K-SET CONTRACTIVE RETRACTIONS IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. Let $X$ be an infinite-dimensional Banach space, and let $B_X$ and $S_X$ be its closed unit ball and unit sphere, respectively. A continuous mapping $R : B_X \to S_X$ is said to be a retraction provided that $x = Rx$ for all $x \in S_X$. It is well known that when $X$ is finite-dimensional there is no retraction from $B_X$ onto $S_X$. We prove that in some Banach spaces of continuous functions for every $\varepsilon > 0$ there exist retractions of the closed unit ball onto the unit sphere being a $(1 + \varepsilon)$-set contraction.

1. Introduction

The Scottish Book [8] contains the following question (Problem 36) raised around 1935 by S. Ulam: "There exists a retraction of the closed unit ball of a Hilbert space onto the unit sphere?" S. Kakutani [6] gave a positive answer to this question. V. Klee [7] proved that answer to Ulam’s question is "yes" in the more general setting of infinite-dimensional Banach spaces. B. Nowak [9] using a complicated construction that was subsequently somewhat simplified by Y. Benyamini and Y. Sternfeld [3] showed that, for any infinite-dimensional Banach space $X$, there is a retraction $R : B_X \to S_X$ satisfying the Lipschitz condition

$$\|Rx - Ry\| \leq k \|x - y\|, \quad \text{for all } x, y \in B_X.$$  

Given an infinite-dimensional Banach space $X$, let $k_0(X)$ denote the infimum of the k’s for which such retraction exists.

Then $k_0(X) \geq 3$ (See [5]). Recall that, if $A$ is a bounded subset of a Banach space $X$, the Hausdorff measure of noncompactness of $A$ is defined by

$$\chi(A) := \inf \{r > 0 : A \text{ can be covered by a finite number of balls centered in } X\}.$$  

A continuous mapping $T : D(T) \subset X \to X$ is said to be a k-set contraction if there exists a constant $k \geq 0$ such that

$$\chi(T(A)) \leq k \chi(A), \quad \text{for all bounded sets } A \subset D(T).$$  

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the maximum norm $|\cdot|_{\infty}$. Throughout this paper we shall use the following notations. $E := (E, \|\cdot\|)$ will denote a finite-dimensional real normed space and $K$ a compact convex subset of $E$ with nonempty interior (Without loss of generality, we can assume that $K$ contains the origin as an interior point). $C(K, \mathbb{R}^n)$ the space of continuous functions on $K$ with values in $\mathbb{R}^n$ equipped with the sup norm $\|\cdot\|_{\infty}$. Let $X$ be an infinite-dimensional Banach space. By $k_1(X)$ denote the infimum of the set of all numbers $k$ for which there is a retraction $R : B_X \to S_X$ satisfying the above condition (2). In this context J. Wosko [10] proved that $k_1(C[0, 1]) = 1$ and that for any infinite-dimensional Banach space $X$ there is no 1-set retraction $R : B_X \to S_X$ being Lipschitzian.

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with some constant $k$. Moreover, he posed the problem to estimate $k_1(X)$ for particular classical Banach spaces and to establish for which spaces is $k_1(X) < k_0(X)$. In this note we extend from $C[0, 1]$ to $C(K, \mathbb{R}^n)$ the Wosko’s result, i.e. we prove that $k_1(C(K, \mathbb{R}^n)) = 1$.

2. Preliminaries

Let $Y$ be a real normed space. We write $B_{Y,r}$ to denote the closed ball of $Y$ centered at the origin with radius $r$. For a set $A \subseteq Y$, $\overline{A}$ it is closure, $\text{int}A$ its interior, $\partial A$ its boundary and $\text{diam}A$ its diameter. Further we set $S_{Y,r} := \partial B_{Y,r}$.

Consider the mapping $\varphi : K \setminus \{0\} \rightarrow \partial K$ defined by $\varphi(t) = w_t$, where $w_t$ is the unique element of $\{\lambda t : \lambda \in [0, +\infty] \cap \partial K\}$. Let $\alpha$ be a positive real number such that $B_{E,\alpha} \subset K$. In this section we prove that $\varphi$ satisfies the Lipschitz condition:

\[(2.1) \|w_t - w_s\| \leq L \|t - s\|, \quad \text{for all } s, t \in K \setminus \text{int}B_{E,\alpha}.
\]

Assume that $\mathbb{R}^n$ is the $n$-dimensional Euclidean space provided with the usual inner product $\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$ where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. $|.|_n$ denotes the Euclidean norm on $\mathbb{R}^n$, $\theta(u, v)$ the angle between two non zero vectors $u$ and $v$ of $\mathbb{R}^n$ such that $0 \leq \theta(u, v) \leq \pi$ and $u_v$ the orthogonal projection of $u$ onto $\langle v \rangle := \{\lambda v : \lambda \in \mathbb{R}\}$. Let $K$ be a compact convex set in $\mathbb{R}^n$ containing the origin as an interior point and let $\alpha$ be a positive real number such that $B_{E,\alpha} \subset K$. In order to prove (2.1) it is sufficient to show that it is true for $\varphi : K \setminus \text{int}B_{E,\alpha} \rightarrow \partial K$.

**Lemma 1.** Let $K$ be a compact convex set in $\mathbb{R}^n$ containing the origin as an interior point. Set $\beta := \min \{|u|_n : u \in \partial K\}$ and $d := \text{diam}K$. Then $\inf \{\cos(\theta(v - u, u_v - u)) : u \neq v, |u|_n \leq |v|_n, |u - v|_n < \frac{\beta}{2} \text{ and } u, v \in \partial K\} \geq \frac{\beta}{\sqrt{2n}}$.

**Proof.** Let $u, v \in \partial K$ with $u \neq v, |u|_n \leq |v|_n$ and $|u - v|_n < \frac{\beta}{2}$. We will prove that $\sin(\theta(0, u - v)) = \sin\left(\frac{\beta}{2} - \theta(v - u, u_v - u)\right) \geq \frac{\beta}{\sqrt{2n}}$. Therefore $\cos(\theta(v - u, u_v - u)) \geq \frac{\beta}{\sqrt{2n}}$. Let $r$ be the straight line through $u$ and $v$. Then $r \cap \text{int}B_{E,\alpha} = \emptyset$. Suppose $r \cap S_{E,\alpha} = \{s, t\}$. We have two possible cases. The segment $[u, v]$ contains $\{s, t\}$. Then $\beta \leq |u|_n \leq |u - s|_n + |s|_n \leq |u - v|_n + |s|_n < \beta$, a contradiction! The segment $[u, v]$ does not contain $\{s, t\}$. Let $\pi$ be the plane containing $0, u$ and $v$; $r_1$ the straight line through $v$ tangent to $B_{E,\alpha} \cap \pi$ in $p$, which lies in the half-plane determined by the straight line through $0$ and $v$ that contains $s$ and $t$; $r_2$ the straight line through $0$ and $u$ (see figure below). Then the segment $[p, v] \setminus r_2 = w \in K$. Hence $u \notin \partial K$, again a contradiction! Therefore $\cos(\theta(v - u, u_v - u)) = \sin\left(\frac{\beta}{2} - \theta(v - u, u_v - u)\right) \geq \sin\left(\frac{\beta}{2} - \theta(v - u, u_v - u)\right) \geq \frac{\beta}{\sqrt{2n}}$. \(\blacksquare\)
Proposition 2. Let $K$ be a compact convex set in $\mathbb{R}^n$ containing the origin as an interior point and $\alpha$ be a positive real number such that $B_{\mathbb{R}^n, \alpha} \subset K$. Set $c := \text{max} \{|w|_n : u \in \partial K\}$. Then the map $\varphi$ is uniformly continuous on $K \setminus \int B_{\mathbb{R}^n, \alpha}$.

Proof. Since $K \setminus \int B_{\mathbb{R}^n, \alpha}$ is compact, it is sufficient to show that $\varphi$ is continuous on $K \setminus \int B_{\mathbb{R}^n, \alpha}$. Our proof will be by way of contradiction. So suppose that there exists a sequence $(t_n)$ of elements of $K \setminus \int B_{\mathbb{R}^n, \alpha}$ such that $t_n \rightarrow t \ (n \rightarrow +\infty)$ and $w_{t_n} \rightarrow w \ (n \rightarrow +\infty)$. By the compactness of $\partial K$ we can find a subsequence $(w_{t_{n_k}})$ of $(w_{t_n})$ convergent to $w \in \partial K \setminus \{w\}$. For all $k \in N$, since $t_{n_k} \in \left[\left(\frac{1}{\alpha}, \left|w_{t_{n_k}}\right|_n\right)\right]$, there is $\lambda_{n_k} \in [1, \left|w_{t_{n_k}}\right|_n / \alpha] \subset [1, c / \alpha]$ such that $w_{t_{n_k}} = \lambda_{n_k} t_{n_k}$. Let $(\lambda_{n_k})$ be a subsequence of $(\lambda_{n_k})$ convergent to $\lambda \in [1, c / \alpha]$. Then, since $w_{t_{n_k}} \rightarrow w \ (s \rightarrow +\infty)$, $\lambda_{n_k} \rightarrow \lambda \ (s \rightarrow +\infty)$ and $t_{n_k} \rightarrow t \ (s \rightarrow +\infty)$, we have that $w = \lambda t$. Therefore $w = w_{t_1}$, a contradiction! 

Proposition 3. Let $K$ be a compact convex set in $\mathbb{R}^n$ containing the origin as an interior point and $\alpha$ be a positive real number such that $B_{\mathbb{R}^n, \alpha} \subset K$. Set $\beta := \text{min} \{|w|_n : u \in \partial K\}$ and $d := \text{diam}K$. Then there exists $L$ such that $|w_1 - w_2|_n \leq L \ |t - s|_n$, for all $s, \ t \in K \setminus \int B_{\mathbb{R}^n, \alpha}.

Proof. By the Proposition 2 there exists a $\delta > 0$ such that $|t - s|_n < \delta \Rightarrow |w_1 - w_2|_n < \frac{\beta}{\alpha}$ for all $s, \ t \in K \setminus \int B_{\mathbb{R}^n, \alpha}$. Moreover $|t - s|_n \geq \delta \Rightarrow |w_1 - w_2|_n \leq \frac{d}{\alpha} |t - s|_n$, for all $s, \ t \in K \setminus \int B_{\mathbb{R}^n, \alpha}$. Now, suppose $s, \ t \in K \setminus \int B_{\mathbb{R}^n, \alpha}$, $|t - s|_n < \delta$, $s \neq t$ ($\Rightarrow w_1 \neq w_2$) and $|w_1|_n \leq |w_2|_n$. By Lemma 1 it follows that $\cos \theta := \cos \left(\theta \left(w_1 - w_2, w_{\|w_1\|} - w_{\|w_2\|}\right)\right) \geq \frac{\beta}{\alpha}$. On the other hand it is easy to see that $|w_{s(w_1)} - w_{s(w_2)}|_n \leq \frac{d}{\alpha} |t - s|_n$. Therefore $|w_1 - w_2|_n = \left|w_{s(w_1)} - w_{s(w_2)}\right|_n \leq \frac{d}{\alpha} |t - s|_n$. Set $L := \text{max} \left\{\frac{d}{\alpha}, \frac{2d^2}{\alpha^2}\right\}$. It follows that $|w_1 - w_2|_n \leq L |t - s|_n$, for all $s, \ t \in K \setminus \int B_{\mathbb{R}^n, \alpha}$. 

Corollary 4. Let $K \subset E$ and let $\alpha$ be a positive real number such that $B_{E, \alpha} \subset K$. Then there exists $L$ such that $\|w_1 - w_2\| \leq L \|t - s\|$, for all $s, \ t \in K \setminus \int B_{E, \alpha}$.

We need the following proposition.
Proposition 5. Let $K \subset E$ and $a_0 \in [0, 1]$. Set $\beta := \min \{ \|u\| : u \in \partial K \}$ and $d := \text{diam} K$. Then there is a constant $L$ such that $\forall \alpha \in [a_0, 1], \forall \varepsilon \in \left[0, \frac{\alpha \beta}{a_0} \right], \forall s \in \alpha K$ and $\forall t \in K \setminus \alpha K$

$$\|t - s\| \leq \varepsilon \Rightarrow \|\alpha^{-1}s - w_t\| \leq \left(2L + \frac{1}{a_0} + \frac{4d}{a_0 \beta} \right) \varepsilon.$$

Proof. Fix $\alpha \in \left[\alpha_0, 1, \varepsilon \in \left[0, \frac{\alpha_0 \beta}{a_0} \right], s \in \alpha K$ and $t \in K \setminus \alpha K$ with $\|t - s\| \leq \varepsilon$. Then $\|s\| \leq \frac{\alpha_0 \beta}{a_0 \varepsilon}$. Infact, if $\|s\| \leq \frac{\alpha \beta}{a_0 \varepsilon}$, we have that $\alpha_0 \beta < \|t\| \leq \|t - s\| + \|s\| \leq \varepsilon + \frac{\alpha_0 \beta}{a_0 \varepsilon} < \alpha_0 \beta$, a contradiction! Therefore, by the Corollary 4, there exists a constant $L$ such that $\|w_t - w_s\| \leq L \|t - s\|$ for all $s, t \in K \setminus \frac{\alpha_0 \beta}{a_0}$.

Suppose $\|w_s\| \leq \|w_t\|$. We prove that $\|w_t - \alpha^{-1}s\| \leq \frac{\varepsilon}{\alpha_0}$. Hence $\|w_t - \alpha^{-1}s\| \leq \|w_t - w_s\| + \|w_s - \alpha^{-1}s\| \leq \left(1 + \frac{1}{\alpha_0} \right) \varepsilon$. Clearly $s \in [0, \alpha w_s]$. Moreover, if $\|s\| \leq \left(\alpha - \frac{\varepsilon}{\alpha_0} \right) \|w_s\|$, we have that $\alpha \|w_t\| < \|t\| \leq \|t - s\| + \|s\| < \alpha \|w_s\| \leq \alpha \|w_t\|$, a contradiction! Therefore $\|w_s - \alpha^{-1}s\| \leq \frac{\alpha}{\alpha_0} \|w_s - \alpha w_t\| \leq \frac{\varepsilon}{\alpha_0}$.

Now assume $\|w_s\| \leq \|w_t\|$ and denote by $w_t$ the element of $[0, w_s]$ such that $\|w_t\| = \|w_s\|$. Define the mapping $T : E \to B_{E, \alpha_0}$ by $T(s) = \frac{\alpha_0 \beta}{a_0 \varepsilon} 1 \leq \|s\| < \left(\alpha - \frac{\varepsilon}{\alpha_0} \right) \|w_s\|$, we have that $\frac{\alpha_0 \beta}{a_0 \varepsilon} \|w_s - w_t\| = \|T(w_t) - T(w_s)\| = \|T(t) - T(s)\| \leq 2 \|t - s\|$ for all $s, t \in E$. Therefore, since $s, t \notin B_{E, \alpha_0}$, we have that $\frac{\alpha_0 \beta}{a_0 \varepsilon} \|w_s - w_t\| < \left(\alpha - \frac{\varepsilon}{\alpha_0} \right) \|w_s\| \leq \frac{\alpha_0 \beta}{a_0 \varepsilon} \|w_t - w_s\| + \|w_s - \alpha^{-1}s\| \leq \|w_t - \alpha^{-1}s\| + \|w_t - w_s\| + \|w_t - w_s\| \leq \left(2L + \frac{4d \beta}{a_0 \varepsilon} \right) \varepsilon$. If $\|\alpha^{-1}s - w_t\| > \|w_t - w_s\|$, then $\|\alpha^{-1}s - w_t\| + \|w_t - w_s\| + \|w_t - w_s\| \leq \left(2L + \frac{4d \beta}{a_0 \varepsilon} \right) \varepsilon$. Hence $\|\alpha^{-1}s - w_t\| \leq \|w_t - w_s\|$. Now we prove that $\|\alpha^{-1}s - w_t\| \leq \frac{\varepsilon}{\alpha_0}$. Therefore $\|\alpha^{-1}s - w_t\| < \frac{\alpha_0 \beta}{a_0 \varepsilon} \|w_s - w_t\| < \frac{\alpha_0 \beta}{a_0 \varepsilon} \|w_s - \alpha w_t\| = \alpha \|w_t\| - \varepsilon$. 

3. MAIN RESULTS

Set $C := C(K, \mathbb{R}^n)$. We start to define a mapping $Q : B_C \to B_C$ by

$$(Qf)(t) := \left\{ \begin{array}{ll} f(\frac{t}{\|t\|}) & \text{if } t \in K_f := \frac{1 + \|t\|}{2} K_f \\ f(w_t) & \text{if } t \in K \setminus K_f \end{array} \right.$$

By the continuity of $f$ and by the Proposition 2 it is very simple to prove that $Qf$ is continuous on $K$. Moreover we have that $\|f\|_\infty = \|Qf\|_\infty = \max \{\|Qf(t)\|_\infty : t \in K_f\}$ for all $f \in B_C$ and $Qf = f$ for all $f \in S_C$.

Proposition 6. The mapping $Q$ is continuous.

Proof. Let $(f_n)$ be a sequence in $B_C$ such that $f_n \xrightarrow{\|\|_\infty} f \ (n \to +\infty)$. Fix $\varepsilon$. Then $\exists n_1 \in \mathbb{N}$ : $\forall n \geq n_1 \|f_n - f\|_\infty \leq \frac{\varepsilon}{2}$ (1). Since $f$ is uniformly continuous on $K$, we have that $\exists \delta > 0 : \forall s, t \in K \|t - s\| \leq \delta \Rightarrow \|f(t) - f(s)\|_\infty \leq \frac{\varepsilon}{2}$ (2). Choose $n_2 \in \mathbb{N}$ : $\forall n \geq n_2$

$$\left| 1 + \frac{2}{\|f_n\|_\infty} - 1 + \frac{2}{\|f\|_\infty} \right| \leq \frac{\delta}{\varepsilon} \ (3).$$
where \( c := \max_{t \in K} \| t \| \). Now we show that \( \forall n \geq \pi := \max \{ n_1, n_2 \} \) and \( \forall t \in K \) we have that 
\[
\| (Qf_n)(t) - (Qf)(t) \|_\infty \leq \varepsilon, \quad \text{so that } \| Qf_n - Qf \|_\infty \leq \varepsilon. \]
Let \( t \in K_f \cap K_{f_n} \) and \( n \geq \pi. \) By (1), (2) and (3) it follows that 
\[
\| (Qf_n)(t) - (Qf)(t) \|_\infty = \left| f_n\left( \frac{2}{1 + \| f_n \|_\infty} \right) - f\left( \frac{2}{1 + \| f \|_\infty} \right) \right|_\infty \leq \varepsilon. \]
Let \( t \in K_f \triangle K_{f_n} \) (where \( \triangle \) denotes the symmetric difference) and \( n \geq \pi. \) Then 
\[
\| (Qf_n)(t) - (Qf)(t) \|_\infty = \left| f_n(w_t) - f(w_t) \right|_\infty \leq \varepsilon. \]
If (4) holds. We have, by (1), (2) and (3), that 
\[
\left| f_n\left( \frac{2}{1 + \| f_n \|_\infty} \right) - f(w_t) \right|_\infty \leq \left| f_n\left( \frac{2}{1 + \| f_n \|_\infty} \right) - f\left( \frac{2}{1 + \| f \|_\infty} \right) \right|_\infty + \left| f\left( \frac{2}{1 + \| f \|_\infty} \right) - f(w_t) \right|_\infty \leq \varepsilon. \]
If (5) is true. Analogously we obtain 
\[
\left| f_n(w_t) - f\left( \frac{2}{1 + \| f \|_\infty} \right) \right|_\infty \leq \varepsilon. \]
Let us recall [2] that there is an explicite formula for the Hausdorff measure of noncompactness in \( C. \) For any bounded set \( A \subset C \) we have 
\[ (\ast) \quad \chi(A) = \frac{1}{2} \omega_0(A) = \frac{1}{2} \lim_{\varepsilon \to 0^+} \omega(A, \varepsilon) = \frac{1}{2} \sup_{\varepsilon \to 0^+} \chi(A) = \frac{1}{2} \lim_{\varepsilon \to 0^+} \omega(f, \varepsilon), \]
where \( \omega(f, \varepsilon) = \sup \{ |f(t) - f(s)|_\infty : s, t \in K, \| t - s \| \leq \varepsilon \}. \)

**Proposition 7.** The mapping \( Q \) is a 1-set contraction.

**Proof.** By Proposition 5 and Corollary 4 we can find a constant \( M \) such that \( \forall \varepsilon \in [0, \frac{1}{2} \beta] \) (where \( \beta := \min \{ \| u \| : u \in \partial K \} \)), \( \forall f \in BC \) and \( \forall s, t \in K \) we have \( \| t - s \| \leq \varepsilon \Rightarrow \| (Qf)(t) - (Qf)(s) \|_\infty \leq M \varepsilon. \) Therefore for any \( \varepsilon \in [0, \frac{1}{2} \beta] \) and any \( f \in BC \)
\[
\omega(Qf, \varepsilon) = \sup \{ \| (Qf)(t) - (Qf)(s) \|_\infty : s, t \in K, \| t - s \| \leq \varepsilon \} \leq \sup \{ \| f(t) - f(s) \|_\infty : s, t \in K, \| t - s \| \leq \varepsilon \} \leq \omega(f, M \varepsilon). \]
In view of (\ast) this implies \( \omega_0(QA) \leq \omega_0(A) \) for any \( A \subset BC. \) Therefore \( \chi(QA) \leq \chi(A), \) i.e. \( Q \) is a 1-set contraction. \[ \square \]

For any \( u \in [0, +\infty] \) define the mapping \( P_u : f \in BC \to P_u f \in C \) putting 
\[
(P_u f)_i(t) := \max \left\{ 0, \frac{u}{2} \left( \frac{\| f \|}{\| w_i \|} - \| f \|_\infty - 1 \right) \right\} (i = 1, \ldots, n). \]

**Remark 8.** For all \( f \in BC \) and for all \( t \in K_f \) we have that \( (P_u f)_i(t) = 0 \) for \( i = 1, \ldots, n. \)
follows that $\forall n \geq \pi \|f_n - f\|_\infty \leq \frac{\varepsilon}{2} (1).$ Now we prove that $\forall n \geq \pi$ and $\forall t \in K$ \((P_u f_n)(t) - (P_u f)(t)\) is totally bounded. We start by observing that $f_n$ is continuous, hence $\forall t \in K$ \((P_u f_n)(t) - (P_u f)(t)\) is totally bounded. Let $t \in K_f \cap K_{f_n}$ and $n \geq \pi.$ Then \((P_u f_n)(t) - (P_u f)(t)\) is compact. Then
\[
\| (P_u f_n)(t) - (P_u f)(t)\|_\infty = \frac{u}{2} 2 \left| 2 - \frac{\|f\|_{w^*}}{\|w_t\|} - \|f_n\|_\infty - 1 \right| (2)
\]

or
\[
\| (P_u f_n)(t) - (P_u f)(t)\|_\infty = \frac{u}{2} 2 \left| 2 - \frac{\|f\|_{w^*}}{\|w_t\|} - \|f\|_\infty - 1 \right| (3).
\]

If (2) is true. We have, since $\|t\| \leq \frac{1+\|f\|_{w^*}}{2} \|w_t\|,$
\[
\frac{u}{2} \left| 2 - \frac{\|f\|_{w^*}}{\|w_t\|} - \|f_n\|_\infty - 1 \right| \leq \frac{u}{2} \|f_n\|_\infty - \|f\|_\infty \leq \frac{u}{2} \|f_n - f\|_\infty \leq \varepsilon.
\]

If (3) holds. Analogously we obtain $\frac{u}{2} \left| 2 - \frac{\|f\|_{w^*}}{\|w_t\|} - \|f\|_\infty - 1 \right| \leq \varepsilon.$

Let $t \in K \setminus (K_f \cup K_{f_n})$ and $n \geq \pi.$ Then we have
\[
\| (P_u f_n)(t) - (P_u f)(t)\|_\infty = \frac{u}{2} \|f_n - f\|_\infty \leq \varepsilon.
\]

(iii) \: Let $u \in [0, +\infty[.$ Since $C$ is a Banach space, it is sufficient to show that $P_u(B_C)$ is totally bounded. We start by observing that $\|P_u f\|_\infty = \frac{u}{2} (1 - \|f\|_\infty)$ for any $f \in B_C.$ Therefore, by $0 \leq \|f\|_\infty \leq 1,$ it follows that $\|P_u f\|_\infty \in [0, \frac{u}{2}]$ for any $f \in B_C.$ Now we prove that
\[
\|P_u f\|_\infty - \|P_u g\|_\infty \leq \varepsilon \Rightarrow \|P_u f - P_u g\|_\infty \leq \varepsilon \quad (4).
\]

Let $t \in K_f \cap K_g.$ Then $|(P_u f)(t) - (P_u g)(t)|_\infty = 0.$

Let $t \in K \setminus (K_f \cup K_g).$ Then $|(P_u f)(t) - (P_u g)(t)|_\infty = \frac{u}{2} \|f\|_\infty - \|f\|_\infty = \|P_u f\|_\infty - \|P_u g\|_\infty.$

Let $t \in K_f \triangle K_{f_n}.$ Then
\[
| (P_u f)(t) - (P_u g)(t) |_\infty = \| (P_u f)(t) \|_\infty = \frac{u}{2} \left| 2 - \frac{\|f\|_{w^*}}{\|w_t\|} - \|f\|_\infty - 1 \right| (5) \text{ or } \| (P_u f)(t) - (P_u g)(t) \|_\infty = \| (P_u g)(t) \|_\infty \quad (6).
\]

If (5) holds. For all $t \in K_g \setminus K_f,$ we have $\|t\| \leq \frac{1+\|f\|_{w^*}}{2} \|w_t\|.$ Therefore
\[
| (P_u f)(t) |_\infty = \frac{u}{2} \left| 2 - \frac{\|f\|_{w^*}}{\|w_t\|} - \|f\|_\infty - 1 \right| \leq \frac{u}{2} \|f\|_\infty - \|g\|_\infty = \|P_u f\|_\infty - \|P_u g\|_\infty.
\]

If (6) holds. Analogously we obtain $| (P_u g)(t) |_\infty \leq \|P_u f\|_\infty - \|P_u g\|_\infty.$ Hence the inequality (4) is true.

Let $\varepsilon > 0.$ Fix an $\varepsilon$-net $\{\alpha_1, ..., \alpha_m\}$ in $[0, \frac{u}{2}],$ choose $\{f_1, ..., f_m\} \subset B_C$ such that $\|P_u f_j\|_\infty = \alpha_j$ for $j = 1, ..., m.$ Then $\{P_u f_1, ..., P_u f_m\}$ is an $\varepsilon$-net in $P_u(B_C).$ Infact, for any $f \in B_C$ there exists $j \in \{1, ..., m\}$ such that $\|P_u f\|_\infty - \|P_u f_j\|_\infty \leq \varepsilon.$ By (4) it follows that $\|P_u f - P_u f_j\|_\infty \leq \varepsilon.$ Hence $P_u(B_C)$ is totally bounded. \[\square\]

Now consider the mapping $T_u : B_C \to C$

$T_u f = Qf + P_u f.$
Clearly, the mapping $T_u$ is a 1-set contraction, and $T_u f = f$ for any $f \in S_C$. Moreover, for any $f \in B_C$, we have that
\[
\|T_u f\|_\infty = \|Qf + P_u f\|_\infty = \max \{ ((Qf)(t) + (P_u f)(t))_\infty : t \in K \} \\
\geq \max \left\{ \max_{t \in K_f} \|Qf(t)\|_\infty, \max_{t \in K \setminus K_f} \|P_u f(t)\|_\infty \right\} \\
\geq \max \left\{ \|f\|_\infty, \max_{t \in K \setminus K_f} \|Qf(w_t) + (P_u f)(w_t)\|_\infty \right\} \\
\geq \max \left\{ \|f\|_\infty, \max_{t \in K \setminus K_f} \|f(w_t) + (P_u f)(w_t)\|_\infty \right\} \\
= \max \left\{ \|f\|_\infty, \max_{t \in K \setminus K_f} \left\{ \max_{i=n} f_i(w_t) + \frac{u}{2} (1 - \|f\|_\infty) \right\} \right\} \\
\geq \max \left\{ \|f\|_\infty, \max_{t \in K \setminus K_f} \left\{ \max_{i=n} f_i(w_t) + \frac{u}{2} (1 - \|f\|_\infty) \right\} \right\} \\
\geq \max \left\{ \|f\|_\infty, \frac{u}{2} (1 - \|f\|_\infty) - \|f\|_\infty \right\}.
\]
The last term attains its minimum $\frac{u}{u+4}$ for functions $f$ with $\|f\|_\infty = \frac{u}{u+4}$. Therefore $\|T_u f\|_\infty \geq \frac{u}{u+4}$ for all $f \in B_C$. Set
\[
R_u f = \frac{1}{\|T_u f\|_\infty} T_u f
\]
for all $f \in B_C$ we have
\[
\omega(R_u f, \varepsilon) = \frac{1}{\|T_u f\|_\infty} \omega(T_u f, \varepsilon) \leq \frac{u+4}{u} \omega(T_u f, \varepsilon).
\]
Hence for any set $A \subset B_C$
\[
\omega_0(R_u A) \leq \frac{u+4}{u} \omega_0(A).
\]
Therefore
\[
\chi(R_u A) \leq \frac{u+4}{u} \chi(A).
\]
Since $\lim_{u \to \infty} \frac{u+4}{u} = 1$, the following result holds.

**Theorem 10.** $k_1(C) = 1$.

**References**


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