TANGENTIAL BOUNDARY BEHAVIOR OF THE POISSON INTEGRALS OF FUNCTIONS IN THE POTENTIAL SPACE WITH THE ORLICZ NORM

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Abstract. Nagel, Rudin and Shapiro (1982) investigated the tangential boundary behavior of the Poisson integrals of functions in the potential space $L^p_K(\mathbb{R}^n) = \{K*F : F \in L^p(\mathbb{R}^n)\}$ with a kernel $K : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$ which is positive, integrable, radial and decreasing. In this paper, we extend the result to $L^\Phi_K(\mathbb{R}^n) = \{K * F : F \in L^\Phi(\mathbb{R}^n)\}$, where $L^\Phi(\mathbb{R}^n)$ is the Orlicz space. Moreover we introduce Ω_R-limit for a continuous increasing function $R : [0,+\infty) \to [0,+\infty)$ with $R(y) \to 0$ as $y \to 0$. The tangential approach region is defined by the function R. We give a relation between $R(y)$ for which all functions in $L^\Phi_K(\mathbb{R}^n)$ have the Ω_R-limit and the $L^{\widetilde{\Phi}}$-norm of $P_y * K$, where $\widetilde{\Phi}$ is the complementary function of Φ, and calculate $R(y)$ precisely.

1. Introduction

It is well known that, for $f \in L^p(\mathbb{R}^n)$, its Poisson integral $u(x,y) = P_y * f(x)$, $x \in \mathbb{R}^n$, $y > 0$, converges nontangentially to $f(x)$ a.e. when y tends to 0. It is also well known that, for general $f \in L^p(\mathbb{R}^n)$, convergence fails when the approach regions have a certain degree of tangentiality. The tangential boundary behavior of the Poisson integrals of functions in subspaces of $L^p(\mathbb{R}^n)$ was studied by Nagel, Rudin and Shapiro [4], Nagel and Stein [5], Dorronsoro [1], etc.

Nagel, Rudin and Shapiro [4] investigated the potential space $L^p_K(\mathbb{R}^n) = \{K * F : F \in L^p(\mathbb{R}^n)\}$ with a kernel $K : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$ which is positive, integrable, radial and decreasing. We note that $L^p_K(\mathbb{R}^n)$ is a subspace of $L^p(\mathbb{R}^n)$. They [4] gave the relation between the geometric properties of approach regions on which the tangential limit of $P_y * f$ exists for all $f \in L^p_K$ and the $L^{p'}$-norm of $P_y * K$ with $K \notin L^{p'}$, where $1/p + 1/p' = 1$.

In this paper, we extend the result in [4] to $L^\Phi_K(\mathbb{R}^n) = \{K * F : F \in L^\Phi(\mathbb{R}^n)\}$, where $L^\Phi(\mathbb{R}^n)$ is the Orlicz space, and give the relation between the geometric properties of approach regions and the $L^{\widetilde{\Phi}}$-norm of $P_y * K$ with $K \notin L^{\widetilde{\Phi}}$, where $\widetilde{\Phi}$ is the complementary function of Φ. However, the $L^{\widetilde{\Phi}}$-norm of $P_y * K$ is not simple. So we introduce Ω_R-limit for a continuous increasing function $R : [0,+\infty) \to [0,+\infty)$ with $R(y) \to 0$ as $y \to 0$ (see Definition 3.2). The tangential approach region is defined by the function R. We give a relation between $R(y)$ for which the Poisson integrals of all functions in $L^\Phi_K(\mathbb{R}^n)$ have the Ω_R-limit and the $L^{\widetilde{\Phi}}$-norm of $P_y * K$, and calculate $R(y)$ precisely for kernels K of the form

$$K(x) = K_\rho(x) = \frac{\rho(|x|)}{|x|^n},$$

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where the function $\rho : (0,+\infty) \to (0,+\infty)$ satisfies that $\rho(r)/r^n$ is decreasing and

$$\int_0^{+\infty} \frac{\rho(r)}{r}\,dr < +\infty.$$

If $K \in L^{\widetilde{\Phi}}(\mathbb{R}^n)$, then $K * F$ is continuous for every $F \in L^{\Phi}(\mathbb{R}^n)$. In this case, the tangential limit of the Poisson integral of $f \in L^{\Phi}_K$ exists trivially. So we are interested in the case $K \notin L^{\widetilde{\Phi}}(\mathbb{R}^n)$.

The Bessel kernel J_α, $0 < \alpha < n$, is the function on $\mathbb{R}^n$ whose Fourier transform is $\widehat{J_\alpha}(\xi) = (1+|\xi|^2)^{-\alpha/2}$, $\xi \in \mathbb{R}^n$. Then $J_\alpha(x) \sim K_\rho(x)$ for small $|x|$ with $\rho(r) = r^\alpha$ for small $r > 0$. This case was studied in [4] and [5].

If $\rho(r) = r^\alpha$ for small $r > 0$ with $0 < \alpha < n/p$, then the Hardy-Littlewood-Sobolev theorem shows that

$$L^p_{K_\rho}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n),$$

where $-n/p + \alpha = -n/q$. In this case the Poisson integrals of all functions $f \in L^p_{K_\rho}$ have the Ω_R-limit with $R(y) = y^{1-\alpha p/n}$. As α is bigger, the Ω_R-limit gets more tangential. If $\alpha = n/p$, then

$$L^p_{K_\rho}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n).$$

In this case $R(y) = (\log(1/y))^{-(p-1)/n}$ (see Theorem 3.7 and Example 3.1). We note that there is a larger class of functions than $L^p_{J_\alpha}(\mathbb{R}^n)$ such that all functions in the class have the Ω_R-limit with the above R. (see Dorronsoro [1]). If $\alpha > n/p$, then

$$L^p_{K_\rho}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \cap \mathrm{Lip}_\beta(\mathbb{R}^n),$$

where $\beta = -p/n + \alpha$. In this case, the tangential limit of the Poisson integral of $f \in L^p_{K_\rho}$ exists trivially.

One of the authors [7, 8, 9] showed that, if

$$\Phi^{-1}\left(\frac{1}{r^n}\right) \int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right) \quad \text{for} \quad r > 0,$$

then

$$L^\Phi_{K_\rho}(\mathbb{R}^n) \subset L^\Phi(\mathbb{R}^n) \cap L^\Psi(\mathbb{R}^n).$$

If $\phi : (0,+\infty) \to (0,+\infty)$ is increasing and

$$\Phi^{-1}\left(\frac{1}{r^n}\right) \int_0^r \frac{\rho(t)}{t}\,dt \le C\phi(r) \quad \text{for} \quad r > 0,$$

then

$$L^\Phi_{K_\rho}(\mathbb{R}^n) \subset L^\Phi(\mathbb{R}^n) \cap \mathrm{BMO}_\phi(\mathbb{R}^n).$$

If $\phi \equiv 1$, then BMO_ϕ is the usual BMO. In these cases, see Theorem 3.8, Remark 3.4 and Example 3.2.

Let $\Phi \in \nabla_2$. Then, for all functions F such that $\Phi(k|F(x)|)$ is integrable for all $k > 0$, the Poisson integrals of $f = K_\rho * F$ have the Ω_R-limit with at least $R(y) = y/\left(\int_0^y (\rho(r)/r)\,dr\right)^{1/n}$, for every kernel K_ρ (see Theorem 3.9).

Notations and definitions of function spaces are in the next section. We state main results and examples in Section 3 and proofs of main results in Sections 4–6.

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2. Notations and definitions

In this section we state some notations and definitions. We also state properties of the N-function and the Orlicz space.

2.1. $\mathbb{R}^n$ is the n-dimensional Euclidean space, with norm $|x| = \sqrt{\sum x_i^2}$, $x = (x_1, ..., x_n)$ and

$$\mathbb{R}_+^{n+1} = \{(x, y) : x \in \mathbb{R}^n, y > 0\}.$$

Let $B(a, r)$ be the ball $\{x \in \mathbb{R}^n : |x - a| < r\}$ with center a and of radius $r > 0$. We denote the measure of a measurable set $E \subset \mathbb{R}^n$ by $|E|$. Let $\sigma_n = |B(0, 1)|$. Then $|B(0, r)| = \sigma_n r^n$.

2.2. A function $\Phi : [0, +\infty) \to [0, +\infty)$ is called an N-function if Φ is continuous, convex, strictly increasing, $\lim_{r\to+0} \Phi(r)/r = 0$ and $\lim_{r\to+\infty} \Phi(r)/r = +\infty$. For an N-function Φ, the complementary function is defined by

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \geq 0\}, \quad r \geq 0.$$

Then $\widetilde{\Phi}$ is also an N-function, $\widetilde{\widetilde{\Phi}} = \Phi$, and,

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r. \tag{2.1}$$

2.3. A function $\Phi : [0, +\infty) \to [0, +\infty)$ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \quad r \geq 0,$$

for some $C > 0$. This condition is also called the doubling condition. A function Φ : $[0, +\infty) \to [0, +\infty)$ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \quad r \geq 0,$$

for some $k > 1$.

Let Φ is an N-function. Then $\Phi \in \Delta_2$ if and only if $\widetilde{\Phi} \in \nabla_2$.

2.4. For an N-function Φ, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)\, dx < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$M^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)\, dx < +\infty \text{ for all } k > 0 \right\},$$

$$\|f\|_\Phi = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

Then $L^\Phi(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{L^\Phi}$. $M^\Phi(\mathbb{R}^n)$ is a closed subspace of $L^\Phi(\mathbb{R}^n)$. If and only if $\Phi \in \Delta_2$, then $L^\Phi(\mathbb{R}^n) = M^\Phi(\mathbb{R}^n)$.

Let $C_{\mathrm{comp}}(\mathbb{R}^n)$ be the set of all continuous functions with compact supports. Then $C_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $M^\Phi(\mathbb{R}^n)$.

We have Hölder's inequality for Orlicz spaces:

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq 2\|f\|_\Phi \|g\|_{\widetilde{\Phi}}. \tag{2.2}$$

We also have the following equivalence:

$$\|f\|_\Phi \leq \sup\left\{ \int_{\mathbb{R}^n} |f(x)g(x)|\, dx : \int_{\mathbb{R}^n} \widetilde{\Phi}(|g(x)|)\, dx \leq 1 \right\} \leq 2\|f\|_\Phi. \tag{2.3}$$

If and only if $\Phi \in \Delta_2$, then

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_\Phi}\right) dx = 1 \quad \text{for all } f \in L^\Phi \text{ with } \|f\|_\Phi \neq 0. \tag{2.4}$$

Let $\{f_j\}_j \subset L^\Phi$. If $f_j \to 0$ in L^Φ as $j \to +\infty$, then

$$\int_{\mathbb{R}^n} \Phi(|f_j(x)|)\, dx \to 0 \quad \text{as} \quad j \to +\infty. \tag{2.5}$$

If and only if $\Phi \in \Delta_2$, the converse is true.

2.5. The letter K and the word *kernel* denote a nonnegative L^1-function on $\mathbb{R}^n$ which is *radial* and *decreasing*; i.e., $K(x) = K(x')$ if $|x| = |x'|$ and $K(x) \le K(x')$ if $|x| \ge |x'|$. Also, $K(0) = +\infty$ (we are not interested in bounded K), and we usually normalize so that $\|K\|_1 = 1$. Let

$$L_K^\Phi = L_K^\Phi(\mathbb{R}^n) = \{f = K * F : F \in L^\Phi(\mathbb{R}^n)\},$$
$$M_K^\Phi = M_K^\Phi(\mathbb{R}^n) = \{f = K * F : F \in M^\Phi(\mathbb{R}^n)\}.$$

2.6. The Poisson kernel for $\mathbb{R}_+^{n+1}$ is

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{(n+1)/2}} \quad (x \in \mathbb{R}^n, y > 0),$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right) \pi^{-(n+1)/2}$ is so chosen that $\|P_y\|_1 = 1$ for $0 < y < +\infty$. $(P_y * K)(x)$ is the harmonic extension of K to $\mathbb{R}_+^{n+1}$. For $f \in L_K^\Phi$, the Poisson integral $u = P[f]$ means that

$$u(x, y) = P[f](x, y) = (P_y * f)(x) = (P_y * K * F)(x),$$

for some $F \in L^\Phi(\mathbb{R}^n)$.

2.7. For a function $\phi : (0, +\infty) \to (0, +\infty)$, let

$$\mathrm{BMO}_\phi(\mathbb{R}^n) = \left\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{BMO}_\phi} < +\infty\right\},$$
$$\text{where} \quad \|f\|_{\mathrm{BMO}_\phi} = \sup_{B = B(a,r)} \frac{1}{\phi(r)} \frac{1}{|B|} \int_B |f(x) - f_B|\, dx,$$
$$\text{and} \quad f_B = \frac{1}{|B|} \int_B f(x)\, dx.$$

If $\phi(r) \equiv 1$, then $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha$, $0 < \alpha \le 1$, then it is known that $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$. All functions in BMO_ϕ are continuous, if and only if

$$\int_0^1 \frac{\phi(t)}{t}\, dt < +\infty \tag{2.6}$$

(see [2] and [6]).

2.8. For functions $\theta, \kappa : (0, +\infty) \to (0, +\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r), \quad r > 0.$$

A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that $\theta(r) \le C\theta(s)$ $(\theta(r) \ge C\theta(s))$ for $r \le s$.

The letter C shall always denote a constant, not necessarily the same one.

3. Main results and examples

For an N-function Φ and for a ball B, let

$$\|f\|_{\Phi,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\lambda}\right)\, dx \le 1\right\}.$$

For a kernel K and for an N-function Φ, let

$$k(r,y) = |B(0,r)|\|P_y * K\|_{\widetilde{\Phi},B(0,r)}, \quad r > 0,\ y > 0,$$

where $\widetilde{\Phi}$ is the complementary N-function of Φ. Let $0 < \beta < +\infty$. We define the *approach region*

$$\Omega^{\Phi}_{K,\beta}(x_0) = \{(x,y) \in \mathbb{R}^{n+1}_+ : k(|x-x_0|,y) < \beta\}, \quad x_0 \in \mathbb{R}^n,$$

and the associated maximal function

$$(\mathfrak{M}[\Omega^{\Phi}_{K,\beta}]f)(x_0) = \sup\{|u(x,y)| : (x,y) \in \Omega^{\Phi}_{K,\beta}(x_0)\},$$

where $u = P[f]$.

We assume that $\Phi \in \nabla_2$ and $K \notin L^{\widetilde{\Phi}}(\mathbb{R}^n)$. Let

$$\tau_\beta(y) = \sup\{r > 0 : k(r,y) < \beta\}. \tag{3.1}$$

Then

$$\begin{cases} 0 < \tau_\beta(y) < +\infty, \\ \tau_\beta \text{ is continuous and increasing}, \\ \tau_\beta(y) \to 0 \text{ as } y \to 0, \\ k(\tau_\beta(y),y) = \beta, \\ k(r,y) < \beta \Leftrightarrow r < \tau_\beta(y). \end{cases} \tag{3.2}$$

Hence we can write

$$\begin{aligned} \Omega^{\Phi}_{K,\beta}(x_0) &= \{(x,y) \in \mathbb{R}^{n+1}_+ : k(|x-x_0|,y) < \beta\} \\ &= \{(x,y) \in \mathbb{R}^{n+1}_+ : |x-x_0| < \tau_\beta(y)\}, \quad x_0 \in \mathbb{R}^n. \end{aligned}$$

Moreover, the *approach region* $\Omega^{\Phi}_{K,\beta}(x_0)$ is tangential to the boundary of $\mathbb{R}^{n+1}_+$, i.e.

$$\begin{cases} \tau_\beta(y)/y \ge c_0 > 0 \quad \text{for all } y > 0, \\ \tau_\beta(y)/y \to +\infty \quad \text{as} \quad y \to 0. \end{cases} \tag{3.3}$$

The properties (3.2) and (3.3) will be proved in the next section.

If $K \in L^{\widetilde{\Phi}}$ and $F \in L^{\Phi}$, then $f = K * F$ is continuous, so that $u = P[f]$ is continuous on the closure of $\mathbb{R}^{n+1}_+$. In this case the tangential limit of u exists trivially. Therefore we are interested in the case $K \notin L^{\widetilde{\Phi}}$.

Our main results are following.

Theorem 3.1. *Let Φ be an N-function and $\Phi \in \nabla_2$. Then there exists a constant $C > 0$ such that, for all $f = K * F \in L^{\Phi}_K$ and for all $t > 0$,*

$$\left|\{x \in \mathbb{R}^n : (\mathfrak{M}[\Omega^{\Phi}_{K,\beta}]f)(x) > t\}\right| \le \int_{\mathbb{R}^n} \Phi\left(\frac{C(\beta + \|K\|_1)|F(x)|}{t}\right) dx,$$

where C is independent of K, β, F and t.

Definition 3.1. A function u on $\mathbb{R}^{n+1}_+$ is said to have Ω^{Φ}_K-limit L at a point $x_0 \in \mathbb{R}^n$ if it is true for every $0 < \beta < +\infty$ that $u(x,y) \to L$ as $(x,y) \to (x_0,0)$ within $\Omega^{\Phi}_{K,\beta}(x_0)$.

Remark 3.1. Since $k(r,y)$ is increasing with respect to r (see (4.6)), $\tau_\beta(y)$ is increasing with respect to β. Hence $\Omega^\Phi_{K,\beta} \subset \Omega^\Phi_{K,\beta'}$ if $\beta \le \beta'$.

Theorem 3.2. *Let Φ be an N-function and $\Phi \in \nabla_2$. If $f \in M^\Phi_K$ and $u = P[f]$, then, for almost all $x_0 \in \mathbb{R}^n$, the Ω^Φ_K-limit of u exists at x_0 and equals $f(x_0)$.*

Corollary 3.3. *Let Φ be an N-function and $\Phi \in \Delta_2 \cap \nabla_2$. If $f \in L^\Phi_K$ and $u = P[f]$, then, for almost all $x_0 \in \mathbb{R}^n$, the Ω^Φ_K-limit of u exists at x_0 and equals $f(x_0)$.*

The next proposition shows that Theorem 3.2 is optimal with regard to the size of the approach regions. To formulate this precisely, compare

$$\Omega^\Phi_{K,\beta}(x_0) = \{(x,y) \in \mathbb{R}^{n+1}_+ : |x - x_0| < \tau_\beta(y)\}$$

with another region

$$\Omega(x_0) = \{(x,y) \in \mathbb{R}^{n+1}_+ : |x - x_0| < \omega(y)\}, \tag{3.4}$$

where ω is some positive continuous function. Let

$$(\mathfrak{M}[\Omega]f)(x_0) = \sup\{|u(x,y)| : (x,y) \in \Omega(x_0)\}, \quad x_0 \in \mathbb{R}^n, \tag{3.5}$$

where $u = P[f]$.

Proposition 3.4. *Let Ω and $\mathfrak{M}[\Omega]$ be as in* (3.4) *and* (3.5)*, respectively. If there exists $c_* > 0$ such that, for all $f = K * F \in L^\Phi_K(\mathbb{R}^n)$ and for all $t > 0$,*

$$|\{x \in \mathbb{R}^n : (\mathfrak{M}[\Omega]f)(x) > t\}| \le \int_{\mathbb{R}^n} \Phi\left(\frac{c_*|F(x)|}{t}\right) dx, \tag{3.6}$$

then there exists $\beta > 0$ such that, for all $y > 0$, $\omega(y) \le \tau_\beta(y)$.

Theorems 3.1, 3.2, Corollary 3.3 and Proposition 3.4 are generalization of the results of Nagel, Rudin and Shapiro [4]. However the definition of τ_β is not simple and τ_β is difficult to calculate. To investigate the geometric properties of approach regions, we introduce Ω_R-limit.

Definition 3.2. For $R : (0,+\infty) \to (0,+\infty)$ and for $0 < b < +\infty$, let

$$\Omega_{R,b}(x_0) = \{(x,y) \in \mathbb{R}^{n+1}_+ : |x - x_0| < bR(y)\}, \quad x_0 \in \mathbb{R}^n.$$

A function u on $\mathbb{R}^{n+1}_+$ is said to have Ω_R-limit L at a point $x_0 \in \mathbb{R}^n$ if it is true for every $0 < b < +\infty$ that $u(x,y) \to L$ as $(x,y) \to (x_0, 0)$ within $\Omega_{R,b}(x_0)$.

In the case $\Phi(r) = r^p$ with $1 < p < \infty$ and $K \notin L^{p'}$, we have

$$\begin{aligned} k(r,y) = |B(0,r)| \|P_y * K\|_{\widetilde{\Phi},B(0,r)} &= |B(0,r)|^{1/p} \|P_y * K\|_{L^{p'}(B(0,r))} \\ &\le |B(0,r)|^{1/p} \|P_y * K\|_{p'}. \end{aligned}$$

This implies

$$\left(\frac{\beta}{{\sigma_n}^{1/p} \|P_y * K\|_{p'}}\right)^{p/n} \le \tau_\beta(y).$$

Hence, for $R(y) = {\|P_y * K\|_{p'}}^{-p/n}$, if $f \in L^p_K$ and $u = P[f]$, then, for almost all $x_0 \in \mathbb{R}^n$, the Ω_R-limit of u exists at x_0 and equals $f(x_0)$. This is a result of Nagel, Rudin and Shapiro [4]. They [4] also showed that, if (3.6) holds, then $\omega(y) \le C{\|P_y * K\|_{p'}}^{-p/n}$. We extend this result to the following.

Theorem 3.5. *Let Φ be an N-function, $\Phi \in \nabla_2$ and $K \notin L^{\widetilde{\Phi}}(\mathbb{R}^n)$. Let τ_β, $0 < \beta < +\infty$, be as in* (3.1). *Then there exists a continuous increasing function $R : (0, +\infty) \to (0, +\infty)$ with $R(y) \to 0$ as $y \to 0$ such that*

$$\forall b > 0 \; \exists \beta > 0 \; \forall y > 0 : bR(y) \le \tau_\beta(y).$$

In this case the Ω_K^Φ-limit is also the Ω_R-limit.

Moreover, if $\Phi \in \Delta_2 \cap \nabla_2$, then the function R also have the property

$$\forall \beta > 0 \; \exists b > 0 \; \forall y > 0 : \tau_\beta(y) \le bR(y).$$

In this case the Ω_K^Φ-limit and the Ω_R-limit are the same.

Remark 3.2. We can choose $R \sim \tau_{\beta_0}$ for any fixed $\beta_0 > 0$ (see Proof of Theorem 3.5).

In the following, we calculate the function $\tau = \tau_1 \sim R$. For a function $\rho : (0, +\infty) \to (0, +\infty)$, let

$$K_\rho(x) = \frac{\rho(|x|)}{|x|^n}.$$

We assume that $\rho(r)/r^n$ is decreasing and that

$$\int_0^{+\infty} \frac{\rho(r)}{r} \, dr < +\infty.$$

Then K_ρ is a kernel. From the decreasingness of $\rho(r)/r^n$ it follows that $\rho(2r) \le 2^n \rho(r)$ for $r > 0$. Let

$$\bar{\rho}(r) = \int_0^r \frac{\rho(t)}{t} \, dt.$$

Then $\rho(r) \le C\bar{\rho}(r)$ for all $r > 0$. If $\rho(r)/r^\alpha$ is almost increasing for small $r > 0$ with $\alpha > 0$, then $\bar{\rho} \sim \rho$ for small $r > 0$. If $\rho(r)/r^\beta$ is almost decreasing for small $r > 0$ with $\beta > 0$, then $\bar{\rho}(r)/r^\beta$ is also almost decreasing for small $r > 0$ (see Lemma 6.2 (iii), (iv)).

Proposition 3.6. *Let $\tau = \tau_1$ be as in* (3.1) *for a kernel K_ρ and an N-function Φ with $K_\rho \notin L^{\widetilde{\Phi}}(\mathbb{R}^n)$.*

(i) *If $\Phi \in \nabla_2$, then*

$$C^{-1} \le \tau(y)^{-n} \int_y^{\tau(y)} \widetilde{\Phi}\left(\frac{\tau(y)^n \rho(t)}{t^n} + \frac{y\tau(y)^n \bar{\rho}(t)}{t^{n+1}} \right) t^{n-1} \, dt \le C \tag{3.7}$$

for small $y > 0$.

(ii) *If $\Phi \in \Delta_2 \cap \nabla_2$, then*

$$C^{-1} \le \tau(y)^{-n} \int_y^{1} \widetilde{\Phi}\left(\frac{\tau(y)^n \rho(t)}{t^n} + \frac{y\tau(y)^n \bar{\rho}(t)}{t^{n+1}} \right) t^{n-1} \, dt \le C \tag{3.8}$$

for small $y > 0$.

(iii) *If $\Phi \in \nabla_2$, $\Phi(r)/r^p$ is almost decreasing with $1 < p < \infty$, and, $\rho(r)/r^\beta$ is almost decreasing for small $r > 0$ with $0 < \beta < n/p$, then*

$$C^{-1} \le \left(\frac{y}{\tau(y)} \right)^n \widetilde{\Phi}\left(\frac{\tau(y)^n \bar{\rho}(y)}{y^n} \right) \le C \quad \textit{for small } y > 0. \tag{3.9}$$

Theorem 3.7. *Let $\Phi(r) = r^p$ with $1 < p < \infty$. Let $\rho(r)/r^\alpha$ be almost increasing and $\rho(r)/r^\beta$ be almost decreasing for small $r > 0$, with $0 \le \alpha \le n/p$ and $\alpha \le \beta < n$.*

(i) *In the case that* $\alpha > 0$*, let* $1/p + 1/p' = 1$ *and*

$$R(y) = \left(\int_y^1 \left(\frac{\rho(t)}{t^{n/p}} \right)^{p'} t^{-1}\, dt \right)^{-p/(np')} \quad \text{for small } y > 0. \tag{3.10}$$

(ii) *In the case that* $0 < \beta < n/p$*, let*

$$R(y) = \begin{cases} y/\rho(y)^{p/n} & (\alpha > 0) \\ y/\bar{\rho}(y)^{p/n} & (\alpha = 0) \end{cases} \quad \text{for small } y > 0. \tag{3.11}$$

If $f \in L^p_{K_\rho}$ *and* $u = P[f]$*, then, for almost all* $x_0 \in \mathbb{R}^n$*, the* Ω_R*-limit of* u *exists at* x_0 *and equals* $f(x_0)$*. Moreover, this is optimal in the sense of Proposition 3.4.*

Remark 3.3. If the integral in (3.10) is finite as $y \to 0$, then $K_\rho \in L^{p'}(\mathbb{R}^n)$.

Let SV be the set of all continuous functions $\ell : (0, +\infty) \to (0, +\infty)$ satisfying, for some constant $C > 0$,

$$C^{-1} \le \frac{\ell(s)}{\ell(r)} \le C \quad \text{for} \quad \frac{1}{2} \le \log_r s \le 2,\ r \ne 1,\ s \ne 1.$$

Then, for every $\epsilon > 0$, $\ell(r)r^\epsilon$ is almost increasing and $\ell(r)/r^\epsilon$ is almost decreasing.

Theorem 3.8. *Let N-function* $\Phi(r)$ *be of the form* $r^p\ell(r)$ *with* $1 < p < \infty$ *and* $\ell \in SV$*, and* $\rho(r)$ *be of the form* $r^\alpha m(r)$ *with* $0 \le \alpha \le n/p$ *and* $m \in SV$*.*

(i) *In the case that* $\alpha = n/p$*, let* $1/p + 1/p' = 1$ *and*

$$R(y) = \left(\int_y^1 m(t)^{p'} \ell\left(\frac{1}{t}\right)^{-p'/p} t^{-1}\, dt \right)^{-p/(np')} \quad \text{for small } y > 0. \tag{3.12}$$

(ii) *In the case that* $0 < \alpha < n/p$*, let*

$$R(y) = y^{1-\alpha p/n} \frac{\ell(1/y)^{1/n}}{m(y)^{p/n}} \quad \text{for small } y > 0. \tag{3.13}$$

(iii) *In the case that* $\alpha = 0$*, let*

$$R(y) = y\, \frac{\ell(1/\bar{m}(y))^{1/n}}{\bar{m}(y)^{p/n}} \quad \text{for small } y > 0. \tag{3.14}$$

If $f \in L^\Phi_{K_\rho}$ *and* $u = P[f]$*, then, for almost all* $x_0 \in \mathbb{R}^n$*, the* Ω_R*-limit of* u *exists at* x_0 *and equals* $f(x_0)$*. Moreover, this is optimal in the sense of Proposition 3.4.*

Remark 3.4. Let

$$\phi(r) = m(r)\ell\left(\frac{1}{r}\right)^{-1/p}.$$

If $\phi(r)$ is almost increasing for small $r > 0$, then

$$L^\Phi_{K_\rho}(\mathbb{R}^n) \subset L^\Phi(\mathbb{R}^n) \cap \mathrm{BMO}_\phi(\mathbb{R}^n).$$

If $\int_y^1 \phi(t)^{p'}/t\, dt < +\infty$ as $y \to 0$, then $K_\rho \in L^{\widetilde{\Phi}}(\mathbb{R}^n)$. If $\int_y^1 \phi(t)^{p'}/t\, dt \to +\infty$ as $y \to 0$, then (2.6) fails.

The following result is not necessarily optimal.

Theorem 3.9. *Let* $\Phi \in \nabla_2$ *and*

$$R(y) = \frac{y}{\bar{\rho}(y)^{1/n}}.$$

If $f \in M^{\Phi}_{K_\rho}$ *and* $u = P[f]$*, then, for almost all* $x_0 \in \mathbb{R}^n$*, the* Ω_R*-limit of* u *exists at* x_0 *and equals* $f(x_0)$*.*

At the end of this section, we state examples. Examples 3.1 and 3.2 follow immediately from Theorems 3.7 and 3.8, respectively. Example 3.3 is for the case of $\Phi \in \nabla_2 \setminus \Delta_2$. A proof of Example 3.3 is in Section 6. Example 3.3 is not necessarily optimal.

Example 3.1. Let $1 < p < \infty$, $0 \le \alpha \le n/p$, $-\infty < \beta < +\infty$, $-\infty < \gamma < +\infty$, $\Phi(r) = r^p$, and, $\rho(r) = r^\alpha(\log(1/r))^{-\beta}(\log\log(1/r))^{-\gamma}$ for small $r > 0$. Let

$$R(y) = \begin{cases} y\left(\log\log\dfrac{1}{y}\right)^{(\gamma-1)p/n} & \text{when} \quad \alpha = 0, \beta = 1, \gamma > 1, \\ y\left(\log\dfrac{1}{y}\right)^{(\beta-1)p/n}\left(\log\log\dfrac{1}{y}\right)^{\gamma p/n} & \text{when} \quad \alpha = 0, \beta > 1, \\ y^{1-\alpha p/n}\left(\log\dfrac{1}{y}\right)^{\beta p/n}\left(\log\log\dfrac{1}{y}\right)^{\gamma p/n} & \text{when} \quad 0 < \alpha < n/p, \\ \left(\log\dfrac{1}{y}\right)^{-(1-1/p-\beta)p/n}\left(\log\log\dfrac{1}{y}\right)^{\gamma p/n} & \text{when} \quad \alpha = n/p, \beta < 1 - 1/p, \\ \left(\log\log\dfrac{1}{y}\right)^{-(1-1/p-\gamma)p/n} & \text{when} \quad \alpha = n/p, \beta = 1 - 1/p, \gamma < 1 - 1/p, \\ \left(\log\log\log\dfrac{1}{y}\right)^{-(p-1)/n} & \text{when} \quad \alpha = n/p, \beta = 1 - 1/p, \gamma = 1 - 1/p. \end{cases}$$

If $f \in L^p_{K_\rho}$ and $u = P[f]$, then, for almost all $x_0 \in \mathbb{R}^n$, the Ω_R-limit of u exists at x_0 and equals $f(x_0)$. Moreover, this is optimal in the sense of Proposition 3.4. (If $\alpha > n/p$, or if $\alpha = n/p$ and $\beta > 1 - 1/p$, or if $\alpha = n/p$, $\beta = 1 - 1/p$ and $\gamma > 1 - 1/p$, then $K_\rho \in L^{\widetilde{\Phi}}(\mathbb{R}^n)$.)

Example 3.2. Let $1 < p < \infty$, $-\infty < \theta < +\infty$ and

$$\Phi(r) = \begin{cases} r^p(\log r)^{\theta p} & \text{for large } r > 0, \\ r^p(\log(1/r))^{-\theta p} & \text{for small } r > 0. \end{cases}$$

For constants α and β with $0 \le \alpha \le n/p$ and $-\infty < \beta < +\infty$, let $\rho(r) = r^\alpha(\log(1/r))^{-\beta}$ for small $r > 0$. Let

$$R(y) = \begin{cases} y\left(\log\dfrac{1}{y}\right)^{(\beta-1)p/n}\left(\log\log\dfrac{1}{y}\right)^{\theta p/n} & \text{when} \quad \alpha = 0,\ \beta > 1, \\ y^{1-\alpha p/n}\left(\log\dfrac{1}{y}\right)^{(\beta+\theta)p/n} & \text{when} \quad 0 < \alpha < n/p, \\ \left(\log\dfrac{1}{y}\right)^{-(1-1/p-\beta-\theta)p/n} & \text{when} \quad \alpha = n/p,\ 1 - 1/p > \beta + \theta, \\ \left(\log\log\dfrac{1}{y}\right)^{-(p-1)/n} & \text{when} \quad \alpha = n/p,\ 1 - 1/p = \beta + \theta. \end{cases}$$

If $f \in L^{\Phi}_{K_\rho}$ and $u = P[f]$, then, for almost all $x_0 \in \mathbb{R}^n$, the Ω_R-limit of u exists at x_0 and equals $f(x_0)$. Moreover, this is optimal in the sense of Proposition 3.4. (If $\alpha > n/p$, or if $\alpha = n/p$ and $1 - 1/p < \beta + \theta$, then $K_\rho \in L^{\widetilde{\Phi}}(\mathbb{R}^n)$.)

Example 3.3. Let

$$\Phi(r) = \begin{cases} \exp r & \text{for large } r > 0, \\ 1/\exp(1/r) & \text{for small } r > 0, \end{cases}$$

and $\rho(r) = (\log(1/r))^{-2}$ for small $r > 0$. Let

$$R(y) = y^{1-\epsilon}\left(\log\frac{1}{y}\right)^{1/n} \quad \text{for small } y > 0,$$

where $\epsilon > 0$ is small enough. If $f \in M^{\Phi}_{K_\rho}$ and $u = P[f]$, then, for almost all $x_0 \in \mathbb{R}^n$, the Ω_R-limit of u exists at x_0 and equals $f(x_0)$.

4. Proofs of the properties (3.2) and (3.3)

To show the properties (3.2) and (3.3), we investigate the properties of $k(r, y)$. First we state two lemmas.

Lemma 4.1. *Let Φ be an N-function and $\|g\|_1 \le 1$. Then, for $z_0 \in \mathbb{R}^n$ and $r > 0$,*

$$\int_{B(z_0,r)} \Phi(|f * g(x)|)\,dx \le \|g\|_1 \sup_{z\in\mathbb{R}^n} \int_{B(z,r)} \Phi(|f(x)|)\,dx.$$

Proof. Let

$$\alpha = \|g\|_1, \quad \beta = \sup_{z\in\mathbb{R}^n} \int_{B(z,r)} \Phi(|f(x)|)\,dx.$$

If $\alpha = 0$, then this inequation is clear. We assume $\alpha \neq 0$. Let $\mu(A) = \int_A |g(x)|\,dx/\alpha$ for $A \subset \mathbb{R}^n$. Then μ is a probability measure. We note by χ_E the characteristic function of $E \subset \mathbb{R}^n$. Then we have

$$\begin{aligned} \int_{B(z_0,r)} \Phi(|f * g(x)|)\,dx &\le \int_{\mathbb{R}^n} \Phi\left(\int_{\mathbb{R}^n} |f(x-x')||g(x')|\,dx'\right) \chi_{B(z_0,r)}(x)\,dx \\ &= \int_{\mathbb{R}^n} \Phi\left(\alpha \int_{\mathbb{R}^n} |f(x-x')|\,d\mu(x')\right) \chi_{B(z_0,r)}(x)\,dx \\ &\le \alpha \int_{\mathbb{R}^n} \Phi\left(\int_{\mathbb{R}^n} |f(x-x')|\,d\mu(x')\right) \chi_{B(z_0,r)}(x)\,dx \\ &\le \alpha \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \Phi(|f(x-x')|)\,d\mu(x')\right) \chi_{B(z_0,r)}(x)\,dx \\ &\le \alpha \int_{\mathbb{R}^n} \beta\,d\mu(x') = \alpha\beta. \quad \square \end{aligned}$$

Lemma 4.1 shows that, if $|f|$ is radial and decreasing and $\|g\|_1 \le 1$, then

$$\|f * g\|_{\Phi,B(0,r)} \le \|g\|_1 \|f\|_{\Phi,B(0,r)}. \tag{4.1}$$

Lemma 4.2. *Let K be a kernel and Φ be an N-function. Then, for all $r > 0$,*

$$\|P_y * K\|_{\Phi,B(0,r)} \le \|P_y\|_{\Phi,B(0,r)} \le \frac{c_n}{\Phi^{-1}(1)y^n} \quad \textit{for } y > 0. \tag{4.2}$$

$$\|P_{y_2} * K\|_{\Phi,B(0,r)} \le \|P_{y_1} * K\|_{\Phi,B(0,r)} \quad \textit{for } y_1 < y_2. \tag{4.3}$$

If $K \notin L^\Phi(\mathbb{R}^n)$, then, for every $r > 0$,

$$\|P_y * K\|_{\Phi,B(0,r)} \to +\infty \textit{ as } y \to 0. \tag{4.4}$$

If $\Phi \in \Delta_2$, then, for $r > 0$, $y > 0$ and $t > 1$,

$$\|P_y * K\|_{\Phi,B(0,tr)} \le \|P_y * K\|_{\Phi,B(0,r)} < t^n \|P_y * K\|_{\Phi,B(0,tr)}. \tag{4.5}$$

Proof. P_y and K are radial and decreasing. Hence $P_y * K$ is also radial and decreasing. By $\|P_y\|_1 = \|K\|_1 = 1$, $P_{y+y_1} = P_y * P_{y_1}$ and $P_y(x) \le c_n/y^n$, using (4.1), we have (4.2) and (4.3). If $K \in L^1(\mathbb{R}^n) \setminus L^\Phi(\mathbb{R}^n)$, then

$$\int_{B(0,r)} \Phi\left(\frac{K(x)}{\lambda}\right) dx = +\infty \quad \text{for all } r > 0, \lambda > 0.$$

Since $P_y * K \to K$ a.e. as $y \to 0$, we have (4.4). By the inequality

$$\frac{1}{|B(0,tr)|}\int_{B(0,tr)} \Phi\left(\frac{P_y * K(x)}{\lambda}\right) dx \le \frac{1}{|B(0,r)|}\int_{B(0,r)} \Phi\left(\frac{P_y * K(x)}{\lambda}\right) dx,$$

we have the first inequality in (4.5). If $\Phi \in \Delta_2$, then, for $\lambda = \|P_y * K\|_{\Phi,B(0,r)}$,

$$\begin{aligned}\frac{1}{|B(0,tr)|}\int_{B(0,tr)} \Phi\left(\frac{P_y * K(x)}{\lambda/t^n}\right) dx &\ge \frac{1}{|B(0,r)|}\int_{B(0,tr)} \Phi\left(\frac{P_y * K(x)}{\lambda}\right) dx \\ &> \frac{1}{|B(0,r)|}\int_{B(0,r)} \Phi\left(\frac{P_y * K(x)}{\lambda}\right) dx = 1.\end{aligned}$$

Hence $\|P_y * K\|_{\Phi,B(0,tr)} > \lambda/t^n$, this is the second inequality in (4.5). □

We assume that $\Phi \in \nabla_2$ and $K \notin L^{\widetilde{\Phi}}$. Then $\widetilde{\Phi} \in \Delta_2$. We apply Lemma 4.2 to $\widetilde{\Phi}$. Then (4.3) and (4.4) imply

$$k(r,y_1) \ge k(r,y_2) \text{ for } y_1 < y_2 \quad \text{and} \quad k(r,y) \to +\infty \text{ as } y \to 0. \tag{4.6}$$

Let $\ell(\lambda,y) = \int_B \widetilde{\Phi}(P_y * K(x)/\lambda)\,dx$. Then $\ell(\lambda,y)$ is continuous with respect to λ and y, strictry decreasing with respect to λ, $\ell(\lambda,y) \to +\infty$ as $\lambda \to 0$, and, $\ell(\lambda,y) \to 0$ as $\lambda \to +\infty$. Hence $\|P_y * K\|_{\widetilde{\Phi},B(0,r)}$ is continuous with respect to y, and so is $k(r,y)$. By (4.5) and (4.2) we have that $k(r,y)$ is continuous with respect to r and

$$k(r_1,y) < k(r_2,y) \text{ for } r_1 < r_2 \quad \text{and} \quad k(r,y) \to 0 \text{ as } r \to 0. \tag{4.7}$$

Let $m = P_y * K(x)$ for $|x| = 1$. For all $\lambda > 0$, there exists $\lambda' > 0$ such that

$$\frac{1}{\lambda'}\widetilde{\Phi}(t) \le \widetilde{\Phi}\left(\frac{m}{\lambda}t\right) \quad \text{for all } t > 0.$$

Then, for $B = B(0,r)$,

$$\begin{aligned}\frac{1}{|B|}\int_B \widetilde{\Phi}\left(\frac{|B|(P_y * K)(x)}{\lambda}\right) dx &\ge \frac{1}{|B|}\int_{B(0,1)} \widetilde{\Phi}\left(\frac{|B|m}{\lambda}\right) dx \\ &\ge \frac{\sigma_n}{\lambda'}\frac{\widetilde{\Phi}(|B|)}{|B|} > 1 \quad \text{for large } r > 0.\end{aligned}$$

Hence

$$k(r,y) = \|\,|B|(P_y * K)\,\|_{\widetilde{\Phi},B} \to +\infty \quad \text{as} \quad r \to +\infty. \tag{4.8}$$

These properties (4.6), (4.7), (4.8) and the continuity of $k(r,y)$ yield (3.2).

Next we show (3.3). We note that

$$\left(\frac{\tau_\beta(y)}{y}\right)^n = \frac{k(\tau_\beta(y),y)}{\sigma_n y^n \|P_y * K\|_{\widetilde{\Phi},B(0,\tau_\beta(y))}} = \frac{\beta}{\sigma_n y^n \|P_y * K\|_{\widetilde{\Phi},B(0,\tau_\beta(y))}},$$

and we show

$$y^n \|P_y * K\|_{\widetilde{\Phi},B(0,\tau_\beta(y))} \le c_1 < +\infty \quad \text{for all } y > 0, \tag{4.9}$$

$$y^n \|P_y * K\|_{\widetilde{\Phi},B(0,\tau_\beta(y))} \to 0 \quad \text{as} \quad y \to 0. \tag{4.10}$$

By (4.2) we have (4.9). For all $\varepsilon > 0$, let

$$K = G_\varepsilon + H_\varepsilon, \quad G_\varepsilon \in L^\infty, \quad \|G_\varepsilon\|_1 < 1, \quad \|H_\varepsilon\|_1 < \varepsilon.$$

For $C_\varepsilon = \|G_\varepsilon\|_\infty / \widetilde{\Phi}^{-1}(1)$, using Lemma 4.1, we have

$$\int_{B(0,r)} \widetilde{\Phi}\left(\frac{|P_y * G_\varepsilon(x)|}{C_\varepsilon}\right) dx \le \sup_{z\in\mathbb{R}^n} \int_{B(z,r)} \widetilde{\Phi}\left(\frac{|G_\varepsilon(x)|}{C_\varepsilon}\right) dx \le |B(0,r)|.$$

Then

$$\|P_y * G_\varepsilon\|_{\widetilde{\Phi},B(0,r)} \le C_\varepsilon \quad \text{for all } r > 0.$$

By (4.1) and (4.2) we have

$$\|P_y * H_\varepsilon\|_{\widetilde{\Phi},B(0,r)} \le \frac{\varepsilon c_n}{\widetilde{\Phi}^{-1}(1) y^n} \quad \text{for all } r > 0.$$

Hence, for $y^n \le \varepsilon / C_\varepsilon$ and $r = \tau_\beta(y)$,

$$y^n \|P_y * K\|_{\widetilde{\Phi},B(0,\tau_\beta(y))} \le \frac{\varepsilon c_n}{\widetilde{\Phi}^{-1}(1)} + C_\varepsilon y^n \le \varepsilon \left(\frac{c_n}{\widetilde{\Phi}^{-1}(1)} + 1\right).$$

Therefore we have (3.3).

5. Proofs of Theorem 3.1–3.5

Let

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|} \int_B |f(z)|\,dz \quad \text{and} \quad M_\Phi f(x) = \sup_{B\ni x} \|f\|_{\Phi,B},$$

where the supremum is taken over all balls B containing x. The next two lemmas are [10, Lemma 5.2] and [4, Lemma 2.2].

Lemma 5.1. *There exists $C > 0$ such that, for all $F \in L^\Phi(\mathbb{R}^n)$ and for all $t > 0$,*

$$|\{x \in \mathbb{R}^n : M_\Phi F(x) > t\}| \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|F(x)|}{t}\right) dx.$$

Lemma 5.2. *If $F \in L^1$, and $g \ge 0$ is radial and decreasing, then*

$$\int_{\mathbb{R}^n} |F(x)| g(x)\,dx \le (MF)(0) \int_{\mathbb{R}^n} g(x)\,dx.$$

We have the following:

Theorem 5.3. *Let Φ and $\widetilde{\Phi}$ be a complementary pair of N-functions. Then there exists a constant $C > 0$ such that*

$$|(K * F)(x)| \le C \left((M_\Phi F)(x_0) |x - x_0|^n \|K\|_{\widetilde{\Phi},B(0,|x-x_0|)} + (MF)(x_0) \|K\|_1 \right)$$

whenever $F \in L^\Phi$, K is a nonnegative, radial and decreasing function on $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, $x \in \mathbb{R}^n$.

Proof. Take $x_0 = 0$, without loss of generality. Fix x. Then

$$|(K * F)(x)| \le \int_{\mathbb{R}^n} K(x - z) |F(z)|\,dz = I_1 + I_2,$$

where I_1 and I_2 are integrals over $B = B(0, 2|x|)$ and B^C respectively. Hölder's inequality shows that

$$I_1 = \int_B K(x-z)|F(z)|\,dz \le 2|B| \|K(x - \cdot)\|_{\widetilde{\Phi},B} \|F\|_{\Phi,B}$$
$$\le 2|B| \|K(x - \cdot)\|_{\widetilde{\Phi},B} (M_\Phi F)(0).$$

Since $B(0,|x|) \subset B(x,2|x|)$ and K is radial and decreasing,

$$\frac{1}{|B(0,2|x|)|}\int_{B(0,2|x|)} \widetilde{\Phi}\left(\frac{K(z-x)}{\lambda}\right)dz = \frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)} \widetilde{\Phi}\left(\frac{K(z)}{\lambda}\right)dz \le \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} \widetilde{\Phi}\left(\frac{K(z)}{\lambda}\right)dz.$$

Hence

$$\|K(x-\cdot)\|_{\widetilde{\Phi},B(0,2|x|)} \le \|K\|_{\widetilde{\Phi},B(0,|x|)}.$$

and

$$I_1 \le 2^{n+1}\sigma_n|x|^n\|K\|_{\widetilde{\Phi},B(0,|x|)}(M_\Phi F)(0).$$

If $z \in B^C$, then $|x-z| \ge |z|/2$, hence $K(x-z) \le K(z/2)$, so that

$$I_2 = \int_{B^C} K(x-z)|F(z)|\,dz \le \int_{B^C} K(z/2)|F(z)|\,dz \le MF(0)\|K(\cdot/2)\|_1 = 2^n MF(0)\|K\|_1,$$

by Lemma 5.2. These two estimates prove the theorem. □

Since P_y and K are nonnegative, radial and decreasing, so is $P_y * K$. Hence Theorem 5.3 holds with $P_y * K$ in place of K.

By Hölder's inequality, $MF \le CM_\Phi F$. By Fubini's theorem, $\|P_y * K\|_1 = \|P_y\|_1\|K\|_1 = \|K\|_1$. Hence Theorem 5.3 implies the following:

Theorem 5.4. *Let Φ and $\widetilde{\Phi}$ be a complementary pair of N-functions. If $F \in L^\Phi$, and u is defined in $\mathbb{R}^{n+1}_+$ by*

$$u(x,y) = (P_y * K * F)(x),$$

then

$$|u(x,y)| \le C(M_\Phi F)(x_0)\left(\|K\|_1 + k(|x-x_0|,y)\right),$$

where $k(r,y) = |B(0,r)|\|P_y * K\|_{\widetilde{\Phi},B(0,r)}$ and $x_0 \in \mathbb{R}^n$.

Proof of Theorem 3.1. If $u = P[f]$ and $f = K * F$, Theorem 5.4 shows that

$$|u(x,y)| \le C(M_\Phi F)(x_0)(\beta + \|K\|_1) \tag{5.1}$$

in $\Omega^\Phi_{K,\beta}(x_0)$. Thus

$$\mathfrak{M}[\Omega^\Phi_{K,\beta}]f(x_0) \le C(\beta + \|K\|_1)M_\Phi F(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n.$$

Combining Lemma 5.1, we have Theorem 3.1. □

Proof of Theorem 3.2. Let

$$E = E(\beta,\epsilon) = \left\{x_0 \in \mathbb{R}^n : \limsup_{(x,y)\in\Omega^\Phi_{K,\beta},(x,y)\to(x_0,0)} |u(x,y) - f(x_0)| > \epsilon\right\}.$$

We shall prove that $|E|$=0 for all β and for all ϵ. For each $j \in \mathbb{N}$, there exists $G_j \in C_{\mathrm{comp}}(\mathbb{R}^n)$ such that

$$\|F - G_j\|_\Phi \le 1/j.$$

Let $g_j = K * G_j$ and $v_j(x,y) = P_y * g_j(x)$. Then

$$\|f - g_j\|_\Phi \le \|K\|_1\|F - G_j\|_\Phi \to 0 \quad \text{as} \quad j \to +\infty.$$

We can use (2.5) for $3C(\beta+1)(F-G_j)/\epsilon$ and $3(f-g_j)/\epsilon$, where C is the constant in Theorem 3.1. Let

$$E_{1,j} = \left\{ x_0 \in \mathbb{R}^n : \limsup_{(x,y)\in\Omega^{\Phi}_{K,\beta},(x,y)\to(x_0,0)} |u(x,y)-v_j(x,y)| > \epsilon/3 \right\},$$

$$E_{2,j} = \left\{ x_0 \in \mathbb{R}^n : \limsup_{(x,y)\in\Omega^{\Phi}_{K,\beta},(x,y)\to(x_0,0)} |v_j(x,y)-g_j(x_0)| > \epsilon/3 \right\},$$

$$E_{3,j} = \{ x_0 \in \mathbb{R}^n : |g_j(x_0)-f(x_0)| > \epsilon/3 \}.$$

Then $E \subset E_{1,j} \cup E_{2,j} \cup E_{3,j}$ for $j = 1, 2, \cdots$. By Theorem 3.1 we have

$$|E_{1,j}| \le \int_{\mathbb{R}^n} \Phi\left(\frac{C(\beta+1)|F(x)-G_j(x)|}{\epsilon/3}\right) dx \to 0 \quad \text{as} \quad j \to +\infty.$$

Since v_j is continuous on the closure of $\mathbb{R}^{n+1}_+$,

$$\limsup_{(x,y)\in\Omega^{\Phi}_{K,\beta},(x,y)\to(x_0,0)} |v_j(x,y)-g_j(x_0)| = 0.$$

Hence we have $|E_{2,j}| = 0$. And we have

$$|E_{3,j}| = \int_{E_{3,j}} dx \le \frac{1}{\Phi(1)} \int_{\mathbb{R}^n} \Phi\left(\frac{|g_j(x)-f(x)|}{\epsilon/3}\right) dx \to 0 \quad \text{as} \quad j \to +\infty. \qquad \square$$

Proof of Proposition 3.4. Let $B = B(0, \tau_\beta(y))$. Since $\|P_y * K\|_{\widetilde{\Phi},B}$ equals the norm of $P_y * K$ in the Orlicz space $L^{\widetilde{\Phi}}(B, dx/|B|)$, using (2.3), we have

$$\begin{aligned}\|P_y * K\|_{\widetilde{\Phi},B} &= \|P_y * K\|_{L^{\widetilde{\Phi}}(B,dx/|B|)} \\ &\le \sup\left\{ \left|\int_B (P_y * K)(x)F(x)\,dx/|B|\right| : \int_B \Phi(|F(x)|)dx/|B| \le 1 \right\}.\end{aligned}$$

Then there exists $F \in L^{\Phi}(\mathbb{R}^n)$ such that $F(x) \ge 0$, $F(x) = 0$ for $x \notin B$,

$$\int_B \Phi(|F(x)|)dx/|B| \le 1 \quad \text{and} \quad \|P_y * K\|_{\widetilde{\Phi},B} \le 2\int_B (P_y * K)(x)F(x)\,dx/|B|.$$

Let $u = P[f]$, $f = K * F$. Then

$$u(0,y) = P_y * K * F(0) = \int (P_y * K)(0-x)F(x)\,dx = \int_B (P_y * K)(x)F(x)\,dx.$$

Hence

$$u(0,y) \ge |B|\|P_y * K\|_{\widetilde{\Phi},B}/2 = k(\tau_\beta(y), y)/2 = \beta/2.$$

If $x \in B(0,\omega(y))$, then $(0,y) \in \Omega(x)$, so that $u(0,y) \le \mathfrak{M}[\Omega]f(x)$. Hence

$$B(0,\omega(y)) \subset \{\mathfrak{M}[\Omega]f(x) \ge u(0,y)\} \subset \{\mathfrak{M}[\Omega]f(x) > \beta/4\}.$$

We have, for $\beta \ge 4c_*$,

$$\begin{aligned}(5.2) \quad \sigma_n \omega(y)^n \le |\{\mathfrak{M}[\Omega]f(x) > \beta/4\}| &\le \int \Phi\left(\frac{c_* F(x)}{\beta/4}\right) dx \\ &\le \int \Phi(F(x))\,dx \le |B| \le \sigma_n \tau_\beta(y)^n,\end{aligned}$$

by (3.6). Then we have, for $\beta \ge 4c_*$, $\omega(y) \le \tau_\beta(y)$. $\square$

In (5.2), if $\Phi \in \Delta_2$, then, for all $\beta > 0$,

$$\sigma_n \omega(y)^n \le |\{\mathfrak{M}[\Omega]f(x) > \beta/4\}| \le \int \Phi\left(\frac{c_* F(x)}{\beta/4}\right) dx \le C_{c_*,\beta} \int \Phi(F(x))\, dx \le C_{c_*,\beta} |B| \le C_{c_*,\beta} \sigma_n \tau_\beta(y)^n, \tag{5.3}$$

where $C_{c_*,\beta}$ depends on Φ, c_* and β.

Proof of Theorem 3.5. Fix $\beta_0 > 0$ and let $R = \tau_{\beta_0}$. Since $\tau_\beta(y)$ is increasing with respect to β, and $\|P_y * K\|_{\widetilde{\Phi},B(0,r)}$ is decreasing with respect to r, we have

$$\|P_y * K\|_{\widetilde{\Phi},B(0,\tau_\beta(y))} \le \|P_y * K\|_{\widetilde{\Phi},B(0,\tau_{\beta_0}(y))} \quad \text{for} \quad \beta_0 \le \beta.$$

Hence

$$\left(\frac{\tau_\beta(y)}{\tau_{\beta_0}(y)}\right)^n = \frac{\beta/\|P_y * K\|_{\widetilde{\Phi},B(0,\tau_\beta(y))}}{\beta_0/\|P_y * K\|_{\widetilde{\Phi},B(0,\tau_{\beta_0}(y))}} \ge \frac{\beta}{\beta_0}.$$

For $\beta \ge b^n \beta_0$, we have

$$b\tau_{\beta_0}(y) \le b\left(\frac{\beta_0}{\beta}\right)^{1/n} \tau_\beta(y) \le \tau_\beta(y).$$

Let $\Phi \in \Delta_2 \cap \nabla_2$. By Theorem 3.1

$$\left|\{x \in \mathbb{R}^n : \mathfrak{M}[\Omega^\Phi_{K,\beta}]f(x) > t\}\right| \le \int_{\mathbb{R}^n} \Phi\left(\frac{C(\beta+1)|F(x)|}{t}\right) dx.$$

In Proof of Proposition 3.4 and (5.3), using τ_β, $C(\beta+1)$ and β_0 instead of ω, c_* and β, respectively, we have

$$\tau_\beta(y) \le C_{\beta,\beta_0}\, \tau_{\beta_0}(y),$$

where $C_{\beta,\beta_0} > 0$ depends on Φ, β, β_0. □

6. Proofs of Proposition 3.6 and Theorems 3.7–3.9

To prove Proposition 3.6, we state a proposition and two lemmas.

Proposition 6.1. *Assume that $\Phi \in \nabla_2$, that K and H are kernels, not in $L^{\widetilde{\Phi}}$, and that there are constants a, b, $\epsilon > 0$ such that*

$$0 < a \le \frac{K(x)}{H(x)} \le b < +\infty \quad \textit{if} \quad 0 < |x| < \epsilon. \tag{6.1}$$

Then there are constants a', b', such that

$$0 < a' \le \frac{\|P_y * K\|_{\widetilde{\Phi},B(0,r)}}{\|P_y * H\|_{\widetilde{\Phi},B(0,r)}} \le b' < +\infty \tag{6.2}$$

for $0 < r < +\infty$ and $0 < y < 1$.

Proof. Let $K = K' + K''$, $H = H' + H''$, where K' and H' are the restrictions of K and H to $\{|x| < \epsilon\}$. Then $K'' \in L^\infty \cap L^1$. Hence

$$\|P_y * K''\|_{\widetilde{\Phi},B(0,r)} = \inf\left\{\lambda > 0 : \frac{1}{|B(0,r)|}\int_{B(0,r)} \widetilde{\Phi}\left(\frac{P_y * K''(x)}{\lambda}\right) dx \le 1\right\}$$
$$\le \inf\left\{\lambda > 0 : \frac{1}{|B(0,r)|}\sup_{z\in\mathbb{R}^n}\int_{B(z,r)} \widetilde{\Phi}\left(\frac{K''(x)}{\lambda}\right) dx \le 1\right\} \le \frac{\|K''\|_\infty}{\widetilde{\Phi}^{-1}(1)} < +\infty.$$

The same is true for $\|P_y * H''\|_{\widetilde{\Phi},B(0,r)}$. Since $\|P_y * K\|_{\widetilde{\Phi},B(0,r)}$ and $\|P_y * H\|_{\widetilde{\Phi},B(0,r)}$ tend to $+\infty$ when $y \to 0$ (see (4.4) in Lemma 4.2), it follows that the upper and lower limits of the ratio in (6.2) are unchanged if K and H are replaced by K' and H'. Since $aH' \le K' \le bH'$ and $\widetilde{\Phi} \in \Delta_2$, (6.2) holds. □

Lemma 6.2. (i) *If $\rho(r)$ is almost increasing for small $r > 0$, then*

$$\int_0^r \rho(t)\,dt \sim r\rho(r) \quad \text{for small } r > 0.$$

(ii) *If $\rho(r)$ is almost increasing and $\rho(r)/r^\beta$ is almost decreasing for small $r > 0$ with $0 \le \beta < n$, then*

$$\int_r^1 \frac{\rho(t)}{t^{n+1}}\,dt \sim \frac{\rho(r)}{r^n} \quad \text{for small } r > 0.$$

(iii) *If $\rho(r)/r^\alpha$ is almost increasing for small $r > 0$ with $0 < \alpha \le n$, then*

$$\bar{\rho}(r) = \int_0^r \frac{\rho(t)}{t}\,dt \sim \rho(r) \quad \text{for small } r > 0.$$

(iv) *If $\rho(r)/r^\beta$ is almost decreasing for small $r > 0$ with $\beta > 0$, then $\bar{\rho}(r)/r^\beta$ is also almost decreasing for small $r > 0$.*

Proof. (i) We note that $\rho(r)/r^n$ is decreasing.

$$\frac{1}{n+1} r\rho(r) = \frac{\rho(r)}{r^n}\int_0^r t^n\,dt \le \int_0^r \rho(t)\,dt \le Cr\rho(r).$$

(ii) If $0 < r < 1/2$, then

$$\frac{1}{2n}\frac{\rho(r)}{r^n} \le \rho(r)\int_r^1 \frac{1}{t^{n+1}}\,dt \le C\int_r^1 \frac{\rho(t)}{t^{n+1}}\,dt \le C\frac{\rho(r)}{r^\beta}\int_r^1 \frac{1}{t^{n-\beta+1}}\,dt \le \frac{C}{n-\beta}\frac{\rho(r)}{r^n}.$$

(iii) Using the decreasingness of $\rho(r)/r^n$, we have

$$\frac{1}{n}\rho(r) = \frac{\rho(r)}{r^n}\int_0^r t^{n-1}\,dt \le \int_0^r \frac{\rho(t)}{t}\,dt \le C\frac{\rho(r)}{r^\alpha}\int_0^r \frac{1}{t^{-\alpha+1}}\,dt \le \frac{C}{\alpha}\rho(r).$$

(iv) For $r < s$, let $t = (r/s)u$. Then

$$\int_0^r \frac{\rho(t)}{t}\,dt = \int_0^s \frac{\rho((r/s)u)}{u}\,du \ge \left(\frac{r}{s}\right)^\beta \int_0^s \frac{\rho(u)}{u}\,du. \quad \square$$

Lemma 6.3. *Suppose ρ is almost increasing and $\rho(r)/r^\beta$ is almost decreasing for small $r > 0$ with $0 < \beta < n$. If $\Phi \in \nabla_2$, then*

$$\|P_y * K_\rho\|_{\widetilde{\Phi},B(0,r)} \sim \inf\left\{\lambda > 0 : r^{-n}\int_y^r \widetilde{\Phi}\left(\frac{\rho(t)}{\lambda t^n} + \frac{y\bar{\rho}(t)}{\lambda t^{n+1}}\right) t^{n-1}\,dt \le 1\right\}, \tag{6.3}$$

for $0 < y \le r/2$.

Proof. Let

$$H_\rho(x) = \int_0^1 \frac{\rho(t)}{t} P_t(x)\,dt.$$

Using

$$|x|^2 + t^2 \sim \begin{cases} |x|^2 & (0 \le t \le |x|), \\ t^2 & (|x| < t), \end{cases}$$

we have

$$H_\rho(x) \sim \frac{1}{|x|^{n+1}} \int_0^{|x|} \rho(t)\,dt + \int_{|x|}^1 \frac{\rho(t)}{t^{n+1}}\,dt \quad \text{for} \quad |x| < 1/2.$$

By Lemma 6.2, we have

$$K_\rho(x) \sim H_\rho(x) \quad \text{for} \quad |x| < 1/2.$$

By $P_t * P_y = P_{t+y}$, we have

$$(H_\rho * P_y)(x) = \int_0^1 \frac{\rho(t)}{t} P_{t+y}(x)\,dt.$$

For $|x| + y < 1/2$, let

$$I_1 = \int_0^{|x|+y} \frac{\rho(t)}{t} P_{t+y}(x)\,dt, \quad I_2 = \int_{|x|+y}^1 \frac{\rho(t)}{t} P_{t+y}(x)\,dt.$$

Then

$$I_2 = \int_{|x|+y}^1 \frac{\rho(t)}{t} \frac{c_n(t+y)}{(|x|^2 + (t+y)^2)^{(n+1)/2}}\,dt. \sim \int_{|x|+y}^1 \frac{\rho(t)}{t^{n+1}}\,dt \sim \frac{\rho(|x|+y)}{(|x|+y)^n}.$$

If $|x| \le y$, then

$$I_1 \sim \int_0^{|x|+y} \frac{\rho(t)}{t} \frac{t+y}{(|x|+t+y)^{n+1}}\,dt \sim \frac{1}{y^n}\bar{\rho}(|x|+y) \sim \frac{\bar{\rho}(y)}{y^n}.$$

If $|x| > y$, then

$$\begin{aligned} I_1 &\sim \int_0^{|x|+y} \frac{\rho(t)}{t} \frac{t+y}{(|x|+t+y)^{n+1}}\,dt \sim \frac{1}{|x|^{n+1}} \int_0^{|x|+y} \frac{\rho(t)}{t}(t+y)\,dt \\ &= \frac{1}{|x|^{n+1}}((|x|+y)\rho(|x|+y) + y\bar{\rho}(|x|+y)) \sim \frac{\rho(|x|)}{|x|^n} + \frac{y\bar{\rho}(|x|)}{|x|^{n+1}}. \end{aligned}$$

Hence

$$P_y * H_\rho(x) \sim \begin{cases} \dfrac{\bar{\rho}(y)}{y^n} & (|x| \le y), \\ \dfrac{\rho(|x|)}{|x|^n} + \dfrac{y\bar{\rho}(|x|)}{|x|^{n+1}} & (|x| \ge y), \end{cases} \quad \text{when} \quad |x| + y < 1/2.$$

Therefore

$$\begin{aligned} \int_{B(0,y)} \widetilde{\Phi}\left(\frac{P_y * H_\rho(x)}{\lambda}\right) dx &\sim \widetilde{\Phi}\left(\frac{\bar{\rho}(y)}{\lambda y^n}\right) y^n \\ &\sim \int_y^{2y} \widetilde{\Phi}\left(\frac{\bar{\rho}(t)}{\lambda t^n}\right) t^{n-1}\,dt \sim \int_y^{2y} \widetilde{\Phi}\left(\frac{y\bar{\rho}(t)}{\lambda t^{n+1}}\right) t^{n-1}\,dt, \end{aligned}$$

and

$$\int_{B(0,r)\setminus B(0,y)} \widetilde{\Phi}\left(\frac{P_y * H_\rho(x)}{\lambda}\right) dx \sim \int_y^r \widetilde{\Phi}\left(\frac{\rho(t)}{\lambda t^n} + \frac{y\bar{\rho}(t)}{\lambda t^{n+1}}\right) t^{n-1}\,dt.$$

This shows (6.3). □

Proof of Proposition 3.6. Let $\tau = \tau_1$. From

$$k(\tau(y), y) = |B(0,\tau(y))| \|P_y * K_\rho\|_{\widetilde{\Phi}, B(0,\tau(y))} = 1,$$

it follows that

$$\|P_y * K_\rho\|_{\widetilde{\Phi}, B(0,\tau(y))} = \frac{1}{|B(0,\tau(y))|} \sim \frac{1}{\tau(y)^n}.$$

By Lemma 6.3, we have

$$\|P_y * K_\rho\|_{\widetilde{\Phi},B(0,\tau(y))} \sim \inf\left\{\lambda > 0 : \tau(y)^{-n}\int_y^{\tau(y)} \widetilde{\Phi}\left(\frac{\rho(t)}{\lambda t^n} + \frac{y\bar{\rho}(t)}{\lambda t^{n+1}}\right) t^{n-1}\,dt \le 1\right\}.$$

By (2.4) we have (3.7).

If $\widetilde{\Phi} \in \nabla_2$, then

$$\int_1^{+\infty} \widetilde{\Phi}\left(\frac{1}{s^n}\right) s^{n-1}\,ds < +\infty.$$

Hence

$$\begin{aligned}\tau(y)^{-n}&\int_{\tau(y)}^1 \widetilde{\Phi}\left(\frac{\tau(y)^n\rho(t)}{t^n} + \frac{y\tau(y)^n\bar{\rho}(t)}{t^{n+1}}\right) t^{n-1}\,dt\\ &\le C\tau(y)^{-n}\int_{\tau(y)}^1 \widetilde{\Phi}\left(\frac{\tau(y)^n}{t^n}\right) t^{n-1}\,dt \le C\int_1^{1/\tau(y)} \widetilde{\Phi}\left(\frac{1}{s^n}\right) s^{n-1}\,ds \le C.\end{aligned}$$

Therefore we have (3.8).

If $y \le t$, then

$$\frac{\tau(y)^n\rho(t)}{t^n} + \frac{y\tau(y)^n\bar{\rho}(t)}{t^{n+1}} \le 2\frac{\tau(y)^n\bar{\rho}(t)}{t^n}.$$

From the almost increasingness of $\widetilde{\Phi}(r)/r^{p'}$ and the almost decreasingness of $\bar{\rho}(r)/r^\beta$, it follows that

$$\begin{aligned}\widetilde{\Phi}\left(\frac{\tau(y)^n\rho(t)}{t^n} + \frac{y\tau(y)^n\bar{\rho}(t)}{t^{n+1}}\right) &\le C\widetilde{\Phi}\left(\frac{\tau(y)^n\bar{\rho}(t)}{t^n}\right) \le C\left(\frac{\tau(y)^n\bar{\rho}(t)}{t^n}\right)^{p'} \frac{\widetilde{\Phi}\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right)}{\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right)^{p'}}\\ &\le C\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^\beta}\right)^{p'} \frac{\widetilde{\Phi}\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right)}{\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right)^{p'}}\, t^{-p'(n-\beta)} \quad \text{for} \quad y \le t.\end{aligned}$$

Hence

$$\begin{aligned}\tau(y)^{-n}&\int_y^{\tau(y)} \widetilde{\Phi}\left(\frac{\tau(y)^n\rho(t)}{t^n} + \frac{y\tau(y)^n\bar{\rho}(t)}{t^{n+1}}\right) t^{n-1}\,dt\\ &\le C\tau(y)^{-n}\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^\beta}\right)^{p'} \frac{\widetilde{\Phi}\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right)}{\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right)^{p'}}\, y^{-p'(n/p-\beta)} = C\left(\frac{y}{\tau(y)}\right)^n \widetilde{\Phi}\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right).\end{aligned}$$

On the other hand

$$\begin{aligned}\tau(y)^{-n}&\int_y^{\tau(y)} \widetilde{\Phi}\left(\frac{\tau(y)^n\rho(t)}{t^n} + \frac{y\tau(y)^n\bar{\rho}(t)}{t^{n+1}}\right) t^{n-1}\,dt\\ &\ge \tau(y)^{-n}\int_y^{2y} \widetilde{\Phi}\left(\frac{y\tau(y)^n\bar{\rho}(t)}{t^{n+1}}\right) t^{n-1}\,dt\\ &\sim \tau(y)^{-n}\int_y^{2y} \widetilde{\Phi}\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right) y^{n-1}\,dt = C\left(\frac{y}{\tau(y)}\right)^n \widetilde{\Phi}\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right). \quad \square\end{aligned}$$

Proof of Theorem 3.7. We show that R is equivalent to τ in Proposition 3.6. Then we have the conclusion by Theorem 3.5.

In the case (i), using $\rho \sim \bar{\rho}$, we have

$$\frac{\tau(y)^n \rho(t)}{t^n} + \frac{y\tau(y)^n \bar{\rho}(t)}{t^{n+1}} \sim \frac{\tau(y)^n \rho(t)}{t^n} \quad \text{for} \quad y \le t.$$

By $\widetilde{\Phi}(r) \sim r^{p'}$, we have

$$\tau(y)^{-n} \widetilde{\Phi}\left(\frac{\tau(y)^n \rho(t)}{t^n} + \frac{y\tau(y)^n \bar{\rho}(t)}{t^{n+1}}\right) t^n \sim \tau(y)^{np'/p} \left(\frac{\rho(t)}{t^{n/p}}\right)^{p'}.$$

Using (3.8) in Proposition 3.6, we have that τ is equivalent to R in (3.10).

In the case (ii), it follows from (3.9) in Proposition 3.6. □

Proof of Theorem 3.8. We have that $\widetilde{\Phi}(r) \sim r^{p'} \ell(r)^{-p'/p}$. Actually, $\Phi(r) = r^p \ell(r)$ implies $\Phi^{-1}(r) \sim r^{1/p} \ell(r)^{-1/p}$. From (2.1) it follows that $\widetilde{\Phi}^{-1}(r) \sim r^{1/p'} \ell(r)^{1/p}$. This implies $\widetilde{\Phi}(r) \sim r^{p'} \ell(r)^{-p'/p}$.

In the case (i), using $\rho \sim \bar{\rho}$, we have

$$\frac{\tau(y)^n \rho(t)}{t^n} + \frac{y\tau(y)^n \bar{\rho}(t)}{t^{n+1}} \sim \frac{\tau(y)^n \rho(t)}{t^n} \quad \text{for} \quad y \le t,$$

and

$$\begin{aligned} \widetilde{\Phi}\left(\frac{\tau(y)^n \rho(t)}{t^n} + \frac{y\tau(y)^n \bar{\rho}(t)}{t^{n+1}}\right) &\sim \widetilde{\Phi}\left(\frac{\tau(y)^n \rho(t)}{t^n}\right) \\ &\sim \left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{p'} \ell\left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{-p'/p} \quad \text{for} \quad y \le t. \end{aligned}$$

Let

$$E(y) = \tau(y)^{-n} \int_y^{\tau(y)} \left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{p'} \ell\left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{-p'/p} t^{n-1}\, dt.$$

Then, by (3.7) in Proposition 3.6, we have $C^{-1} \le E(y) \le C$. Choose $\delta > 0$ and $\nu > 1$ so that

$$1 < \frac{p'}{1 - \delta p'/n} < \nu < \frac{1 - \delta p}{2\delta},$$

and let

$$\begin{aligned} B_1(y) =& \tau(y)^{-n} \int_{\tau(y)^\nu}^1 \left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{p'} \ell\left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{-p'/p} t^{n-1}\, dt, \\ B_2(y) =& \tau(y)^{-n} \int_{\tau(y)^\nu}^1 \left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{p'} \ell\left(\frac{1}{t}\right)^{-p'/p} t^{n-1}\, dt. \end{aligned}$$

If we show that

$$y < \tau(y)^\nu < \tau(y) < 1 \quad \text{for small } y > 0, \tag{6.4}$$

$$\ell\left(\frac{\tau(y)^n \rho(t)}{t^n}\right) \sim \ell\left(\frac{1}{t}\right) \quad \text{for } y \le t \le \tau(y)^\nu, \tag{6.5}$$

$$B_1(y),\ B_2(y) \to 0 \quad \text{as } y \to 0, \tag{6.6}$$

then we have

$$\begin{aligned} E(y) &\sim \tau(y)^{-n} \int_y^1 \left(\frac{\tau(y)^n \rho(t)}{t^n}\right)^{p'} \ell\left(\frac{1}{t}\right)^{-p'/p} t^{n-1}\, dt \\ &= \tau(y)^{np'/p} \int_y^1 m(t)^{p'} \ell\left(\frac{1}{t}\right)^{-p'/p} t^{-1}\, dt \quad \text{for small } y > 0, \end{aligned}$$

and then

$$\tau(y) \sim R(y) = \left(\int_y^1 m(t)^{p'} \ell \left(\frac{1}{t} \right)^{-p'/p} t^{-1} \, dt \right)^{-p/(np')}.$$

In the following we show (6.4)–(6.6). From the almost increasingness of $r^{\delta p'} \ell(r)^{-p'/p}$ and

$$\frac{\tau(y)^n \rho(t)}{t^n} \le \frac{C}{t^n},$$

it follows that

$$\begin{aligned} (6.7) \quad \ell \left(\frac{\tau(y)^n \rho(t)}{t^n} \right)^{-p'/p} &= \left(\frac{\tau(y)^n \rho(t)}{t^n} \right)^{-\delta p'} \left(\frac{\tau(y)^n \rho(t)}{t^n} \right)^{\delta p'} \ell \left(\frac{\tau(y)^n \rho(t)}{t^n} \right)^{-p'/p} \\ &\le C \left(\frac{\tau(y)^n \rho(t)}{t^n} \right)^{-\delta p'} \left(\frac{1}{t^n} \right)^{\delta p'} \ell \left(\frac{1}{t} \right)^{-p'/p} \\ &= C \tau(y)^{-\delta n p'} t^{-\delta n p'/p} m(t)^{-\delta p'} \ell \left(\frac{1}{t} \right)^{-p'/p} \quad \text{for } t < 1, \ \tau(y) < 1. \end{aligned}$$

Hence, using the almost increasingness of $t^{\delta np'/p} m(t)^{p' - \delta p'} \ell \left(\frac{1}{t} \right)^{-p'/p}$, we have

$$\begin{aligned} E(y) &= \tau(y)^{np'/p} \int_y^{\tau(y)} m(t)^{p'} \ell \left(\frac{\tau(y)^n \rho(t)}{t^n} \right)^{-p'/p} t^{-1} \, dt \\ &\le C \tau(y)^{np'/p - \delta np'} \int_y^{\tau(y)} t^{-\delta np'/p} m(t)^{p' - \delta p'} \ell \left(\frac{1}{t} \right)^{-p'/p} t^{-1} \, dt \\ &\le C \tau(y)^{np'/p - \delta np'} \int_y^{\tau(y)} t^{-2\delta np'/p} t^{-1} \, dt \le C \tau(y)^{np'/p - \delta np'} y^{-2\delta np'/p}. \end{aligned}$$

From $C^{-1} \le E(y)$ it follows that

$$y \le C \tau(y)^{(1-\delta p)/(2\delta)} < \tau(y)^{\nu} \quad \text{for small } y > 0.$$

Then we have (6.4). For $y \le t \le \tau(y)^{\nu} < 1$, we have

$$\frac{C}{t^n} \ge \frac{\tau(y)^n \rho(t)}{t^n} \ge \frac{t^{n/\nu} \rho(t)}{t^n} = \frac{m(t)}{t^{\delta}} \frac{1}{t^{n - n/p - \delta - n/\nu}} \ge \frac{C}{t^{n/p' - \delta - n/\nu}},$$

We note that $n/p' - \delta - n/\nu > 0$. Hence we have (6.5). By (6.7) we have

$$\begin{aligned} B_1(y) &= \tau(y)^{np'/p} \int_{\tau(y)^{\nu}}^1 m(t)^{p'} \ell \left(\frac{\tau(y)^n \rho(t)}{t^n} \right)^{-p'/p} t^{-1} \, dt \\ &\le C \tau(y)^{np'/p - \delta np'} \int_{\tau(y)^{\nu}}^1 t^{-\delta np'/p} m(t)^{p' - \delta p'} \ell \left(\frac{1}{t} \right)^{-p'/p} t^{-1} \, dt \\ &\le C \tau(y)^{np'/p - \delta np'} \int_{\tau(y)^{\nu}}^1 t^{-2\delta np'/p} t^{-1} \, dt \\ &\le C \tau(y)^{np'/p - \delta np'} \tau(y)^{-2\nu\delta np'/p} \\ &= C \tau(y)^{(np'/p)(1 - \delta p - 2\nu\delta)} \to 0 \quad \text{as } y \to 0. \end{aligned}$$

Using the almost increasingness of $t^{\delta np'/p}m(t)^{p'}\ell\left(\frac{1}{t}\right)^{-p'/p}$, we have

$$\begin{aligned}B_2(y) = \tau(y)^{np'/p}\int_{\tau(y)^\nu}^1 m(t)^{p'}\ell\left(\frac{1}{t}\right)^{-p'/p} t^{-1}\,dt \\ = \tau(y)^{np'/p}\int_{\tau(y)^\nu}^1 t^{-\delta np'/p+\delta np'/p}m(t)^{p'}\ell\left(\frac{1}{t}\right)^{-p'/p} t^{-1}\,dt \\ \le C\tau(y)^{np'/p}\int_{\tau(y)^\nu}^1 t^{-\delta np'/p}t^{-1}\,dt \\ \le C\tau(y)^{np'/p}\tau(y)^{-\nu\delta np'/p} \\ = C\tau(y)^{(np'/p)(1-\nu\delta)} \to 0 \quad \text{as } y \to 0.\end{aligned}$$

In the cases (ii) and (iii), let $W(r) = \widetilde{\Phi}(r)/r \sim r^{p'-1}\ell(r)^{-p'/p}$. Then $W^{-1}(r) \sim r^{1/(p'-1)}\ell(r)$. Using (3.9) in Proposition 3.6, we have

$$W\left(\frac{\tau(y)^n\bar{\rho}(y)}{y^n}\right) \sim \frac{1}{\bar{\rho}(y)},$$

and then

$$\frac{\tau(y)^n\bar{\rho}(y)}{y^n} \sim W^{-1}\left(\frac{1}{\bar{\rho}(y)}\right) \sim \frac{1}{\bar{\rho}(y)^{1/(p'-1)}}\ell\left(\frac{1}{\bar{\rho}(y)}\right).$$

Hence

$$\tau(y) \sim y\,\frac{\ell(1/\bar{\rho}(y))^{1/n}}{\bar{\rho}(y)^{p/n}}.$$

We note that $\bar{\rho}(y) \sim y^\alpha m(y)$ in the case (ii) and that $\bar{\rho}(y) \sim \bar{m}(y)$ in the case (iii). Therefore we have that τ is equivalent to R in (3.13) or in (3.14). □

Proof of Theorem 3.9. Let

$$w(y,t) = \frac{\tau(y)^n\rho(t)}{t^n} + \frac{y\tau(y)^n\bar{\rho}(t)}{t^{n+1}}. \tag{6.8}$$

Then $w(y,t) \le w(y,y)$ for $t > y$. By the increasingness of $\widetilde{\Phi}(r)/r$, we have

$$\begin{aligned}\tau(y)^{-n}\int_y^{\tau(y)} \widetilde{\Phi}(w(y,t))t^{n-1}\,dt \le \tau(y)^{-n}\int_y^{\tau(y)} w(y,t)\frac{\widetilde{\Phi}(w(y,y))}{w(y,y)}t^{n-1}\,dt \\ = \frac{\widetilde{\Phi}(w(y,y))}{w(y,y)}\int_y^{\tau(y)}\left(\frac{\rho(t)}{t} + \frac{y\bar{\rho}(t)}{t^2}\right)dt \le C\frac{\widetilde{\Phi}(w(y,y))}{w(y,y)}\bar{\rho}(\tau(y)).\end{aligned}$$

By Proposition 3.6 (i), we have

$$\frac{\widetilde{\Phi}(w(y,y))}{w(y,y)}\bar{\rho}(\tau(y)) \ge C^{-1}. \tag{6.9}$$

From $\bar{\rho}(\tau(y)) \to 0$ as $y \to 0$, it follows that $w(y,y) \to +\infty$ as $y \to 0$. Therefore we have $R(y) = y/\bar{\rho}(y)^{1/n} \le \tau(y)$ for small $y > 0$. □

Proof of Example 3.3. Let

$$\ell(r) = \begin{cases}\log r & \text{for large } r > 0,\\ 1/\log(1/r) & \text{for small } r > 0.\end{cases}$$

Then $\widetilde{\Phi}(r) \sim r\ell(r)$. Let $w(y,t)$ be as in (6.8). For small $y > 0$, $w(y,y)$ is large. Then $\ell(w(y,y)) = \log w(y,y)$. By (6.9), we have

$$C^{-1} \leq \bar{\rho}(\tau(y))\ell(w(y,y)) \leq \bar{\rho}(\tau(y)) \log\left(\frac{\tau(y)^n \bar{\rho}(y)}{y^n}\right)$$

This shows that

$$C^{-1}\bar{\rho}(\tau(y))^{-1} \leq -n\log\frac{1}{\tau(y)} + n\log\frac{1}{y} + \log\bar{\rho}(y).$$

By $\bar{\rho}(y) \sim (\log(1/y))^{-1}$ for small $y > 0$, we have

$$\frac{1}{\tau(y)} \leq \frac{(\log(1/y))^{-1/n}}{y^{1-\epsilon}} = \frac{1}{R(y)}, \quad \epsilon = 1 - \frac{n}{n + C^{-1}}. \quad \square$$

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