ON A RADICAL APPROACH IN BCI-ALGEBRAS

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Abstract. We define a notion of radical in a BCI-algebra, and some fundamental results concerning this notation are proved. The notion of α-ideal is introduced, and we discuss p-ideals in BCI-algebras and their relations with α-ideals.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki introduced the notion of BCI-algebras which is a generalization of BCK-algebras([Is1]).

As we know, the primary aim of the theory of BCI-algebras is to determine the structure of all BCI-algebras. The main task of a structure theorem is to fine a complete system of invariants describing the BCI-algebra up to isomorphism, or to establish some connection with other mathematics branches. In addition, the ideal theory plays an important role in studying BCI-algebras, and some interesting results have been obtained by several authors.

In this paper, we define a notion of radical in a BCI-algebra, and some fundamental results concerning this notation are proved. The notion of α-ideal is introduced, and we discuss p-ideals in BCI-algebras and their relations with α-ideals.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI-2) $(x * (x * y)) * y = 0$,
(BCI-3) $x * x = 0$,
(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra X satisfying $0 * x = 0$ for all $x \in X$ is called a *BCK-algebra*. In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X is said to be *p-semisimple* if $X_+ = \{0\}$, where X_+ is the *BCK-part* of X, *i.e.*, $X_+ := \{x \in X | 0 \leq x\}$. Note that a BCI-algebra X is p-semisimple if and only if $x * y = 0$ implies $x = y$ for all $x, y \in X$ if and only if $x * y = 0 * (y * x)$ for all $x, y \in X$. A BCI-algebra X is said to be *associative* if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$. Note that a BCI-algebra X is associative if and only if $0 * x = x$ for all $x \in X$.

An element a of a BCI-algebra X is called an *atom* if $z * a = 0$ implies $z = a$ for all $z \in X$. Denote by $L(X)$ the set of all atoms of X. Clearly, $0 \in L(X)$ and $L(X)$ is a subalgebra of

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X, *i.e.*, $L(X)$ is a p-semisimple BCI-algebra. Note that if $a \in L(X)$, then $a * x \in L(X)$ for all $x \in X$.

A BCI-algebra X has the following properties for any $x, y, z \in X$:

(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,
(3) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$,
(4) $(x * z) * (y * z) \leq x * y$,
(5) $x * (x * (x * y)) = x * y$,
(6) $0 * (x * y) = (0 * x) * (0 * y)$.

A nonempty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies

(7) $0 \in I$,
(8) $x * y \in I$ and $y \in I$ imply $x \in I \ \forall x, y \in X$.

In general, an ideal I of a BCI-algebra X need not be a subalgebra. However, if X is a p-semisimple BCI-algebra then any subalgebra of X is an ideal. A nonempty subset I in a BCI-algebra X is called a *p-ideal* of X, if it satisfies (7) and

(9) $(x * z) * (y * z) \in I$ and $y \in I$ imply $x \in I \ \forall x, y, z \in X$.

Note that an ideal I of a BCI-algebra X is a p-ideal if and only if $0 * (0 * x) \in I$ implies $x \in I$ for any $x \in X$. A mapping $f : X \to Y$ of BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Clearly, $f(0) = 0$.

3. A radical approach in BCI-algebras

Definition 1 ([MW]). For any x in a BCI-algebra X and any positive integer n, the *n-th power* x^n of x is defined by

$$x^1 = x \text{ and } x^n = x * (0 * x^{n-1}).$$

Clearly $0^n = 0$.

Definition 2. An element x of a BCI-algebra X is *nilpotent* if $x^n = 0$ for some positive integer n. An ideal R of X is called a *nil ideal* of X if every element of R is nilpotent. In particular, if every x in X is nilpotent, then X is called a *nil algebra*.

The following example shows that there is an element which is not nilpotent.

Example 3. Let $X = \{0, a, b, c\}$ be a BCI-algebra in which $*$-operation is defined by:

*	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

Then, by routine calculations, we can see that $0, a$ and c are nilpotent elements of X, but b is not a nilpotent elements of X.

In the following theorem we give some properties of BCK-algebras.

Theorem 4. *Let X be a BCI-algebra. Then the BCK-part X_+ of X is subset of the set* $\{x \in X | x^2 = x\}$.

Proof. Let $x \in X_+$. Then we have $x^2 = x * (0 * x) = x * 0 = x$, and hence $X_+ \subseteq \{x \in X | x^2 = x\}$. □

Following Theorem 4, we know that there is no nonzero nilpotent element in the BCK-part X_+ of a BCI-algebra X.

Corollary 5. *If X is a BCK-algebra, then $X = \{x \in X | x^2 = x\}$.*

Noticing that a BCI-algebra X is p-semisimple if and only if $X = L(X)$, the following lemma follows from [MW, Theorem 2].

Lemma 6. *Let X be a p-semisimple BCI-algebra. Then for any $a, b \in X$ and any positive integer m, n, we have*
(10) $a^{m+n} = a^m * (0 * a^n)$,
(11) $(a^m)^n = a^{mn}$,
(12) $(a * b)^m = a^m * b^m$.

The following theorem is a generalization of Theorem 7 and Corollary 8 in [M].

Theorem 7. *For any x in a BCI-algebra and any positive integer n, we have*

$$(0 * x)^n = 0 * x^n.$$

Proof. We argue by induction on the positive integer n. For $n = 1$ there is nothing to prove. Assume that the theorem is true for positive integer n. Then using (6) we have

$$\begin{aligned}(0 * x)^{n+1} &= (0 * x) * (0 * (0 * x)^n)\\ &= (0 * x) * (0 * (0 * x^n))\\ &= 0 * (x * (0 * x^n))\\ &= 0 * x^{n+1}. \quad \square\end{aligned}$$

The following corollary is an immediate consequence of Lemma 6 and Theorem 7.

Corollary 8. *For any x in a BCI-algebra X and any positive integer n, we have*
(13) $0 * x^n \in L(X)$,
(14) $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$.

Definition 9. Let R be a non-empty subset of a BCI-algebra X and k a positive integer. Then we define

$$[R; k] := \{x \in R | x^k = 0\},$$

which is called the *radical* of R.

By using radical, we give an equivalent condition in order that a BCI-algebra X would be associative.

Theorem 10. *Let X be a BCI-algebra. Then X is associative if and only if $X = [X; 2]$.*

Proof. Let X be an associative BCI-algebra. Then for all $x \in X$, we have

$$x^2 = x * (0 * x) = (x * 0) * x = x * x = 0,$$

and hence $X = [X; 2]$. Conversely, assume that $x^2 = 0$ for all $x \in X$. Then we have

$$\begin{aligned}(0 * x) * x &= (x^2 * x) * x\\ &= (((x * (0 * x)) * x) * x\\ &= (0 * (0 * x)) * x\\ &= 0,\end{aligned}$$

and hence $0 * x = x$ for all $x \in X$. Therefore X is an associative BCI-algebra. $\square$

Following Theorem 10, we know that every associative BCI-algebra is a nil algebra.

Remark. We know that, in general, the radical of an ideal in a BCI-algebra X may not be an ideal. In fact, taking an ideal $R = X$ in Example 3, then $[R;3] = \{0, a, c\}$ is not an ideal of X since $b * a = c \in [R;3]$ and $b \notin [R;3]$. In the following theorem, we give a condition in order that a radical would be an ideal.

Theorem 11. *Let R be an ideal of a p-semisimple BCI-algebra X. Then the radical of R is an ideal of X.*

Proof. Let $[R;k]$ be a radical of R for some positive integer k. Then clearly $0 \in [R;k]$. Let $x, y \in X$ be such that $x*y \in [R;k]$ and $y \in [R;k]$. Then we have $(x*y)^k = 0, y^k = 0, x*y \in R$ and $y \in R$. Hence using Lemma 6 and R is an ideal of X, we obtain

$$x^k = x^k * y^k = (x * y)^k = 0 \text{ and } x \in R.$$

Therefore $x \in [R;k]$. $\square$

Theorem 12. *Let R be a subalgebra of a p-semisimple BCI-algebra X. Then the radical of R is a subalgebra of X.*

Proof. Assume that $[R;k]$ is a radical of R for some positive integer k. Let $x, y \in X$ be such that $x, y \in [R;k]$. Then $x^k = 0$ and $y^k = 0$. Hence by Lemma 6 we have

$$(x * y)^k = x^k * y^k = 0 \text{ and } x * y \in R,$$

and so $x * y \in [R;k]$. $\square$

Theorem 13. *Let R be a subalgebra of a BCI-algebra X and k a positive integer. If $x \in [R;k]$, then $0 * x \in [R;k]$.*

Proof. Let $x \in [R;k]$. Then $x^k = 0$ and $x \in R$. Thus by Theorem 7 we have

$$(0 * x)^k = 0 * x^k = 0 \text{ and } 0 * x \in R,$$

and hence $0 * x \in [R;k]$. $\square$

This leave open question, if R is a subalgebra of a BCI-algebra X and $0 * x \in [R;k]$, then is x in $[R;k]$? The answer is negative. In Example 3, $[X;3]$ is a subalgebra of X and $0 * b \in [X;3]$, but $b \notin [X;3]$.

It is then natural to ask that given a nonempty subset R of a BCI-algebra X, under which condition of X and R is x in $[R;k]$? Solving this problem, we define the following definition.

Definition 14. If an ideal R of a BCI-algebra X satisfies the condition

(A) $$0 * x \in R \text{ implies } x \in R,$$

then we say that R is an α-*ideal* of X.

Example 15. Let $X = \{0, 1, 2, 3, 4, 5\}$ and $*$ table is given by:

*	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	0	3	3	3
2	2	2	0	5	5	3
3	3	3	3	0	0	0
4	4	3	3	1	0	0
5	5	5	3	2	2	0

Then $(X;*,0)$ is a BCI-algebra. By routine calculations, we can see that $\{0,1,2\}$ is an α-ideal X and $R := \{0,1,3,4\}$ is an ideal of X. But R is not an α-ideal of X because $0*5 \in R$ and $5 \notin R$.

Next, we discuss p-ideals in BCI-algebras and their relation with α-ideals.

Theorem 16. *In a BCI-algebra, every α-ideal is a p-ideal, but the converse does not hold.*

Proof. Suppose that R is an α-ideal of a BCI-algebra X. Let $x \in X$ be such that $0*(0*x) \in R$. Since R is an α-ideal of X, we have $0*x \in R$ and so $x \in R$. Therefore R is a p-ideal of X.

The last part is shown by the following example. □

Example 17. Let $X = \{0,1,2,3\}$ in which $*$-operation is defined by:

*	0	1	2	3
0	0	3	0	3
1	1	0	3	2
2	2	3	0	1
3	3	0	3	0

Then $(X;*,0)$ is a BCI-algebra. By routine calculations, we can see that $\{0,3\}$ is a p-ideal of X, but it is not an α-ideal since $0*1 \in \{0,3\}$ and $1 \notin \{0,3\}$.

The following theorem is a generalization of Theorem 1.3 in [Ho].

Theorem 18. *Let R be an ideal in a BCI-algebra X. Then for any $x,y \in X$, the following are equivalent.*

(15) *$x*y \in R$ implies that $y*x \in R$,*

(16) *$0*x \in R$ implies that $x \in R$.*

Proof. (15) $\Rightarrow$ (16) is obvious. (16) $\Rightarrow$ (15). Let $x,y \in X$ be such that $x*y \in R$. Then by (6) and BCI-2, we have

$$\begin{aligned}(0*(y*x))*(x*y) &= ((0*y)*(0*x))*(x*y)\\ &= ((0*(x*y))*(0*x))*y\\ &= (((0*x)*(0*x))*(0*y))*y\\ &= (0*(0*y))*y\\ &= 0 \in R.\end{aligned}$$

Using (16) and R is an ideal of X, we get $y*x \in R$. □

By applying Theorem 18, we obtain the following theorem.

Theorem 19. *Let R be an α-ideal of a p-semisimple BCI-algebra X and k a positive integer. If $x*y \in [R;k]$, then $y*x \in [R;k]$.*

Proof. Let $x,y \in X$ be such that $x*y \in [R;k]$. Then we have $(x*y)^k = 0$ and $x*y \in R$. Using Theorem 7 and X is p-semisimple, we obtain

$$(y*x)^k = (0*(x*y))^k = 0*(x*y)^k = 0*0 = 0.$$

By Theorem 18, $y*x \in R$ is obvious. Therefore $y*x \in [R;k]$. □

By applying Theorems 11, 18 and 19, we obtain the following corollary, which is the positive answer for the open question.

Corollary 20. *Let R be an α-ideal of a p-semisimple BCI-algebra X and k a positive integer. If $0 * x \in [R; k]$, then $x \in [R; k]$.*

Theorem 21. *Let R be a nonempty subset of a p-semisimple BCI-algebra X and let k and r be positive integers. If $k|r$, then $[R; k] \subseteq [R; r]$.*

Proof. If $k|r$, then $r = kq$ for some positive integer q. Let $x \in [R; k]$. Then by Lemma 6 we have $x^r = x^{kq} = (x^k)^q = 0^q = 0$, and so $[R; k] \subseteq [R; r]$. □

Theorem 22. *Let R be a subalgebra of a p-semisimple BCI-algebra X. Then the set*

$$[R] := \{x \in R | \ x \text{ is a nilpotent element in } X\}$$

is a nil closed ideal of R.

Proof. It is sufficient to show that $[R]$ is a subalgebra of R. Assume that $x, y \in [R]$. Then there exist positive integer k and r such that $x^k = 0, y^r = 0$ and $x, y \in R$. It follows from Theorem 21 that $x^{kr} = 0$ and $y^{kr} = 0$. Hence by Lemma 6 we have

$$(x * y)^{kr} = x^{kr} * y^{kr} = 0 \text{ and } x * y \in R,$$

and so $x * y \in [R]$. □

In the following, we give quotient algebras via ideals. Let I be an ideal of a BCI-algebra X. Define a binary relation $\sim$ on X as follows:

$$x \sim y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

Then $\sim$ is a congruence relation on X. Denote by $[x] := \{y \in X | y \sim x\}$ the equivalence class containing $x \in X$ and $X/I := \{[x] | x \in X\}$. Define $[x] * [y] = [x * y]$. Then $[0]$ is the greatest closed ideal contained in I, and $(X/I; *, 0)$ is a BCI-algebra, called the *quotient algebra* of X by I. But $[0]$ may not equal I. We can easily check that $[0] = I$ if I is a closed ideal.

Theorem 23. *Let R be a subalgebra of a p-semisimple BCI-algebra X, then $X/[R]$ has no nonzero nilpotent element.*

Proof. Let $[x] \in X/[R]$ be a nilpotent element. Then $[x^k] = [x]^k = [0]$ for some positive integer k. Thus we know that x^k is a nilpotent element in X. Hence $x^{kr} = (x^k)^r = 0$ for some positive integer r, and so we get $x \in [R; kr] \subseteq [R]$. Therefore $[x] = [0]$. □

Now we give some properties of radicals related to BCI-homomorphisms.

Theorem 24. *Let X be a BCI-algebra, Y be a p-semisimple BCI-algebra and $f : X \to Y$ be a homomorphism. Then for every subalgebra R of Y, $f^{-1}([R; k])$ is a subalgebra of X containing $[f^{-1}(R); k]$ for any positive integer k.*

Proof. To prove that $[f^{-1}(R); k] \subseteq f^{-1}([R; k])$, let $x \in [f^{-1}(R); k]$. Then $x^k = 0$ and $x \in f^{-1}(R)$. Since f is a homomorphism, we have

$$f(x)^k = f(x^k) = f(0) = 0 \text{ and } f(x) \in R.$$

Thus $f(x) \in [R; k]$, and so $x \in f^{-1}([R; k])$. If $x, y \in f^{-1}([R; k])$, then $f(x), f(y) \in [R; k]$. It follows from Theorem 12 that

$$f(x * y) = f(x) * f(y) \in [R; k],$$

and so $x * y \in f^{-1}([R; k])$. □

Note that the inverse image of an ideal under a BCI-homomorphism is an ideal. Hence we have the following theorem.

Theorem 25. *Let X be a BCI-algebra, Y be a p-semisimple BCI-algebra and $f : X \to Y$ be a homomorphism. If R is an ideal of Y, then $f^{-1}([R;k])$ is an ideal of X containing $[f^{-1}(R);k]$ for any positive integer k.*

Theorem 26. *Let $f : X \to Y$ be a homomorphism of BCI-algebras, R be a subalgebra of X and k be a positive integer. Then*

(17) $f([R;k]) \subseteq [f(R);k]$,

(18) *if f is 1-1, then* $f([R;k]) = [f(R);k]$.

Proof. (17) Let $x \in [R;k]$. Then we have

$$0 = f(0) = f(x^k) = f(x)^k \text{ and } f(x) \in f(R).$$

Hence $f(x) \in [f(R);k]$, and so $f([R;k]) \subseteq [f(R);k]$.

(18) Assume that f is 1-1 and let $y \in [f(R);k]$. Then $y^k = 0$ and $y = f(x)$ for some $x \in R$. It follows that

$$0 = y^k = f(x)^k = f(x^k).$$

Since f is 1-1, we have $x^k = 0$. Thus $x \in [R;k]$, which implies that $y = f(x) \in f([R;k])$. □

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