FOLDING THEORY APPLIED TO SOME TYPES OF POSITIVE IMPLICATIVE HYPER BCK-IDEALS IN HYPER BCK-ALGEBRAS

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Abstract. The foldness of $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \ll)_{BCK}$-ideals is considered. The fuzzy version of such notions is also discussed.

1. INTRODUCTION

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) is introduced in 1934 by F. Marty [9] at the 8th congress of Scandinavian Mathematicians. In [8], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyper BCK-ideal and a weak hyper BCK-ideal, and gave relations between hyper BCK-ideals and weak hyper BCK-ideals. Y. B. Jun et al. [7] gave a condition for a hyper BCK-algebra to be a BCK-algebra, and introduced the notion of a strong hyper BCK-ideal, a weak hyper BCK-ideal and a reflexive hyper BCK-ideal. They showed that every strong hyper BCK-ideal is a hypersubalgebra, a weak hyper BCK-ideal and a hyper BCK-ideal; and every reflexive hyper BCK-ideal is a strong hyper BCK-ideal. In [4], Y. B. Jun and X. L. Xin introduced the notion of an implicative hyper BCK-ideal. They gave the relations among hyper BCK-ideals, implicative hyper BCK-ideals and positive implicative hyper BCK-ideals. They stated some characterizations of implicative hyper BCK-ideals. And they also introduced the notion of implicative hyper BCK-algebras and investigated the relation between implicative hyper BCK-ideals and implicative hyper BCK-algebras. In [5], Y. B. Jun and X. L. Xin introduced the notion of a positive implicative hyper BCK-ideal, and investigated some related properties. Y. B. Jun and W. H. Shim [1] discussed several types of positive implicative hyper BCK-ideals in hyper BCK-algebras, and investigated their relations. In this paper we consider the foldness of $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \ll)_{BCK}$-ideals in hyper BCK-algebras, and discuss their fuzzy version.

2. PRELIMINARIES

We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [3] and [8]. Let $H$ be a nonempty set endowed with a hyper operation $\circ$, that is, $\circ$ is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\emptyset\}$. For

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two subsets $A$ and $B$ of $H$, denote by $A \circ B$ the set \( \bigcup_{a \in A, b \in B} a \circ b \). We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a hyper BCK-algebra we mean a nonempty set $H$ endowed with a hyper operation “$\circ$” and a constant $0$ satisfying the following axioms:

(K1) $x \circ (y \circ z) \leq x \circ y$,
(K2) $x \circ y = (x \circ z) \circ y$,
(K3) $x \circ H \subseteq \{x\}$,
(K4) $x \leq y$ and $y \leq x$ imply $x = y$,

for all $x, y, z \in H$, where $x \leq y$ is defined by $0 \leq x \circ y$ and for every $A, B \subseteq H$, $A \leq B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \leq b$.

In any hyper BCK-algebra $H$, the following hold (see [3] and [8]):

1. $0 \leq x$,
2. $A \subseteq B$ implies $A \leq B$,
3. $x \circ 0 = \{x\}$ and $A \circ 0 = A$

for all $x, y, z \in H$ and for all nonempty subsets $A, B$ and $C$ of $H$. In what follows let $H$ denote a hyper BCK-algebra unless otherwise specified.

**Definition 2.1.** [8] A nonempty subset $A$ of $H$ is called a hyper BCK-ideal of $H$ if it satisfies the following conditions:

1. $0 \in A$,
2. $\forall x, y \in H$ $(x \circ y \leq A$, $y \in A \Rightarrow x \in A)$.

**Definition 2.2.** [8] A nonempty subset $A$ of $H$ is called a weak hyper BCK-ideal of $H$ if it satisfies (I1) and

3. $\forall x, y \in H$ $(x \circ y \subseteq A$, $y \in A \Rightarrow x \in A)$.

**Definition 2.3.** [1] A nonempty subset $A$ of $H$ is called a $PI(\leq, \subseteq, \subseteq)$ BCK-ideal of $H$ if it satisfies (I1) and

4. $\forall x, y, z \in H$ $((x \circ y) \circ z \leq A$, $y \circ z \subseteq A \Rightarrow x \circ z \subseteq A)$.

We place a bar over a symbol to denote a fuzzy set so $\bar{A}$, $\bar{B}$, $\cdots$ all represent fuzzy sets in $H$. We write $\bar{A}(x)$, a number in $[0, 1]$, for the membership function of $\bar{A}$ evaluated at $x \in H$. An $\alpha$-cut of $\bar{A}$, written $\bar{A}[\alpha]$, is defined as

$$\{x \in H \mid \bar{A}(x) \geq \alpha\} \text{ for } 0 < \alpha \leq 1.$$  

We separately specify $\bar{A}[0]$ as the closure of the union of all the $\bar{A}[\alpha]$ for $0 < \alpha \leq 1$.

**Definition 2.4.** [6] A fuzzy set $\bar{A}$ in $H$ is called a fuzzy hyper BCK-ideal of $H$ if it satisfies:

1. $\forall x, y \in H$ $(x \leq y \Rightarrow \bar{A}(x) \geq \bar{A}(y))$
2. $\forall x, y \in H$ $\left(\bar{A}(x) \geq \min_{a \in \pi(x)} \bar{\bar{A}}(a), \bar{A}(y)\right)$.

**Proposition 2.5.** [6] A fuzzy set $\bar{A}$ in $H$ is a fuzzy hyper BCK-ideal of $H$ if and only if the level set $\bar{A}[\alpha]$, $\alpha \in \text{Im}(\bar{A})$, of $\bar{A}$ is a hyper BCK-ideal of $H$.

**Definition 2.6.** [2] A fuzzy set $\bar{A}$ in $H$ is called a fuzzy $PI(\leq, \subseteq, \subseteq)$ BCK-ideal of $H$ if it satisfies (F1) and

3. $\forall x, y, z \in H$ $\left(\inf_{a \in \pi(x \circ y)} \bar{\bar{A}}(a) \geq \min_{b \in \pi(y \circ z)} \bar{\bar{A}}(b), \inf_{c \in \pi(z)} \bar{\bar{A}}(c)\right)$.
3. Folding Theory of Some Types of Positive Implicative Hyper BCK-ideals

For any $x, y \in H$ and any natural number $n$, denote
\[ x \circ y^n = \underbrace{(\cdots ((x \circ y) \circ y) \cdots ) \circ y}_{n\text{-times}} \]

**Definition 3.1.** Let $k$, $m$, and $n$ be natural numbers. A nonempty subset $A$ of $H$ is called a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$ if it satisfies (I1) and (I5) \( \forall x, y, z \in H \) \( ((x \circ y) \circ z^k \ll A, y \circ z^m \subseteq A \Rightarrow x \circ z^n \subseteq A) \).

**Example 3.2.** Let $H = \{0, a, b\}$ be a hyper $BCK$-algebra with the following Cayley table:

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</table>

Then $A = \{0, a\}$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$ for natural numbers $k$, $m$, and $n$.

**Example 3.3.** Let $H = \{0, a, b\}$ be a hyper $BCK$-algebra with the following Cayley table:

<table>
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Then $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$ for natural numbers $k$, $m$, and $n > 2$. But it is not a $(2, 3; 1)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$ since \( (b \circ a) \circ a^2 = \{0\} \ll A \) and \( a \circ a^3 = \{0\} \subseteq A \), but \( b \circ a^1 = \{a\} \not\subseteq A \).

**Theorem 3.4.** Every $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal is a hyper $BCK$-ideal for natural numbers $k$, $m$, and $n$.

**Proof.** Let $A$ be a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$. Let $x, y \in H$ be such that $x \circ y \ll A$ and $y \in A$. Putting $z = 0$ in (I5), we get $(x \circ y) \circ 0^k = x \circ y \ll A$ and $y \circ 0^m = \{y\} \subseteq A$. It follows from (I5) that $\{x\} = x \circ 0^n \subseteq A$, i.e., $x \in A$. Hence $A$ is a hyper $BCK$-ideal of $H$. \( \square \)

The converse of Theorem 3.4 may not be true. In fact, consider the hyper $BCK$-algebra $H$ in Example 3.3. Then $A := \{0\}$ is a hyper $BCK$-ideal of $H$. But it is not a $(k, m; 1)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$ for $k \geq 2$ because \( (b \circ 0) \circ a^k = \{0\} \ll A \) and \( 0 \circ a^m = \{0\} \subseteq A \), but \( b \circ a^1 = \{a\} \not\subseteq A \).

**Theorem 3.5.** For natural number $m$, let $A$ be a $(m, m; m)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$. Then, for $w \in H$, the set
\[ A_w := \{x \in H \mid x \circ w^m \subseteq A\} \]
is a weak hyper $BCK$-ideal of $H$.

**Proof.** Obviously $0 \in A_w$. Let $x, y \in H$ be such that $x \circ y \subseteq A_w$ and $y \in A_w$. Then $(x \circ y) \circ w^m \subseteq A$ and $y \circ w^m \subseteq A$, which implies that $(x \circ y) \circ w^m \ll A$ and $y \circ w^m \subseteq A$. It follows from (I5) that $x \circ w^m \subseteq A$ or equivalently $x \in A_w$. Therefore $A_w$ is a weak hyper $BCK$-ideal of $H$. \( \square \)

**Lemma 3.6.** [3] Let $A$ be a subset of $H$. If $I$ is a hyper $BCK$-ideal of $H$ such that $A \ll I$, then $A$ is contained in $I$. 
Theorem 3.7. Let $A$ be a hyper $BCK$-ideal of $H$. If

$$A_w := \{ x \in X | x \circ w^m \subseteq A \}$$

is a weak hyper $BCK$-ideal of $H$ for all $w \in H$, then $A$ is a $(m, m; m)$-fold $PI(\ll, \subseteq, \subseteq)_BCK$-ideal of $H$.

Proof. Let $x, y, z \in H$ be such that $(x \circ y) \circ z^m \ll A$ and $y \circ z^m \subseteq A$. Then $(x \circ y) \circ z^m \subseteq A$ by Lemma 3.6 and $y \in A_z$. Thus for each $t \in x \circ y$, we have $t \circ z^m \subseteq A$ or equivalently $t \in A_z$. Since $A_z$ is a weak hyper $BCK$-ideal of $H$, we get $x \in A_z$, i.e., $x \circ z^m \subseteq A$. Therefore $A$ is a $(m, m; m)$-fold $PI(\ll, \subseteq, \subseteq)_BCK$-ideal of $H$. \hfill \Box

Definition 3.8. Let $k, m,$ and $n$ be natural numbers. A nonempty subset $A$ of $H$ is called a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_BCK$-ideal of $H$ if it satisfies (I1) and

(I6) $\forall x, y, z \in H (\ (x \circ y) \circ z^k \ll A, y \circ z^m \ll A \Rightarrow x \circ z^n \ll A)$.

The following example shows that there is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_BCK$-ideal which is not a hyper $BCK$-ideal.

Example 3.9. Let $H = \{0, a, b\}$ be a hyper $BCK$-algebra with the following Cayley table:

\[
\begin{array}{c|ccc}
\circ & 0 & a & b \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
a & \{a\} & \{0, a\} & \{0, a\} \\
b & \{b\} & \{a, b\} & \{0, a, b\}
\end{array}
\]

Then $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_BCK$-ideal of $H$ for natural numbers $k$, $m$ and $n$, but not a hyper $BCK$-ideal of $H$ since $a \circ b = \{0, a\} \ll A$ and $b \in A$, but $a \notin A$.

Definition 3.10. [1] A nonempty subset $A$ of $H$ is said to be closed if for every $x, y \in H$, $x \ll y$ and $y \in A$ imply $x \in A$.

Example 3.11. In Example 3.9, $A := \{0, a\}$ is closed, which is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_BCK$-ideal of $H$.

Theorem 3.12. Every closed $(k, m; n)$-fold $PI(\ll, \ll, \ll)_BCK$-ideal is a hyper $BCK$-ideal.

Proof. Let $A$ be a closed $(k, m; n)$-fold $PI(\ll, \ll, \ll)_BCK$-ideal of $H$ and let $x, y \in H$ be such that $x \circ y \ll A$ and $y \in A$. Taking $z = 0$ in (I6) and using (p3), we have $(x \circ y) \circ 0^k = x \circ y \ll A$ and $y \circ 0^n = \{y\} \ll A$. It follows from (I6) and (p3) that $(x) = x \circ 0^n \ll A$ so that there exists $u \in A$ such that $x \ll u$. Since $A$ is closed, we get $x \in A$. Hence $A$ is a hyper $BCK$-ideal of $H$. \hfill \Box

4. FUZZIFICATION OF FOLDING THEORY APPLIED TO SOME TYPES OF POSITIVE IMPLICATIVE HYPER $BCK$-IDEALS

Definition 4.1. Let $k, m,$ and $n$ be natural numbers. A fuzzy set $\tilde{A}$ in $H$ is called a $(k, m; n)$-fold fuzzy positive implicative ideal of $H$ if it satisfies (F1) and

(F4) $\forall x, y, z \in H \left( \inf_{a \in x \circ z^k} \tilde{A}(a) \geq \min \left\{ \inf_{b \in (x \circ y) \circ z^m} \tilde{A}(b), \inf_{c \in y \circ z^n} \tilde{A}(c) \right\} \right)$

Note that $(1, 1; 1)$-fold fuzzy positive implicative ideal is a fuzzy $PI(\ll, \subseteq, \subseteq)_BCK$-ideal.

Example 4.2. Let $H = \{0, a, b\}$ be a hyper $BCK$-algebra in Example 3.2. Define a fuzzy set $\tilde{A}$ in $H$ by $\tilde{A}(0) = \tilde{A}(a) = 0.7$ and $\tilde{A}(b) = 0.07$. It is easily checked that, for natural numbers $k$, $m$, and $n$, $\tilde{A}$ is a $(k, m; n)$-fold fuzzy positive implicative ideal of $H$.

Theorem 4.3. Every $(k, m; n)$-fold fuzzy positive implicative ideal is a fuzzy hyper $BCK$-ideal, where $k, m$ and $n$ are natural numbers.
Proof. Let $\bar{A}$ be a $(k,m;n)$-fold fuzzy positive implicative ideal of $H$ and let $x,y \in H$. Taking $z = 0$ in (F4) and using (p3), we have

\[
\bar{A}(x) = \inf_{a \in \mathbb{R}^{0 \times k}} \bar{A}(a) \\
\geq \min \left\{ \inf_{b \in (x \circ y)^{0 \times m}} \bar{A}(b), \inf_{c \in y^{0 \times n}} \bar{A}(c) \right\} \\
= \min \left\{ \inf_{b \in x \circ y} \bar{A}(b), \bar{A}(y) \right\}.
\]

Hence $\bar{A}$ is a fuzzy hyper $BCK$-ideal of $H$.

The converse of Theorem 4.3 may not be true as seen in the following example.

**Example 4.4.** Let $H = \{0, a, b, c\}$ be a hyper $BCK$-algebra with the following Cayley table:

<table>
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<tr>
<th></th>
<th>0</th>
<th>a</th>
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<th>c</th>
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<td>c</td>
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Define a fuzzy set $\bar{A}$ in $H$ by $\bar{A}(0) = \bar{A}(a) = 0.6$ and $\bar{A}(b) = \bar{A}(c) = 0.07$. Then $\bar{A}$ is a fuzzy hyper $BCK$-ideal. But it is not a $(1,2;3)$-fold fuzzy positive implicative ideal of $H$, since

\[
\inf_{u \in \text{coba}} \bar{A}(u) = 0.07 \nless 0.6 = \min \left\{ \inf_{v \in (\text{coba})^{0 \times m}} \bar{A}(v), \inf_{w \in (\text{coba})^{0 \times n}} \bar{A}(w) \right\}.
\]

**Example 4.5.** Let $H = \{0, a, b\}$ be a hyper $BCK$-algebra in Example 3.3. Define a fuzzy set $\bar{A}$ in $H$ by $\bar{A}(0) = 0.5$ and $\bar{A}(a) = \bar{A}(b) = 0.3$. Then $\bar{A}$ is a fuzzy hyper $BCK$-ideal, but not a $(1, m; n)$-fold fuzzy positive implicative ideal of $H$ for natural numbers $m \geq 2$ and $n$,

\[
\inf_{u \in \text{boaa}} \bar{A}(u) = 0.3 \nless 0.5 = \min \left\{ \inf_{v \in (\text{boaa})^{0 \times m}} \bar{A}(v), \inf_{w \in (\text{boaa})^{0 \times n}} \bar{A}(w) \right\}.
\]

**Theorem 4.6.** If $\bar{A}$ is a $(k, m; n)$-fold fuzzy positive implicative ideal of $H$, then the $\alpha$-cut $\bar{A}[\alpha]$ of $\bar{A}$ is an $(m, n; k)$-fold $P_{I}(\ll, \subseteq, \subseteq)$ $BCK$-ideal of $H$, where $\alpha \in \text{Im} \bar{A}$.

**Proof.** Let $\bar{A}$ be a $(k, m; n)$-fold fuzzy positive implicative ideal of $H$ and let $\alpha \in \text{Im} \bar{A}$. Both (p1) and (F1) induce the inequality $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$, and so $0 \in \bar{A}[\alpha]$. Let $x, y, z \in H$ be such that $(x \circ y) \circ z^{m} \ll \bar{A}[\alpha]$ and $y \circ z^{n} \subseteq \bar{A}[\alpha]$. Then for every $a \in (x \circ y) \circ z^{m}$, there exists $a' \in \bar{A}[\alpha]$ such that $a \ll a'$, and therefore $\bar{A}(a) \geq \bar{A}(a')$ by (F1). Hence $\bar{A}(a) \geq \alpha$ for all $a \in (x \circ y) \circ z^{m}$. It follows from (F4) that, for every $b \in x \circ z^{k}$,

\[
\bar{A}(b) \geq \inf_{c \in x \circ z^{k}} \bar{A}(c) \geq \inf_{u \in (x \circ y)^{0 \times m}} \bar{A}(u), \inf_{v \in y^{0 \times n}} \bar{A}(v) \geq \alpha
\]

so that $b \in \bar{A}[\alpha]$, that is, $x \circ z^{k} \subseteq \bar{A}[\alpha]$. Consequently, $\bar{A}[\alpha]$ is a $(m, n; k)$-fold $P_{I}(\ll, \subseteq, \subseteq)$ $BCK$-ideal of $H$.

We now consider the converse of Theorem 4.6.

**Theorem 4.7.** Let $\bar{A}$ be a fuzzy set in $H$ such that $\bar{A}[\alpha], \alpha \in \text{Im} \bar{A}$, is an $(m, n; k)$-fold $P_{I}(\ll, \subseteq, \subseteq)$ $BCK$-ideal of $H$. Then $\bar{A}$ is a $(k, m; n)$-fold fuzzy positive implicative ideal of $H$.\]
Theorem 4.8. If $\bar{A}[\alpha]$, $\alpha \in \text{Im}(\bar{A})$, is a $(m,n;k)$-fold fuzzy $BCK$-ideal of $H$, then $\bar{A}[\alpha]$ is a hyper $BCK$-ideal of $H$ by Theorem 3.4. It follows from Proposition 2.5 that $\bar{A}$ is a fuzzy hyper $BCK$-ideal of $H$, and so the condition (F1) is valid. Now let $\alpha = \min \left\{ \inf_{a \in (x \circ y) \circ z^m} \bar{A}(u), \inf_{v \in y \circ z^n} \bar{A}(v) \right\}$. Then $\bar{A}(u') \geq \inf_{u \in (x \circ y) \circ z^m} \bar{A}(u)$ and $\bar{A}(v') \geq \inf_{v \in y \circ z^n} \bar{A}(v)$ for all $u' \in (x \circ y) \circ z^m$ and $v' \in y \circ z^n$. Hence $u', v' \in \bar{A}[\alpha]$, which implies that $(x \circ y) \circ z^m \subseteq \bar{A}[\alpha]$, hence $(x \circ y) \circ z^m \ll \bar{A}[\alpha]$, and $y \circ z^n \subseteq \bar{A}[\alpha]$. Since $\bar{A}[\alpha]$ is a $(m,n;k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$, it follows from (I5) that $x \circ z^k \subseteq \bar{A}[\alpha]$ so that $\bar{A}(d) \geq \alpha$ for all $d \in x \circ z^k$. Consequently,

$$\inf_{d \in x \circ z^k} \bar{A}(d) \geq \alpha = \min \left\{ \inf_{u \in (x \circ y) \circ z^m} \bar{A}(u), \inf_{v \in y \circ z^n} \bar{A}(v) \right\}.$$ 

Thus $\bar{A}$ is a $(k,m,n)$-fold fuzzy positive implicative ideal of $H$. $\square$

Theorem 4.8. If $\bar{A}$ is a $(k,m;n)$-fold fuzzy positive implicative ideal of $H$, then $\bar{A}[\alpha]$, $\alpha \in \text{Im}(\bar{A})$, is an $(m,n;k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$.

Proof. Let $\bar{A}$ be a $(k,m;n)$-fold fuzzy positive implicative ideal of $H$ and let $\alpha \in \text{Im}(\bar{A})$. Both (p1) and (F1) induce the inequality $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$, and $0 \in \bar{A}[\alpha]$. Let $x,y,z \in H$ be such that $(x \circ y) \circ z^m \ll \bar{A}[\alpha]$ and $y \circ z^n \ll \bar{A}[\alpha]$. Then for every $a \in (x \circ y) \circ z^m$ and $b \in y \circ z^n$, there exists $a', b' \in \bar{A}[\alpha]$ such that $a \ll a'$ and $b \ll b'$. It follows that $\bar{A}(a) \geq \bar{A}(a') \geq \alpha$ and $\bar{A}(b) \geq \bar{A}(b') \geq \alpha$ for all $a \in (x \circ y) \circ z^m$ and $b \in y \circ z^n$. Hence $\inf_{a \in (x \circ y) \circ z^m} \bar{A}(a) \geq \alpha$ and $\inf_{b \in y \circ z^n} \bar{A}(b) \geq \alpha$. Using (F4), we get for every $c \in x \circ z^k$,

$$\bar{A}(c) \geq \inf_{u \in x \circ z^k} \bar{A}(u) \geq \min \left\{ \inf_{a \in (x \circ y) \circ z^m} \bar{A}(a), \inf_{b \in y \circ z^n} \bar{A}(b) \right\} \geq \alpha,$$

and thus $c \in \bar{A}[\alpha]$. This shows that $x \circ z^k \subseteq \bar{A}[\alpha]$, and thus $x \circ z^k \ll \bar{A}[\alpha]$ by (p2). Therefore $\bar{A}[\alpha]$ is an $(m,n;k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$. $\square$

Now we consider the converse of Theorem 4.8. Let $\bar{A}$ be a fuzzy set in $H$ such that $\bar{A}[\alpha]$, $\alpha \in \text{Im}(\bar{A})$, is a closed $(k,m;n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$. Then $\bar{A}$ is an $(n,k;m)$-fold fuzzy positive implicative ideal of $H$.

Proof. Assume that for $\alpha \in \text{Im}(\bar{A})$, $\bar{A}[\alpha]$ is a closed $(k,m;n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of $H$. Then $\bar{A}[\alpha]$ is a hyper $BCK$-ideal of $H$ (see Theorem 3.12). It follows from Proposition 2.5 that $\bar{A}$ is a fuzzy hyper $BCK$-ideal of $H$ so that the condition (F1) holds. Let $x,y,z \in H$ and let

$$\beta := \min \left\{ \inf_{b \in (x \circ y) \circ z^k} \bar{A}(b), \inf_{c \in y \circ z^n} \bar{A}(c) \right\}.$$ 

Then for each $b' \in (x \circ y) \circ z^k$ and $c' \in y \circ z^n$, we have $\bar{A}(b') \geq \inf_{b \in (x \circ y) \circ z^k} \bar{A}(b) \geq \beta$ and $\bar{A}(c') \geq \inf_{c \in y \circ z^n} \bar{A}(c) \geq \beta$. Hence $b', c' \in \bar{A}[\beta]$, and so $(x \circ y) \circ z^k \subseteq \bar{A}[\beta]$ and $y \circ z^n \subseteq \bar{A}[\beta]$. Using (p2), we get $(x \circ y) \circ z^k \ll \bar{A}[\beta]$ and $y \circ z^n \ll \bar{A}[\beta]$, and therefore $x \circ z^n \ll \bar{A}[\beta]$ by (I6). It follows from Lemma 3.6 that $x \circ z^n \subseteq \bar{A}[\beta]$. Thus $\bar{A}(d) \geq \beta$ for all $d \in x \circ z^n$, and so

$$\inf_{a \in x \circ z^n} \bar{A}(a) \geq \beta = \min \left\{ \inf_{b \in (x \circ y) \circ z^k} \bar{A}(b), \inf_{c \in y \circ z^n} \bar{A}(c) \right\}.$$ 

Consequently, $\bar{A}$ is an $(n,k;m)$-fold fuzzy positive implicative ideal of $H$. $\square$
References


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