L-UP AND MIRROR ALGEBRAS

P. J. Allen, J. Neggers and Hee Sik Kim* 

Received April 16, 2003

Abstract. In this paper we consider several families of abstract algebras including the well-known BCK-algebras and several larger classes including the class of d-algebras which is a generalization of BCK-algebras. For these algebras it is usually difficult and often impossible to obtain a complementation operation and the associated “de Morgan’s laws”. In this paper we construct a “mirror image” of a given algebra which when adjoined to the original algebra permit a natural complementation to take place. The class of BCK-algebras is not closed under this operation but the class of d-algebras is, thus explaining why it may be better to work with this class rather than the class of BCK-algebras. Other classes of interest in this setting are also discussed.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. The present authors ([7]) introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([5]) introduced a new notion, called an BH-algebra, which is a generalization of BCH/BCI/BCK-algebras, and defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. Furthermore, they constructed the quotient BH-algebras via translation ideals and obtained the fundamental theorem of homomorphisms for BH-algebras as a consequence. The present authors ([8]) gave an analytic method for constructing proper examples of a great variety of non-associative algebras of the BCK-type and generalizations of these. In this paper we consider several families of abstract algebras including the well-known BCK-algebras and several larger classes including the class of d-algebras which is a generalization of BCK-algebras. For these algebras it is usually difficult and often impossible to obtain a complementation operation and the associated “de Morgan’s laws”. In this paper we construct a “mirror image” of a given algebra which when adjoined to the original algebra permit a natural complementation to take place. The class of BCK-algebras is not closed under this operation but the class of d-algebras is, thus explaining why it may be better to work with this class rather than the class of BCK-algebras. Other classes of interest in this setting are also discussed.

1991 Mathematics Subject Classification. 06F35.

Keywords and phrases. BCK/BCH/BH/d-algebras, Up-algebras, Mirror-algebras.

* Supported by the research fund of Hanyang University (HY-2002)
2. Up algebras.

Suppose that \((X; \ast, 0)\) is an algebra of type \((2,0)\) with \(T\) a subset of the following axioms:

(I) \(x \ast x = 0\),

(II) \(0 \ast x = 0\),

(III) \(x \ast y = 0\) and \(y \ast x = 0\) imply \(x = y\)

(IV) \(x \ast 0 = x\),

(V) \((x \ast y) \ast z = (x \ast z) \ast y\),

(VI) \((x \ast (x \ast y)) \ast y = 0\),

(VII) \(((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0\),

(VIII) \(x \ast y = 0 \Rightarrow x \ast (y \ast x) = x\),

for any \(x, y, z\) in \(X\).

In such a case we shall refer to \((X; \ast, 0)\) as a \(T\)-algebra. Using this device, we observe that we can deal simultaneously with statements concerning different classes of algebras.

Indeed, note that included are:

(1) \(d\)-algebra, when \(T_1 = \{(I), (II), (III)\}\),

(2) \(BH\)-algebra, when \(T_2 = \{(I), (II), (IV)\}\),

(3) \(d-BH\)-algebra, when \(T_3 = T_1 \cup T_2\),

(4) \(BCH\)-algebra, when \(T_4 = \{(I), (III), (V)\}\),

(5) \(BCI\)-algebra, when \(T_5 = \{(I), (III), (VI), (VII)\}\),

(6) \(BCK\)-algebra, when \(T_6 = \{(I), (II), (III), (VI), (VII)\}\).

The axioms for \(BCK\)-algebras are known to be independent ([6]). The following examples demonstrate further differences among classes of \(T_i\)-algebras for \(i = 1, \cdots, 6\).

**Example 2.1.** Let \(X := \{0, 1, 2, 3\}\) be a set with the following table:

<table>
<thead>
<tr>
<th>(\ast)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

It is easy to verify that \((X; \ast, 0)\) is a \(d-BH\)-algebra, but not a \(BCH\)-algebra, since \((2 \ast 3) \ast 2 = 1 \neq 0 = (2 \ast 2) \ast 3\).

**Example 2.2.** Let \(X = \{0, 1, 2, 3\}\) be a set with the following tables:

\[
\begin{array}{c|cccc}
\ast_1 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 3 & 0 & 2 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 3 \\
3 & 3 & 3 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\ast_2 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 3 \\
3 & 2 & 3 & 1 & 0 \\
\end{array}
\]

Then \((X; \ast_1, 0)\) is a \(BH\)-algebra, but not a \(d\)-algebra. At the same time, \((X; \ast_2, 0)\) is a \(d\)-algebra, but not a \(BH\)-algebra.
We introduce the following notations:

\[(x \wedge y)_L = x \ast (x \ast y)\]

and

\[(x \wedge y)_R = y \ast (y \ast x)\]

noting that in many situations, e.g., in Boolean algebras, \((x \wedge y)_L = (x \wedge y)_R = x \wedge y\) when \(x \ast y = x - y\) is the difference of sets. However, the relation \((x \wedge y)_L = (x \wedge y)_R\) does not hold in general, as follows from the example below:

**Example 2.3.** Let \(X := \{0, 1, 2, 3\}\) be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X; \ast, 0)\) is a \(d\)-algebra, and \((3 \wedge 2)_L = 3\), but \((3 \wedge 2)_R = 2\).

Given a \(T\)-algebra \((X; \ast, 0)\), it is said to be a left (resp., right) \(L\)-up algebra if there is defined an operation \((x \wedge y)_L\) such that \((x \wedge (x \vee y))_L = x\) (resp., \((x \vee y)_L \wedge y = y\)) for any \(x, y \in X\). An \(L\)-up algebra is both a left \(L\)-up algebra and a right \(L\)-up algebra. Similarly, \((X; \ast, 0)\) is said to be a left (resp., right) \(R\)-up algebra if there is defined an operation \((x \vee y)_R\) such that \((x \wedge (x \vee y))_R = x\) (resp., \((x \vee y)_R \wedge y = y\)) for any \(x, y \in X\). An \(R\)-up algebra is both a left \(R\)-up algebra and a right \(R\)-up algebra. An algebra \((X; \ast, 0)\) is a dual \(L\)-up algebra if \((x \wedge y)_L = x\) and \((y \wedge (x \vee y))_L = y\), for any \(x, y \in X\). An algebra \((X; \ast, 0)\) is said to be a dual \(R\)-up algebra if \((x \vee y)_R = x\) and \((y \vee (x \wedge y))_R = y\), for any \(x, y \in X\).

We observe several possibilities at work. First, note that \((x \wedge y)_L = x \ast (x \ast y) = (y \wedge x)_R\) in all cases. Now suppose that \((x \vee y)_L\) or \((x \vee y)_R\) have been obtained in some way. Then we define “conjugate symmetries” as follows:

\[
\begin{align*}
(x \vee y)_L := (x \vee y)_L, & \quad (x \wedge y)_R := (x \wedge y)_R; \\
(x \wedge y)_L := (y \vee x)_L, & \quad (x \vee y)_R := (y \vee x)_R; \\
(x \vee y)_L := (x \vee y)_R, & \quad (x \wedge y)_R := (x \wedge y)_L; \\
(x \wedge y)_L := (y \vee x)_R, & \quad (x \vee y)_R := (y \vee x)_L; \\
\end{align*}
\]

We construct a table for computation of conjugate symmetries as \((x \vee y)_L = (y \vee x)_L = (y \vee x)_R = (y \vee x)_L = (x \vee y)_R\), i.e., \(\vee \cdot \vee = \vee = \vee\) in this “multiplication” to obtain the Klein 4-group as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Suppose now that we start with an $L$-up algebra, i.e.,

$$x = (x \land (x \lor y))_L, \quad y = ((x \lor y)_L \land y)_L$$

for all $x, y \in X$. If we introduce $\frac{1}{\lor}$, then we obtain:

$$x = (x \land \frac{1}{\lor} (x \lor y)_L)_L, \quad y = ((y \lor \frac{1}{\lor} x)_L \land y)_L$$

and interchanging the rules of $x$ and $y$,

$$x = ((x \lor y)_L \land x)_L, \quad y = (y \land (x \lor y)_L)_L,$$

produces a dual $L$-up algebra. If we introduce $\frac{2}{\lor}$, then we obtain:

$$x = (x \land \frac{2}{\lor} y)_L, \quad y = ((x \lor y)_R \land y)_L$$

and thus

$$x = ((x \lor y)_R \land x)_R, \quad y = (y \land (x \lor y)_R)_R,$$

which yields a dual $R$-up algebra. Finally, via the introduction of $\frac{3}{\lor}$ we obtain:

$$x = (x \land \frac{3}{\lor} y)_L, \quad y = ((x \lor y)_R \land y)_L,$$

i.e.,

$$x = ((y \lor \frac{3}{\lor} x)_R \land x)_R, \quad y = (y \land (y \lor x)_R)_R,$$

and interchanging the roles of $x$ and $y$ we obtain:

$$x = (x \land \frac{3}{\lor} y)_R, \quad y = ((\frac{3}{\lor} y)_R \land y)_R,$$

which are precisely the conditions for an $R$-up algebra. Thus, we may construct a “symmetry diagram”:

```
            L - up  \quad \frac{1}{\lor} \quad \text{dual}L - up
```

```
            \frac{3}{\lor} \quad \frac{2}{\lor} \quad \frac{3}{\lor}  \\
```

```
            \frac{1}{\lor} \quad \text{dual}R - up \quad \text{R - up}
```

This does not mean that an $L$-up algebra is necessarily an $R$-up algebra or one of the other types of algebras. On the other hand, theorems and statements for $L$-up algebras have corresponding statements for $R$-up, dual $L$-up and dual $R$-up algebras via the scheme outlined above.
Proposition 2.4. Every bounded implicative $BCK$-algebra is an $L$-up algebra.

Proof. Since any bounded implicative $BCK$-algebra is a Boolean algebra (see [6, pp. 34]), $x \land y = \inf\{x, y\}$ and $x \lor y = \sup\{x, y\}$. Hence $x \land (x \lor y) = \inf\{x, \sup\{x, y\}\} = x$ and $(x \lor y) \land y = \inf\{\sup\{x, y\}, y\} = y$. □

Example 2.5. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; *, 0)$ is a $BCK$-algebra. If we define an $\land_L$-table and an $\lor_L$-table as follows:

<table>
<thead>
<tr>
<th>$\land_L$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lor_L$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

then it is an $L$-up algebra.

Example 2.6. Consider the following $BH$-algebra, which is not a $BCK/BCI$-algebra.

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

If we define an $\land_L$-table and an $\lor_L$-table as follows:

<table>
<thead>
<tr>
<th>$\land_L$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lor_L$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

then $(X; *, 0)$ is an $L$-up algebra.

3. Mirror Algebras

Suppose $(X; *, 0)$ is a $T$-algebra. Let $M(X) := X \times \{0, 1\}$ and define a binary operation “$*$” on $M(X)$ as follows:
\[(m_1). \quad (x,0) \ast (y,0) := (x \ast y, 0),\]
\[(m_2). \quad (x,1) \ast (y,1) := (y \ast x, 0),\]
\[(m_3). \quad (x,0) \ast (y,1) := ((x \land y)_L, 0) = (x \ast (x \ast y), 0),\]
\[(m_4). \quad (x,1) \ast (y,0) := \begin{cases} (y,1) & \text{when } x \ast y = 0, \\
(x,1) & \text{when } x \ast y \neq 0.
\end{cases}\]

Then we say that \(M(X) := (M(X); \ast, (0, 0))_L\) is a left mirror algebra of the \(T\)-algebra \(X\).

Similarly, if we define
\[(x,0) \ast (y,1) := ((x \land y)_R, 0) = (y \ast (y \ast x), 0)\]

then \(M(X) := (M(X); \ast, (0, 0))_R\) is a right mirror algebra of the \(T\)-algebra \(X\).

**Example 3.1.** Let \(X := \{0, 1, 2\}\) be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then we construct the mirror algebra \(M(X)\) of \(X\) as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
<td>e</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>0</td>
<td>d</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>e</td>
<td>e</td>
<td>0</td>
<td>e</td>
<td>b</td>
<td>e</td>
<td>0</td>
</tr>
</tbody>
</table>

where \(0 := (0,0), a := (0,1), b := (1,0), c := (1,1), d := (2,0)\) and \(e := (2,1)\).

**Proposition 3.2.** If \((X; \ast, 0)\) is a \(d\)-algebra then its mirror algebra \(M(X)\) is also a \(d\)-algebra.

**Proof.** Since \((x,1) \ast (y,0) \in \{(x,1), (y,1)\}, (x,1) \ast (y,0) = (0,0) = (y,0) \ast (x,1)\) is impossible. Hence \((x,i) \ast (y,j) = (y,j) \ast (x,i) = (0,0)\) means \(i = j\) and thus \(x \ast y = y \ast x = 0\) so that \(x = y\) as well. Hence, the condition (III) for \(d\)-algebras holds. Other conditions are easy to be checked, and omit the proof. It follows that \((M(X); \ast, (0,0))_L\) is a \(d\)-algebra. \(\Box\)

Similar argument can be used to demonstrate that \((M(X); \ast, (0,0))_R\) is also a \(d\)-algebra. We can easily prove the following proposition.

**Proposition 3.3.** If \((X; \ast, 0)\) is a \(d-BH\)-algebra then its mirror algebra \(M(X)\) is also a \(d-BH\)-algebra.

**Remark.** The mirror algebra \(M(X)\) of a \(BCK\)-algebra \((X; \ast, 0)\) need not be a \(BCK\)-algebra. Consider a \(BCK\)-algebra with the following table:
algebra. Moreover, the mirror algebra \( M \) that the conditions (I) and (IV) hold. If as follows:

\[
\text{BCH-} \text{algebra also. Consider a BCH-algebra also.}
\]

\[
\text{Since (I), (II) and (V) hold. Then by routine computation,}
\]

\[
\text{Theorem 3.4. Let (X; *, 0) be an algebra satisfying at least the conditions (I), (II), (IV), (V) and (VIII). Then its mirror algebra (X; *, 0) is not a BCH-algebra.}
\]

\[
\text{Proof. Given elements (x, i), (y, j) } \in M(X), \text{ it is enough to show that ((x, i) } \wedge (x, i) \lor (y, j))_L = (x, i), \text{ where i, j } \in \{0, 1\}. \text{ We consider 4 cases. Case(1). i = j = 0. We assume that the conditions (I) and (IV) hold. If x * (y * x) = 0 then ((x, 0) } \wedge (x, 0) \lor (y, 0))_L = ((x, 0) \wedge (y * x, 0))_L = (x, 0) * (x, 0) * (y * x, 0) = (x, 0) * (x * (y * x), 0) = (x, 0) * (0, 0) = (x, 0).}
\]

\[
\text{If x * (y * x) } \neq 0, \text{ then ((x, 0) } \wedge (x, 0) \lor (y, 0))_L = ((x, 0) \wedge (x, 0))_L = (x, 0) * (x, 0) * (x, 0) = (x, 0) * (x * x, 0) = (x, 0) * (0, 0) = (x, 0). \text{ Case(2). i = j = 1. We assume the conditions (I), (II), (IV) and (V) hold. Then, by routine computation,}
\]

\[
\text{((x, 1) } \wedge (x, 1) \lor (y, 1))_L =
\]

\[
\begin{cases}
(x, 1) & \text{if } x \neq 0, \\
(x, 1) & \text{otherwise.}
\end{cases}
\]

\[
\text{We know that}
\]

\[
x * [(x * (x * y)) * x] = x * [(x * x) * (x * y)] \quad \text{[by (V)]}
\]

\[
= x * (0 * (x * y)) \quad \text{[by (I)]}
\]

\[
= x * 0 \quad \text{[by (II)]}
\]

\[
= x. \quad \text{[by (IV)]}
\]

\[
\text{Hence ((x, 1) } \wedge (x, 1) \lor (y, 1))_L = (x, 1) \text{ in any cases. Case (3). i = 1 and j = 0. Assume the conditions (I), (II) and (V) hold. Then}
\]

\[
x * [(x \wedge y)_L * x] = x * [(x * (x * y)) * x
\]

\[
= x * ((x * x) * (x * y))
\]

\[
= x * (0 * (x * y))
\]

\[
= x.
\]
Hence
\[(x, 1) \land ((x, 1) \lor (y, 0))_L = (x, 1) \ast ((x \land y)_L \ast x, 0)\]
\[\begin{cases}
(x \land y)_L \ast x, 1 & \text{if } x \ast [(x \land y)_L \ast x] = 0,
(x, 1) & \text{otherwise}
\end{cases}\]
\[\begin{cases}
(0 \land y)_L \ast 1, 1 & \text{if } x = 0,
(x, 1) & \text{otherwise}
\end{cases}\]
\[= (x, 1).
\]

Case (4). \(i = 0\) and \(j = 1\). Assume the conditions (I), (IV) and (VIII) hold. If \(x \ast y = 0\), then by (VIII) \(x = x \ast (y \ast x)\), and hence \(x \ast [(x \ast (y \ast x))] = x \ast (x \ast x) = x \ast 0 = x\). It follows that
\[(x, 0) \land ((x, 0) \lor (y, 1))_L = \begin{cases}
(x, 0) \land (y \ast x, 0) & \text{if } x \ast y = 0,
(x, 1) \land (0, 1) & \text{otherwise}
\end{cases}\]
\[\begin{cases}
(x \ast [x \ast (y \ast x)], 0) & \text{if } x \ast y = 0,
(x \ast (x \ast 0), 0) & \text{otherwise}
\end{cases}\]
\[= (x, 0),
\]
proving the theorem. \(\square\)

Since every implicative BCK-algebra satisfies all conditions described in Theorem 3.4, we give the following corollary:

**Corollary 3.5.** If \((X; \ast, 0)\) is an implicative BCK-algebra, then its mirror algebra \(M(X)\) is a left \(L\)-up algebra.

**References**


P. J. Allen and J. Neggers
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350
U. S. A.
{pjallen}{jneggers}@gp.as.ua.edu

Hee Sik Kim
Department of Mathematics
Hanyang University
Seoul 133-791, Korea
heekim@hanyang.ac.kr