ON THE KP-SEMISIMPLE PART IN BCI-ALGEBRAS

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Abstract. In this paper, we introduce the concept of kp-semisimple part in BCI-algebras and give some characterization of such algebras.

1. Introduction and Preliminaries

A BCI-algebra is an algebra \((X, *, 0)\) of type \((2, 0)\) with the following conditions:
1. \(((x * y) * (x * z)) * (z * y) = 0\)
2. \((x * (x * y)) * y = 0\)
3. \(x * x = 0\)
4. \(x * y = y * x = 0\) implies \(x = y\).

A partial ordering \(\leq\) on \(X\) can be defined by \(x \leq y\) if and only if \(x * y = 0\).

The following identities hold for any BCI-algebra \(X\):
1. \(x * 0 = x\),
2. \((x * y^k) * z^k = (x * z^k) * y^k\),
3. \(0 * (x * y)^k = (0 * x)^k * (0 * y)^k\),
4. \(0 * (0 * x)^k = 0 * (0 * x)^k\).

where \(k\) is any positive integer.

A nonempty subset \(I\) of a BCI-algebra \(X\) is called an ideal if \(0 \in I\) and if \(x * y, y \in I\) then \(x \in I\). For any BCI-algebra \(X\), the set \(P(X) = \{x | 0 * x = 0\}\) is called the BCK-part of \(X\). If \(P(X) = 0\), then we say that \(X\) is a p-semisimple BCI-algebra.

Definition 1.1 ([1]). A nonempty subset \(I\) of a BCI-algebra \(X\) is called a \(k\)-ideal of \(X\) if
1. \(0 \in I\)
2. \(x * y^k \in I\) and \(y \in I\) imply \(x \in I\).

Definition 1.2. Let \(X\) be a BCI-algebra and \(k\) a positive integer, we define

\[ SP_k(X) = \{ x \in X \mid 0 * (0 * x)^k = x \} \]

We say that \(SP_k(X)\) is the kp-semisimple part of \(X\). In particular, if \(k = 1\), the \(SP(X)\) is called the p-semisimple part of \(X\) ([2]).

Proposition 1.3. \(SP_k(X) \cap P(X) = 0\)

Proof. If \(x \in SP_k(X) \cap P(X)\), then \(0 * x = 0\) and \(0 * (0 * x)^k = x\). Hence \(x = 0\) and that \(SP_k(X) \cap P(X) = 0\).

Proposition 1.4. For any BCI-algebra \(X\), \(SP_k(X)\) is a subalgebra of \(X\).
Proof. Let \( x, y \in SP_k(X) \), then \( 0 \ast (0 \ast x)^k = x \) and \( 0 \ast (0 \ast y)^k = y \).

\[
0 \ast (0 \ast (x \ast y)^k) = 0 \ast (0 \ast (x \ast y)^k) = (0 \ast (0 \ast x^k)) \ast (0 \ast (0 \ast y^k)) = x \ast y
\]

Hence \( x \ast y \in SP_k(X) \).

2. Main Results

Theorem 2.1. For any BCI-algebra X, \( SP_k(X) \) is a k-ideal if and only if for \( x, y \in P(X) \) and \( u, v \in SP_k(X) \), then \( x \ast u^k = y \ast u^k \) implies \( x = y \) and \( u = v \).

Proof. If \( SP_k(X) \) is a k-ideal of X and \( x \ast u^k = y \ast u^k \) for any \( x, y \in P(X) \) and \( u, v \in SP_k(X) \), then \( 0 \ast (x \ast u^k) = 0 \ast (y \ast u^k) \) and thus \( 0 \ast (0 \ast x^k) \ast (0 \ast u^k) = (0 \ast y) \ast (0 \ast u^k) \). Hence \( 0 \ast (0 \ast u^k) = 0 \ast (0 \ast v^k) \) since \( x, y \in P(X) \). It follows that \( u = v \) since \( u, v \in SP_k(X) \). From this, we have \( x \ast u^k = y \ast u^k \) and thus \( (x \ast y) \ast u^k = (x \ast u^k) \ast y = (y \ast u^k) \ast y = (y \ast u^k) \ast y = 0 \ast u^k \in SP_k(X) \) by proposition 1.4.

Hence \( x \ast y \in SP_k(X) \) since \( SP_k(X) \) is a k-ideal. Therefore \( x \ast y = 0 \) since \( x \ast y \in SP_k(X) \cap P(X) \). Similarly, we have \( y \ast x = 0 \) and thus \( x = y \).

Conversely, if \( y, x \ast u^k \in SP_k(X) \), then \( x \ast u^k = y \ast u^k \) implies \( x = y \) for any \( x, y \in X \).

(\( x \ast y \)) \( \ast a^k = (x \ast a^k) \ast y = (y \ast a^k) \ast y = 0 \ast a^k \in SP_k(X) \). From this it follows that \( x \ast y \in SP_k(X) \) since \( SP_k(X) \) is a k-ideal of X, and

\[
a = 0 \ast (0 \ast a^k) = 0 \ast (x \ast y)^k \ast (0 \ast a^k) = (0 \ast (0 \ast a^k)) \ast (x \ast y)^k = a \ast (x \ast y)^k
\]

In particular, \( 0 \ast (x \ast y)^k = 0 \), and thus \( x \ast y = 0 \ast (0 \ast (x \ast y)^k) = 0 \).

Similarly, we have \( y \ast x = 0 \). Therefore \( x = y \) Hence \( a^k \) is injective. On the other hand, for any

\[
x \in X, ((x \ast a^k) \ast (0 \ast a)) \ast x = ((x \ast a^k) \ast x) \ast (0 \ast a) =
\]

\[
(0 \ast a^k) \ast (0 \ast a) = 0 \ast a^k \ast a = 0 \ast (0 \ast a^k) \ast a = 0 \ast (0 \ast a^k) \ast a^k = 0 \text{ and}
\]

\[
(0 \ast a^k) \ast a^k = ((x \ast a^k) \ast (0 \ast a)) = ((x \ast (a^k) \ast (0 \ast a)) \ast a^k) \ast (0 \ast a) =
\]

\[
((x \ast (a^k) \ast (0 \ast a)) \ast a^k) \ast (0 \ast a) = ((x \ast a^k) \ast (0 \ast a)) \ast a^k \ast (0 \ast a) =
\]

\[
((x \ast a^k) \ast (0 \ast a)) \ast a = a \ast a = 0 = 0 \ast a \ast (0 \ast a) = (0 \ast (0 \ast a^k) \ast a^k = (0 \ast a) \ast a^k
\]

Since \( (0 \ast a^k) \) and \( a^k \) are injective, we have

\[
x \ast ((x \ast a^k) \ast (0 \ast a)) = 0
\]

Hence \( x = (x \ast a^k) \ast (0 \ast a) = (x \ast (0 \ast a)) \ast a^k = a^k(x \ast (0 \ast a)) \).

Therefore \( a^k \) is surjective.

Conversely if \( a^k \) is bijective for any \( a \in SP_k(X) \), then \( SP_k(X) \) is a k-ideal of X by Theorem 2.1.
From the proof of Theorem 2.1, it’s easy to see that $(0 * a)_{p}^{k}$ is the inverse of $a_{p}^{k}$.

**Theorem 2.3.** Let $X$ be a $BCI$-algebra. If $SP_{k}(X)$ is a k-ideal of $X$, then $a_{p}^{k}b_{r}^{k} = (a * (0 * b^{k}))_{r}^{k}$ for any $a, b \in SP_{k}(X)$.

**Proof.** For any $x \in X$. $(a * (0 * b^{k}))_{r}^{k}((x * b^{k}) * a^{k}) * (x * (a * (0 * b^{k}))_{r}^{k}) =

((x * b^{k}) * (x * (a * (0 * b^{k}))_{r}^{k})) * (a * (0 * b^{k}))_{r}^{k} = (0 * b^{k}) * a^{k} = 0 * (a * (0 * b^{k}))_{r}^{k} = (a * (0 * b^{k}))_{r}^{k}$.

and $a_{p}^{k}b_{r}^{k}((x * (a * (0 * b^{k}))_{r}^{k}) * (x * b^{k}) * a^{k}) = ((x * (a * (0 * b^{k}))_{r}^{k}) * (x * b^{k})) * a^{k}) * a^{k} = 0 * (a * (0 * b^{k}))_{r}^{k} = (0 * a^{k}) * (0 * (0 * b^{k}))_{r}^{k} = 0 * (0 * a^{k}) * b^{k} = a_{p}^{k}b_{r}^{k}(0)$

Since $(a * (0 * b^{k}))_{r}^{k}$ and $a_{p}^{k}b_{r}^{k}$ are injective, we have $(x * b^{k}) * a^{k}) * (x * (a * (0 * b^{k}))_{r}^{k}) = 0$ and $(x * (a * (0 * b^{k}))_{r}^{k}) * (x * b^{k}) * a^{k} = 0$. Hence $x * (a * (0 * b^{k}))_{r}^{k} = (x * b^{k}) * a^{k}$ and that $a_{p}^{k}b_{r}^{k}(x) = (a * (0 * b^{k}))_{r}^{k}(x)$ for any $x \in X$. Hence $a_{p}^{k}b_{r}^{k} = (a * (0 * b^{k}))_{r}^{k}$.

**References**


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