INTUITIONISTIC FUZZY $K$-IDEALS OF $IS$-ALGEBRAS

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Abstract. In this paper, we introduce the notion of intuitionistic fuzzy $K$-ideals of $IS$-algebras and investigate some of their properties.

1. Introduction and Preliminaries

In 1966, Iseki [1] introduced the notion of $BCI$-algebras. For the general development of $BCK/BCI$-algebras, the ideal theory plays an important role. In 1993, Jun et al. [2] introduced a new class of algebras related to $BCI$-algebras and semigroups, called a $BCI$-semigroup. In 1998, for the convenience of study, Jun et al. [3] renamed the $BCI$-semigroups as the $IS$-algebra and studied further properties. In [4], we introduced the concept of $K$-ideals of $BCI$-algebras. In this paper, we consider the fuzzification of $K$-ideals of $IS$-algebras and study their properties.

By a $BCI$-algebra we mean algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(I) $(x * y) * (x * z) = (z * y) * (z * x)$

(II) $(x * (x * y)) * y = 0$

(III) $x * x = 0$

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In any $BCI$-algebra $X$ one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$.

A nonempty subset $I$ of a $BCI$-algebra $X$ is called an ideal of $X$ if it satisfies (i) $0 \in I$, (ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in I$.

By an $IS$-algebra we mean a nonempty set $X$ with two binary operation $\cdot$ and $\ast$ and constant 0 satisfying the axioms:

(I) $I(X) = (X; *, 0)$ is a $BCI$-algebra.

(II) $S(X) = (X; \cdot)$ is a semigroup.

(III) The operation $\ast$ is distribute over the operation $\cdot$, i.e., $x \cdot (y \ast z) = (x \cdot y) \ast (x \cdot z)$ and $(x \cdot y) \cdot z = (x \cdot z) \ast (y \cdot z)$ for all $x, y, z \in X$.

A nonempty subset $A$ of a semigroup $S(X) = (X; \cdot)$ is said to be stable if $xa \in A$ whenever $x \in S(X)$ and $a \in A$.

We now review some fuzzy logic concepts. A fuzzy set in a set $X$ is a function $\mu : X \rightarrow [0, 1]$ and the complement of $\mu$, denoted by $\overline{\mu}$, is the fuzzy set in $X$ given by $\overline{\mu}(x) = 1 - \mu(x)$. For $t \in [0, 1]$, the set $U(\mu; t) = \{x \in X | \mu(x) \geq t\}$ is called an upper $t$-level cut of and the

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set \( I(\mu; t) = \{ x \in X \mid \mu(x) \leq t \} \) is called a lower t-level cut of \( \mu \). We shall write \( a \wedge b \) for
\( \min \{ a, b \} \) and \( a \vee b \) for \( \max \{ a, b \} \), where \( a \) and \( b \) are any real numbers.

An intuitionistic fuzzy set (briefly, IFS) \( A \) in a nonempty set \( X \) is an object having the form
\[
A = \{ (x, a_A(x), \beta_A(x)) \mid x \in X \}
\]
where the functions \( a_A : X \to [0, 1] \) and \( \beta_A : X \to [0, 1] \) denote the degree of membership
and the degree of non-membership respectively, and \( 0 \leq a_A(x) + \beta_A(x) \leq 1, \ \forall x \in X \).

An intuitionistic fuzzy set \( A = \{ (x, a_A(x), \beta_A(x)) \mid x \in X \} \) in \( X \) can be identified to
an ordered pair \( (a_A, \beta_A) \) in \( I^X \times I^X \). For the sake of simplicity, we shall use the symbol
\( A = (a_A, \beta_A) \) for the IFS \( A = \{ (x, a_A(x), \beta_A(x)) \mid x \in X \} \).

2. Intuitionistic Fuzzy K-ideals

**Definition 2.1** ([4]). Let \( k \) be any positive integer. A nonempty subset \( I \) of a \( BCI \)-algebra
\( X \) is called a K-ideal of \( X \) if
(i) \( 0 \in I \),
(ii) \( x \ast y^k \in I \) and \( y \in I \) imply \( x \in I \).

**Definition 2.2.** A nonempty subset \( I \) of an \( IS \)-algebra \( X \) is called a K-ideal of \( X \) if
(i) \( xa \in I \) for any \( x \in S(X) \) and \( a \in I \)
(ii) \( x \ast y^k \in I \) and \( y \in I \) imply \( x \in I \)

**Definition 2.3.** A fuzzy set \( \mu \) in an \( IS \)-algebra \( X \) is called a fuzzy K-ideal (briefly, FK-ideal)
of \( X \) if
(i) \( \mu(x \cdot y) \geq \mu(y) \),
(ii) \( \mu(x) \geq \mu(x \ast y^k) \wedge \mu(y) \)
for all \( x, y \in X \).

**Definition 2.4.** An IFS \( \mu = (a_A, \beta_A) \) in an \( IS \)-algebra \( X \) is called an intuitionistic fuzzy
K-ideals (briefly, IFK-ideal) of \( X \) if
(i) \( a_A(x \cdot y) \geq a_A(y) \),
(ii) \( \beta_A(x \cdot y) \leq \beta_A(y) \),
(iii) \( a_A(x) \geq a_A(x \ast y^k) \wedge a_A(y) \),
(iv) \( \beta_A(x) \leq \beta_A(x \ast y^k) \vee \beta_A(y) \)
for all \( x, y \in X \).

**Example 2.5.** Consider an \( IS \)-algebra \( X = \{0, a, b, c\} \) with Cayley tables as follows:

\[
\begin{array}{cccc}
  \ast & 0 & a & b & c \\
 0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
  \cdot & 0 & a & b & c \\
 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & a & b & c \\
c & 0 & 0 & 0 & 0 \\
\end{array}
\]
Define an $IFSA = (\alpha_A, \beta_A)$ in X as follows:

$\alpha_A(0) = \alpha_A(a) = 1$ and $\alpha_A(b) = \alpha_A(c) = t$

$\alpha_A(0) = \beta_A(a) = 0$ and $\beta_A(b) = \beta_A(c) = s$

where $t, s \in [0, 1]$ and $t + s \leq 1$.

Hence $A = (\alpha_A, \beta_A)$ is an IFK-ideal of X.

**Lemma 2.6.** An $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of IS-algebra X if and only if the fuzzy sets $\alpha_A$ and $\beta_A$ are a FK-ideal of X.

**Proof.** Let $IFSA = (\alpha_A, \beta_A)$ be an IFK-ideal of X. Clearly $\alpha_A$ is a FK-ideal of X. For any $x, y \in X$, we have $\beta_A(x \cdot y) \geq 1 - \beta_A(x \cdot y) = 1 - \beta_A(y) = \overline{\beta_A}(y)$ and $\overline{\beta_A}(x) \geq 1 - \overline{\beta_A}(x \cdot y) \lor \overline{\beta_A}(y) = (1 - \overline{\beta_A}(x \cdot y)) \land (1 - \overline{\beta_A}(y)) = \overline{\beta_A}(x \cdot y) \lor \overline{\beta_A}(y)$. Hence $\overline{\beta_A}$ is a FK-ideal of X.

Conversely, assume that $\alpha_A$ and $\beta_A$ are FK-ideal of X. For any $x, y \in X$, we get $\overline{\beta_A}(x \cdot y) \geq \overline{\beta_A}(y)$ and that $\beta_A(x \cdot y) \leq \beta_A(y)$. Moreover, $\overline{\beta_A}(x) \geq \overline{\beta_A}(x \cdot y) \land \overline{\beta_A}(y)$ and that $1 - \beta_A(x) \geq (1 - \beta_A(x \cdot y)) \land (1 - \beta_A(y)) = 1 - \beta_A(x \cdot y) \lor \beta_A(y)$, that is, $\beta_A(x) \leq \beta_A(x \cdot y) \lor \beta_A(y)$. Hence $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of X.

**Theorem 2.7.** $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of IS-algebra X if and only if $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\Box A = (\overline{\beta}_A, \beta_A)$ are IFK-ideals of X.

**Proof.** If $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of X, then $\alpha_A = \overline{\alpha}_A A$ and $\beta_A$ are FK-ideals of X from Lemma 2.6, hence $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\Box A = (\overline{\beta}_A, \beta_A)$ are IFK-ideals of X. Conversely, if $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\Box A = (\overline{\beta}_A, \beta_A)$ are IFK-ideals of X, then $\alpha_A$ and $\overline{\alpha}_A$ are FK-ideals of X, hence $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of X.

**Theorem 2.8.** An $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of IS-algebra X if and only if for all $s, t \in [0, 1]$, the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are K-ideals of X.

**Proof.** Let $x \in S(X)$ and $y \in U(\alpha_A; t)$. If $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of X, then $\alpha_A(y) \geq t$ and that $\alpha_A(x \cdot y) \geq \alpha_A(y) \geq t$, which implies that $x \cdot y \in U(\alpha_A; t)$. Let $x, y \in I(X)$ be such that $x \cdot y \in U(\alpha_A; t)$ and $y \in U(\alpha_A; t)$. Then $\alpha_A(x \cdot y) \geq t$ and $\alpha_A(y) \geq t$. It follows that $\alpha_A(x) \geq \alpha_A(x \cdot y) \land \alpha_A(y) \geq t$, so that $x \in U(\alpha_A; t)$. Hence $U(\alpha_A; t)$ is a K-ideal of X. Now let $x \in S(X)$ and $y \in L(\beta_A; s)$, then $\beta_A(y) \leq s$ and so $\beta_A(x \cdot y) \leq \beta_A(y) \leq s$, which implies that $x \cdot y \in L(\beta_A; s)$. Let $x, y \in I(X)$ be such that $x \cdot y \in L(\beta_A; s)$ and $y \in L(\beta_A; s)$, then $\beta_A(x \cdot y) \leq s$ and $\beta_A(y) \leq s$. It follows that $\beta_A(x) \leq \beta_A(x \cdot y) \lor \beta_A(y) \leq s$, so that $x \in L(\beta_A; s)$. Hence $L(\beta_A; s)$ is a K-ideal of X.

Conversely, assume that for each $s, t \in [0, 1]$, the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are K-ideals of X. If there are $x_0, y_0 \in S(X)$ such that $\alpha_A(x_0 \cdot y_0) < \alpha_A(y_0)$, then taking $t_0 = (\alpha_A(x_0 \cdot y_0) + \alpha_A(y_0))/2$, we have $\alpha_A(x_0 \cdot y_0) < t_0 < \alpha_A(y_0)$. It follows that $y_0 \in U(\alpha_A; t_0)$ and $x_0 \cdot y_0 \notin U(\alpha_A; t_0)$. This is a contradiction. Therefore $\alpha_A$ is a fuzzy stable set in S(X). If there are $x_0, y_0 \in S(X)$ such that $\beta_A(x_0 \cdot y_0) < \beta_A(y_0)$, then taking $s_0 = (\beta_A(x_0 \cdot y_0) + \beta_A(y_0))/2$, we have $\beta_A(x_0 \cdot y_0) > s_0 > \beta_A(y_0)$, it follows that $y_0 \in L(\beta_A; s_0)$ and $x_0 \cdot y_0 \notin U(\beta_A; s_0)$. This is a contradiction. Therefore $\alpha_A$ is a fuzzy stable set in S(X). Suppose that $\alpha_A(x_0) < \alpha_A(x_0 \cdot y_0) \land \beta_A(y_0)$ for some $x_0, y_0 \in X$, putting $t_0 = (\alpha_A(x_0) + \alpha_A(x_0 \cdot y_0) \land \beta_A(y_0))/2$, we have $\alpha_A(x_0) < t_0 < \alpha_A(x_0 \cdot y_0) \land \beta_A(y_0)$, which shows that $x_0 \cdot y_0, y_0 \in U(\alpha_A; t_0)$ and $x_0 \notin U(\alpha_A; t_0)$. This is impossible. Finally, assume that $a, b \in X$ such that $\beta_A(a) > \beta_A(a \cdot b) \lor \beta_A(b)$. Taking $s_0 = (\beta_A(a) + \beta_A(a \cdot b) \lor \beta_A(b))/2$, then $\beta_A(a \cdot b) \lor \beta_A(b) < s_0 < \beta_A(a)$. Therefore $a \cdot b$ and $b \in L(\beta_A; s_0)$, but $a \notin L(\beta_A; s_0)$, which is a contradiction. This completes the proof.
3. On homomorphism of IS-algebras

Definition 3.1. ([4]) A mapping $f : X \rightarrow Y$ of IS-algebras is called a homomorphism if

(i) $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in I(X)$;
(ii) $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in S(X)$.

For any $IFSA = (A, \beta_A)$ in $Y$, we define a new $IFSA' = (A', \beta_A')$ in $X$ by $A'_A(x) = a_A(f(x))$, $\beta_A'(x) = \beta_A(f(x))$ \quad $\forall x \in X$

Theorem 3.2. Let $f : X \rightarrow Y$ be a homomorphism of IS-algebras. If an $IFSA = (A, \beta_A)$ is an $IFK$-ideal of $Y$, then $IFSA' = (A', \beta_A')$ in $X$ is an $IFK$-ideal of $X$.

Proof. Suppose an $IFSA = (A, \beta_A)$ is an $IFK$-ideal of $Y$, then $A_A(x \cdot y) = a_A(f(x \cdot y)) = a_A(f(x) \cdot f(y)) \geq a_A(f(y)) = A_A(y)$ and $\beta_A(x \cdot y) = \beta_A(f(x \cdot y)) = \beta_A(f(x) \cdot f(y)) \leq \beta_A(f(y)) = \beta_A(y)$. Now let $x, y, z \in X$, then $a_A(x) = a_A(f(x)) \geq a_A(f(x) \ast f(g)^k) \land a_A(f(y)) = a_A(f(x \ast y^k)) \land a_A(y)$ and $\beta_A(x) = \beta_A(f(x)) \leq \beta_A(f(x) \ast y^k) \lor \beta_A(y)$. This completes the proof.

If we strengthen the condition $f$, then the converse of Theorem 3.2 is obtained as follows:

Theorem 3.3. Let $f : X \rightarrow Y$ be an epimorphism of IS-algebras and let $IFSA = (A, \beta_A)$ be in $Y$. If $IFSA' = (A', \beta_A')$ is an $IFK$-ideal of $X$, then $IFSA = (A, \beta_A)$ is an $IFK$-ideal of $Y$.

Proof. For any $x, y \in Y$, there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Then $A_A(x \ast y) = a_A(f(a) \ast f(b)) = a_A(f(a) \ast b) \geq a_A(b) = A_A(f(b)) = A_A(y)$ and $\beta_A(x \ast y) = \beta_A(f(a) \ast f(b)) = \beta_A(f(a) \ast f(b)) = \beta_A(f(b)) = \beta_A(y)$. Moreover, $a_A(x) = a_A(f(a)) = A_A(a \ast b^k) \land A_A(b) = a_A(f(a \ast b^k)) \land A_A(f(b)) = a_A(f(a) \ast f(b)^k) \land a_A(f(b)) = a_A(x \ast y^k) \land A_A(y)$ and $\beta_A(x) = \beta_A(f(a)) = \beta_A(a) \leq \beta_A(a \ast b^k) \lor \beta_A(b) = \beta_A(f(a \ast b^k)) \lor \beta_A(f(b)) = \beta_A(f(a) \ast f(b^k)) \lor \beta_A(f(b)) = \beta_A(f(a) \ast f(b^k)) \lor \beta_A(f(b)) = \beta_A(x \ast y^k) \lor \beta_A(y)$. This completes the proof.

References


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