
Department of Mathematics
Faculty of Science
Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku
Tokyo, 162 Japan
e-mail: saitoi@rs.kagu.sut.ac.jp