VARIETIES AND QUASIVARIETIES OF MONADIC TARSKI ALGEBRAS

Manuel ABAD, José P. DIAZ VARELA and Marta ZANDER

Received March 1, 2002

Abstract. In this paper we give a description of the lattice $\Lambda(\mathcal{MT})$ of subvarieties of monadic Tarski algebras introduced in [13], and prove that quasivarieties coincide with varieties. We also investigate some properties of the lattice $L(\mathcal{MB})$ of quasivarieties of monadic Boolean algebras determined in [3], showing the difference when a constant is added to the language of monadic Tarski algebras, and we give a quasiidentity for each member of $L(\mathcal{MB})$.

1 Introduction and Tarski algebras. Implicative structures are particularly common among algebras associated with logical systems, although they arise in many other areas of mathematics. In general, they consist of an ordered set in which the ordering is characterized by a binary operation of implication $\to$. If the partial order is a semilattice order we have the Brouwerian semilattices, that are the models of the $\{\wedge, \to\}-$fragment of the intuitionistic propositional calculus. If the semilattice satisfies the property that every filter $[p)$ is a Boolean algebra, we obtain the Tarski algebras [2] - the variety of $\{\vee, \to\}-$subreducts of Boolean algebras.

In this work we study the varieties and quasivarieties of the variety of all monadic Tarski algebras. The notion of *monadic Tarski algebra* was introduced by A. Monteiro and L. Iturrioz [13] as a generalization of the concept of monadic Boolean algebra. In [9], A. Figallo and independently, in [21] L. Monteiro et al. determined the free monadic Tarski algebra with a finite set of n free generators and calculated its number of elements.

We start with the notion of Tarski algebras. These algebras have been introduced by J. C. Abbott in [2] and have been studied by several authors. Recently, Davey et al. [5] proved that no non-trivial Tarski algebra, termed also implication algebras, is dualisable. Endoprimality in the variety of Tarski algebras has been considered in [22].

Definition 1.1 *An algebra $(A, \to, 1)$ of type $(2, 0)$ is said to be a Tarski algebra if:*

(T1) $x \to (y \to x) \approx 1$.

(T2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) \approx 1$.

(T3) $x \to 1 \approx 1$.

(T4) *If* $x \to y \approx 1$ *and* $y \to x \approx 1$ *then* $x \approx y$.

(T5) $(x \to y) \to x \approx x$.

2000 *Mathematics Subject Classification.* 08B15, 08C15, 03G25, 03C05, 96D99.
Key words and phrases. Varieties, Quasivarieties, Tarski algebras, monadic algebras.

Boolean algebras are the simplest examples of Tarski algebras: if $(A, \wedge, \vee, -, 0, 1)$ is a Boolean algebra and we define $x \to y = -x \vee y$ for $x, y \in A$, then $(A, \to, 1)$ is a Tarski algebra.

Recall that an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying properties T1 to T4 is called a *Hilbert algebra* [1], [2], [6], [7], [16], [17], [19]. Axiom (T5) is the characteristic identity for semisimple Hilbert algebras, so that the class of Tarski algebras is the class of semisimple Hilbert algebras [19].

The following set of axioms can be found among the many handwritten results that A. Monteiro left without publishing (see [18]). Observe that (M1), (M2) and (M3) are the equations that characterize the variety of Hilbert algebras.

Theorem 1.2 *An algebra $(A, \to, 1)$ of type $(2, 0)$ is a Tarski algebra if:*

(M1) $1 \to x \approx x$.

(M2) $x \to x \approx 1$.

(M3) $x \to (y \to z) \approx (x \to y) \to (x \to z)$.

(M4) $(x \to y) \to y \approx (y \to x) \to x$.

Throughout this paper, $\mathcal{B}$, $\mathcal{T}$, $\mathcal{MB}$ and $\mathcal{MT}$ will denote the equational classes of all Boolean algebras, all Tarski algebras, all monadic Boolean algebras and all monadic Tarski algebras, respectively.

If $\mathcal{K}$ is a class of similar algebras we will use the following notation: $H(\mathcal{K})$ for the class of algebras that are homomorphic images of algebras in $\mathcal{K}$; $I(\mathcal{K})$ for the class of algebras that are isomorphic copies of algebras in $\mathcal{K}$ and $S(\mathcal{K})$ for the class of algebras that are subalgebras of algebras in $\mathcal{K}$. The lattice of congruences of an algebra $A \in \mathcal{K}$ is denoted by $Con(A)$.

Let $A \in \mathcal{T}$. Given $x, y \in A$ we denote $x \leq y$ whenever $x \to y = 1$. It is well known that A is an ordered set with last element 1, that A is a join-semilattice and that the supremum of two elements a and b is $a \vee b = (a \to b) \to b$ [2].

The following result can also be found in [2].

Lemma 1.3 *If a Tarski algebra A has least element 0, then A is a Boolean algebra, where the Boolean complement of $a \in A$ is $-a = a \to 0$ and the infimum of the elements a and $b \in A$ is $a \wedge b = -(b \to -a)$.*

Lemma 1.4 *If A is a Tarski algebra, then the set $[p) = \{x \in A : p \leq x\}$ is a Boolean algebra, for every $p \in A$.*

Observe that if $x \in [p)$ then the complement of x in $[p)$ is $x \to p$, so the infimum of two elements x, $y \in [p)$ is $x \wedge y = (y \to (x \to p)) \to p$.

Now suppose that R is a join-semilattice with last element 1, in which $[r) = \{y \in R : r \leq y\}$ is a Boolean algebra, for every $r \in R$. J. C. Abbott [1], [2] proved that R is a Tarski algebra. Hence, there is a bijective correspondence between the variety of Tarski algebras and the class of all upper-bounded join-semilattices for which every principal filter is a Boolean lattice.

Definition 1.5 *A subset D of a Tarski algebra A is called a deductive system if:*

(D_1) $1 \in D$.

(D_2) *If* $x,\ x \to y \in D$ *then* $y \in D$.

For $H \subseteq A$, the intersection of all deductive systems of A containing H is called the *deductive system generated* by H. If $H \neq \emptyset$, we say that an element $x \in A$ is a *consequence* of H if there exist $h_1, h_2, \dots, h_n \in H$ such that $h_1 \to (h_2 \to (\dots(h_n \to x)\dots)) = 1$. If $H = \emptyset$ we say that x is a *consequence* of H if $x = 1$. The set of all consequences of H will be denoted $C(H)$.

Lemma 1.6 [19] *Let A be a Tarski algebra and $H \subseteq A$. The deductive system generated by H is $C(H)$.*

Corollary 1.7 *Let A be a Tarski algebra and $a \in A$. Then the deductive system generated by a is $C(a) = \{x \in A : a \to x = 1\} = \{x \in A : a \leq x\} = [a)$.*

Define a *filter* in a Tarski algebra A as a non-empty increasing set D such that if $x, y \in D$ and there exists $x \wedge y$ in A, then $x \wedge y \in D$. Then D is a filter if and only if D is a deductive system. Indeed, if D is a filter, then clearly $1 \in D$, and if $x,\ x \to y \in D$, then, as $x \leq x \vee y$, $x \vee y \in D$. Since $(x \vee y) \wedge (x \to y) = [(x \to y) \to ((x \vee y) \to y)] \to y = y$, it follows that $y \in D$. Conversely, suppose that D is a deductive system. Then D is clearly increasing. Let $x, y \in D$ be such that $x \wedge y$ exists. Then $[x \wedge y)$ is a deductive system that is also a Boolean algebra. If $E = D \cap [x \wedge y)$, then $E \subseteq [x \wedge y)$ and E is a filter of the Boolean algebra $[x \wedge y)$ and contains x and y. So E contains $x \wedge y$, and consequently $E = [x \wedge y) \subseteq D$.

It is known [2] that every congruence on a Tarski algebra A is determined by a deductive system D where the relation is $a \equiv b \pmod D$ if and only if $a \to b$ and $b \to a \in D$. Thus the lattice of deductive systems is isomorphic to the lattice of congruence relations. From this it is clear that the 2-element Tarski algebra $\mathbf{2} = \{0, 1\}$ is the only simple algebra in $\mathcal{T}$.

On the other hand, the intersection of all maximal deductive systems of A is $\{1\}$ [13], so the mapping $A \to \prod_{i \in I} A/D_i$, where $\{D_i\}_{i \in I}$ is the family of all deductive systems of A, is a subdirect embedding. In addition, if D is a maximal deductive system of A, A/D is simple. Hence, if A is subdirectly irreducible, A is simple.

Theorem 1.8 *The only subdirectly irreducible algebra in $\mathcal{T}$ is the simple algebra $\mathbf{2}$.*

In the rest of this section, A will be a finite non-trival Tarski algebra. $Ant(A)$ will denote the set of all antiatoms (dual atoms) of A. Observe that if $z \in A$, since $[z)$ is a Boolean algebra, $Ant([z)) \subseteq Ant(A)$ and $z = \bigwedge Ant([z))$. Thus, in a finite Tarski algebra, every element different from 1 is an infimum of a non-empty set of antiatoms.

The next lemma gives a characterization of maximal deductive systems of a finite Tarski algebra A.

Lemma 1.9 *Let A be a finite non-trivial Tarski algebra, and $n = |Ant(A)|$. If $a \in Ant(A)$ then $A \setminus (a]$ is a maximal deductive system of A, where $(a] = \{x \in A : x \leq a\}$. Moreover, every maximal deductive system in A is of the form $A \setminus (a]$, with $a \in Ant(A)$, that is, A has exactly n maximal deductive systems.*

Proof We first prove that $A \setminus (a]$ is a deductive system. Clearly, $1 \in A \setminus (a]$. Let $x, x \to y \in A \setminus (a]$ and let us prove that $y \in A \setminus (a]$. Suppose that $y \notin A \setminus (a]$, then $y \in (a]$.

So $y \leq a$. On the other hand, $a \leq x \to a$ and thus $x \to a = a$ as $x \not\leq a$ and a is a dual atom. Then

$$1 = x \to 1 = x \to (y \to a) = (x \to y) \to (x \to a) = (x \to y) \to a,$$

and hence $x \to y \leq a$, a contradiction. Let us see that $A \setminus (a]$ is maximal. Let $y \in (a]$. Then $a \to y \in A \setminus (a]$, since otherwise $a \to y \leq a$ and $a = (a \to y) \to a = 1$ which is a contradiction. So if D is a deductive system such that $A \setminus (a] \subset D$ and $A \setminus (a] \neq D$ then $a \in D$ and $a \to y \in A \setminus (a] \subset D$ for all $y \in (a]$, hence $A = D$.

Let D a maximal deductive system and suppose that $Ant(A) \subseteq D$. By the remark preceding this lemma, $A = D$, a contradiction. So there exists $a \in Ant(A)$ such that $a \notin D$ and consequently $D = A \setminus (a]$. ■

Suppose that $Ant(A) = \{a_1, \ldots, a_n\}$. Let $D_i = A \setminus (a_i]$ and let $j : A \to \prod_{i=1}^n A/D_i$ be a subdirect embedding. For $x \in A$, $(j(x))_i = 1$ if $x \notin (a_i]$ and $(j(x))_i = 0$ if $x \in (a_i]$. Hence $j(a_i)$ is an antiatom of $\prod_{i=1}^n A/D_i$, for $i = 1, \ldots, n$, and consequently, j induces a bijection between the set of antiatoms of A and the set of antiatoms of $\prod_{i=1}^n A/D_i$. Since every element of a finite Tarski algebra is a meet of antiatoms, it follows that $j : A \to [Min(j(A)))$ is an isomorphism, where $Min(j(A))$ denotes the set of minimal elements in $j(A)$.

2 Varieties of Monadic Tarski algebras.

The aim of this section is to give an equational basis with a minimum number of variables for each subvariety of monadic Tarski algebras.

Definition 2.1 [13] *An algebra $(A, \to, \forall, 1)$ of type $(2, 1, 0)$ is said to be a monadic Tarski algebra if $(A, \to, 1)$ is a Tarski algebra and:*

(Q_1) $\forall 1 \approx 1$.

(Q_2) $\forall x \to x \approx 1$.

(Q_3) $\forall((x \to \forall y) \to \forall y) \approx (\forall x \to \forall y) \to \forall y$.

(Q_4) $\forall(x \to y) \to (\forall x \to \forall y) \approx 1$.

Taking into account that $x \vee y = (x \to y) \to y$, then Q_3 can be written:
$\forall(x \vee \forall y) \approx \forall x \vee \forall y$.

In a monadic Tarski algebra A the following properties hold (see [21]):

(Q_5) $\forall \forall x \approx \forall x$.

(Q_6) If $x \leq y$ then $\forall x \leq \forall y$.

(Q_7) If $x \approx \forall x$ and $y \approx \forall y$, then $x \to y \approx \forall(x \to y)$.

Recall that the variety $\mathcal{T}$ is congruence distributive. Since algebras in $\mathcal{MT}$ have Tarski algebra reducts and congruence distributivity is a Mal'cev condition, it follows that $\mathcal{MT}$ is congruence distributive.

Let $\forall A = \{\forall x : x \in A\}$, then from Q_1, Q_5 and Q_7 it follows that $\forall A$ is a Tarski subalgebra of A. If A is a monadic Tarski algebra with least element 0, we know that A is a Boolean algebra. In that case, if we put by definition $\exists x = -\forall - x$, then $(A, \wedge, \vee, -, \exists, 0, 1)$ is a monadic Boolean algebra [11], [12]. On the other hand, if A is a monadic Boolean algebra and we put $x \to y = -x \vee y$ and $\forall x = -\exists - x$, then $(A, \to, \forall, 1)$ is a monadic Tarski algebra.

Definition 2.2 *A subset D of a monadic Tarski algebra A is said to be a monadic deductive system if D is a deductive system satisfying: (D_3) For $x \in D$, $\forall x \in D$.*
The notion of monadic deductive system generated by $X \subseteq A$ is defined in the usual way.

Lemma 2.3 *Let A be a monadic Tarski algebra and D a monadic deductive system of A, then the relation $x \equiv y$ if and only if $x \to y \in D$ and $y \to x \in D$, is a congruence.*

Lemma 2.4 *Let A be a monadic Tarski algebra. If $\equiv$ is a congruence defined on A then $|1| = \{x \in A : x \equiv 1\}$ is a monadic deductive system, and $x \equiv y$ if and only if $x \to y \in |1|$ and $y \to x \in |1|$.*

Lemma 2.5 ([8]) *Let A be a monadic Tarski algebra and $H \subseteq A$. Then the monadic deductive system generated by H is $C(\forall H)$.*

Let $\mathfrak{D}(A)$ be the lattice of monadic deductive systems of A. Observe that if $M \in \mathfrak{D}(A)$, $\forall M$ is a deductive system of $\forall A$, and if D' is a deductive system of $\forall A$, then $C(D') \in \mathfrak{D}(A)$.

¿From the previous lemmas we obtain

Corollary 2.6 *The lattices $Con(A)$, $\mathfrak{D}(A)$ and the lattice of deductive systems of $\forall A$ are all isomorphic.*

A non-trivial monadic Tarski algebra A is simple if and only if the only monadic deductive systems in A are A and $\{1\}$.

Lemma 2.7 ([8], [21]) *A is a subdirectly irreducible (simple) monadic Tarski algebra if and only if A is a simple monadic Boolean algebra.*

If B_n denotes the n-atom simple monadic Boolean algebra, then $B_k \in IS(B_l)$ if and only if $k \leq l$. From this and the fact that $\mathcal{MT}$ is congruence distributive and locally finite [9, 21] we have the following result.

Theorem 2.8 *The lattice $\Lambda(\mathcal{MT})$ of subvarieties of the variety $\mathcal{MT}$ is isomorphic to a chain of type $\omega + 1$:*

$$T \subset T_1 \subset T_2 \subset T_3 \subset \ldots \subset \mathcal{MT},$$

where T is the trivial variety and T_p is the variety generated in $\mathcal{MT}$ by the simple monadic Tarski algebra B_p.

Observe that the lattice of subvarieties of $\mathcal{MB}$ is

$$T \subset M_1 \subset M_2 \subset M_3 \subset \ldots \subset \mathcal{MB},$$

where M_p is the variety generated in $\mathcal{MB}$ by the simple monadic Boolean algebra B_p.

Below we will determine a characteristic equation with a minimum number of variables for each subvariety of $\mathcal{MT}$.

Consider the following term:

$$\gamma_p(x_0, \ldots, x_{p+1}) = \bigvee_{i=0}^{p} \forall \left(x_{i+1} \to \bigvee_{j=0}^{i} x_j \right).$$

Let us see that the identity $\gamma_p \approx 1$ characterizes the variety T_p generated by B_p.

If $p = 1$, $\gamma_1(x_0, x_1, x_2) = \forall(x_1 \to x_0) \vee \forall(x_2 \to (x_0 \vee x_1))$, and it is immediate that $\gamma_1(x_0, x_1, x_2) \approx 1$ holds in B_1.

Suppose that $p > 1$ and let $a_0, \ldots, a_{p+1} \in B_p$. Consider the elements $b_0 = a_0$, $b_1 = a_0 \vee a_1, \ldots, b_p = \bigvee_{j=0}^{p} a_j$. It is clear that $b_0 \leq b_1 \leq \ldots \leq b_p$. If $b_i < b_{i+1}$ for $i = 0, \ldots, p-1$, then $b_p = 1$, as B_p is a p-atom Boolean algebra. So $\forall(a_{p+1} \to \bigvee_{j=0}^{p} a_j) = \forall(a_{p+1} \to b_p) = \forall(a_{p+1} \to 1) = \forall 1 = 1$, and consequently, $\gamma_p(a_0, \ldots, a_{p+1}) = 1$. If $b_i = b_{i+1}$, for some i, then $a_{i+1} \leq b_i = \bigvee_{j=0}^{i} a_i$. So $\forall(a_{i+1} \to \bigvee_{j=0}^{i} a_j) = \forall 1 = 1$. Thus $\gamma_p(a_0, \ldots, a_{p+1}) = 1$. Therefore $\gamma_p(x_0, \ldots, x_{p+1}) \approx 1$ holds in B_p.

Let A be a finite (recall that $\mathcal{MT}$ is locally finite) subdirectly irreducible algebra in $\mathcal{MT}$ and suppose that the identity $\gamma_p(x_0, \ldots, x_{p+1}) \approx 1$ holds in A. Observe that $A \cong B_q$. Suppose that $q > p$ and let $a_1, \ldots, a_q$ be the atoms of A. Consider the elements $b_0 = 0$, $b_1 = a_1$, $b_2 = a_1 \vee a_2, \ldots, b_p = \bigvee_{i=1}^{p} a_i$ and $b_{p+1} = 1$. We have that $b_p \neq 1$, as $p < q$. Since $b_{i+1} \to b_i \neq 1$, it follows that $\forall(b_{i+1} \to b_i) = 0$. Thus $\gamma_p(b_0, \ldots, b_{p+1}) = \bigvee_{i=0}^{p} \forall(b_{i+1} \to \bigvee_{j=0}^{i} b_j) = \bigvee_{i=0}^{p} \forall(b_{i+1} \to b_i) = 0$, a contradiction. So $q \leq p$, and $A \in T_p$. Then we have the following theorem

Theorem 2.9 *$\gamma_p \approx 1$ is an equational basis for T_p.*

Now, from the identity $\gamma_p \approx 1$ and the results of Cignoli and Petrovich [4], we will determine a characteristic equation for T_p with a minimum number of variables.

Observe that if B is a simple monadic Boolean algebra and G is a generating set of B, then B can be generated by G as a Boolean algebra. If B_{2^n} is the free Boolean algebra over an $n-$element set G, then the atoms of B_{2^n} can be obtained as $\bigwedge\{G \setminus G_i\} \cup \{-G_i\}$ where $G_i \subseteq G$ and $-G_i = \{-x : x \in G_i\}$. Finally, if $f : B_{2^n} \to B_m$ $(m \leq 2^n)$ is an epimorphism, then $f(G)$ is a generating set for B_m. In addition, there exist sets $f(G_i)$ with $i = 1, \ldots, m$ and $|f(G_i)| \leq n$ such that for every atom $a_i \in B_m$, $a_i = \bigwedge f(G_i)$.

Theorem 2.10 ([4]) *If V is a congruence distributive variety and $\Lambda(V) \cong \omega + 1$, then for every $n \in \omega$, the minimum number of variables needed in an identity to characterize the subvariety T_n is the same as the minimum number of generators of the algebra B_{n+1} (the algebra that generates T_{n+1}).*

Lemma 2.11 ([4]) *In the variety $\mathcal{MT}$, the minimum number of generators for B_m is the smallest p such that $m \leq 2^p$.*

Consequently, in the variety $\mathcal{MT}$, the minimum number of variables needed to characterize T_m is the least p such that $m + 1 \leq 2^p$.

Consider the terms

$$x \wedge_{\forall z} y = (y \to (x \to \forall z)) \to \forall z \quad \text{and} \quad \exists x = \forall(x \to \forall x) \to \forall x.$$

When evaluated in B_m, we obtain

$$x \wedge_{\forall z} y = \begin{cases} x \wedge y & \text{if } \forall z = 0 \\ 1 & \text{if } \forall z = z = 1 \end{cases} \quad \text{and} \quad \exists x = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Consider now a set of variables $S = \{y_1, \ldots, y_p\}$, where p is the least positive integer such that $m + 1 \leq 2^p$, and let $H_i \subset S \cup -S$, $1 \leq i \leq m$, $|H_i| = p$ and such that $y_j \in H_i$ if and only if $-y_j \notin H_i$. Let $z_i = \bigwedge_{\forall y_1} H_i$. Consider the identity

$$\gamma^m_{min}(y_1, \ldots, y_p) = \left[\begin{array}{l} m \\ \bigwedge_{\forall y_1} \exists(z_i) \\ i = 1 \end{array} \right] \to \gamma_m(\forall y_1, z_1, \ldots, z_m, 1) \approx 1.$$

Observe that $\gamma^m_{min}(y_1, \ldots, y_p) \approx 1$ is a $p-$variable identity in the language of $\mathcal{MT}$.

Since $\gamma_m \approx 1$ holds in B_m, it is clear that $\gamma^m_{min}(y_1, \ldots, y_p) \approx 1$ holds in B_m.

Since $m+1 \leq 2^p$, there exists a generating set $G = \{g_1, \ldots, g_p\}$ of B_{m+1} and $G_i \subset G \cup -G, i = 1, \ldots, m+1$ such that if $\{a_1, \ldots a_{m+1}\} = At(B_{m+1})$ then $\bigwedge_{\forall g_1} G_i = \bigwedge G_i = a_i$. Then

$$\gamma^m_{min}(g_1, \ldots, g_p) = \left[\begin{array}{l} m \\ \bigwedge_{\forall g_1} \exists(a_i) \\ i = 1 \end{array} \right] \to \gamma_m(\forall g_1, a_1, \ldots, a_m, 1) = 1 \to 0 = 0.$$

So $\gamma^m_{min}(y_1, \ldots, y_p) \approx 1$ does not hold in B_{m+1}.

So we have proved

Theorem 2.12 *The identity $\gamma^m_{min}(y_1, \ldots, y_p) \approx 1$ is an equational basis for the subvariety T_m of $\mathcal{MT}$ with a minimum number of variables p, where p is the least positive integer such that $m+1 \leq 2^p$.*

3 Quasivarieties.

A class of algebras of similar type that is closed under isomorphisms, subalgebras, direct products, and ultraproducts is called a *quasivariety*. If V is a variety, $L(V)$ will denote the lattice of quasivarieties contained in V.

In this section we will prove that $L(\mathcal{MT}) = \Lambda(\mathcal{MT})$.

Remark 3.1 Let A be a finite monadic Tarski algebra, $a_1, \ldots, a_n$ the antiatoms of A and $b_1, \ldots b_m$ the antiatoms of $\forall A$. We know that for every $x \in A$, $x = \bigwedge\{a \in Ant(A) : x \leq a\}$. In particular, $b_k = \bigwedge_{j=1}^{t_k} a^k_j$, for $k = 1, \ldots, m$. If $Ant(b_k) = \{a \in Ant(A) : b_k \leq a\} = \{a^k_1, \ldots, a^k_{t_k}\}$, then $\{Ant(b_k)\}_{k=1}^m$ is a partition of $Ant(A)$. If $x = \bigwedge S$, where $S \subseteq Ant(A)$, then $\forall x = \bigwedge\{Ant(b_k) : S \cap Ant(b_k) \neq \emptyset\}$.

We know that the lattice $\mathcal{D}(A)$ of all monadic deductive systems of A and the lattice of all deductive systems of $\forall A$ are isomorphic, and, for finite A, the maximal deductive systems of $\forall A$ are of the form $\forall A \setminus (b]$, b an antiatom of $\forall A$ (Lemma 1.9). Consequently, the maximal monadic deductive systems of A are of the form $C(\forall A \setminus (b])$, the deductive system generated in A by $\forall A \setminus (b]$, b an antiatom of $\forall A$. We now characterize the maximal monadic deductive systems of A.

Proposition 3.2 *Let A be a finite monadic Tarski algebra and let b be an antiatom of $\forall A$, $b = \bigwedge_{i=1}^n a_i$, a_i antiatom of A, $1 \leq i \leq n$. Then $C(\forall A \setminus (b]) = \bigcap_{i=1}^n (A \setminus (a_i])$.*

Proof Let $x \in C(\forall A \setminus (b])$. If we suppose that $x \notin \bigcap_{i=1}^n (A \setminus (a_i])$, then there exists i such that $x \notin A \setminus (a_i]$, that is, $x \leq a_i$, and then $a_i \in C(\forall A \setminus (b])$, for some i. Hence there exist $h_1, \ldots, h_s \in \forall A \setminus (b]$ such that

$$h_1 \to (h_2 \to \ldots (h_s \to a_i) \ldots) = 1 \in \forall A \setminus (b].$$

Since $h_1 \in \forall A \setminus (b]$, $h_2 \to (\ldots (h_s \to a_i) \ldots) \in \forall A \setminus (b]$. Continuing with this procedure we obtain that $a_i \in \forall A \setminus (b]$, which is not possible.

For the converse, suppose that $x \in \bigcap_{i=1}^{n}(A \setminus (a_i])$. Then $x \not\leq a_i$ for every i. From $b = \bigwedge_{i=1}^{n} a_i$, we have from Remark 3.1 that $\forall x \not\leq b$. So $\forall x \in \forall A \setminus (b]$. Since $\forall x \to x = 1$, $x \in C(\forall A \setminus (b]))$. ■

The following lemma is the key to prove that the varieties and quasivarieties in $\mathcal{MT}$ coincide.

Lemma 3.3 *Let A be a finite monadic Tarski algebra and let b be an antiatom of $\forall A$, $b = \bigwedge_{i=1}^{n} a_i$, a_i antiatom of A, $1 \leq i \leq n$. Then $A/\bigcap_{i=1}^{n}(A \setminus (a_i]) \cong [b) \cong B_n$.*

Proof If $x, y \in [b)$, $x \neq y$, as $[b)$ is a Tarski subalgebra of A, $x \to y \in [b)$ and $y \to x \in [b)$. Since $x \neq y$, it follows that $x \to y \neq 1$ or $y \to x \neq 1$, so $x \to y \notin \bigcap_{i=1}^{n}(A \setminus (a_i])$ or $y \to x \notin \bigcap_{i=1}^{n}(A \setminus (a_i])$, being that $\bigcap_{i=1}^{n}(A \setminus (a_i]) \cap [b) = \{1\}$. Hence $|x| \neq |y|$, where $|z|$ stands for the equivalence class of an element z in the quotient.

Now, let $y \in A$, $y \notin [b)$, and let us prove that $|y| = |y \vee b|$. Since $y \to (y \vee b) = 1 \in \bigcap_{i=1}^{n}(A \setminus (a_i])$, we just have to prove that $(y \vee b) \to y \in \bigcap_{i=1}^{n}(A \setminus (a_i])$. Suppose on the contrary that $(y \vee b) \to y \notin \bigcap_{i=1}^{n}(A \setminus (a_i])$. Then there exists i, $1 \leq i \leq n$, such that $(y \vee b) \to y \in (a_i]$, that is, $(y \vee b) \to y \leq a_i$. In addition, $(y \vee b) \to y = (b \vee y) \to y = ((b \to y) \to y) \to y = b \to y$. So $b \to y \leq a_i$. Then $b \leq a_i \to b \leq (b \to y) \to b = b$, that is, $a_i \to b = b$. Since $b \to a_i = 1$, it follows that $a_i = (a_i \to b) \to a_i = b \to a_i = 1$, a contradiction.

The second isomorphism is clear. ■

Corollary 3.4 *Every simple homomorphic image of a finite algebra A is a retract of A.*

The quasivariety generated by a class K, which we denote by $Q(K)$, is the least quasivariety containing K. Every variety is a quasivariety.

A *critical* algebra is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras.

Theorem 3.5 (See [10]) *Every non-trivial locally finite quasivariety is generated by its critical algebras.*

Since $\mathcal{MT}$ is locally finite, if $W \in L(\mathcal{MT})$ then W is the quasivariety generated by the critical algebras contained in W.

Theorem 3.6 *The set of critical algebras of $\mathcal{MT}$ is the set of finite simple algebras of $\mathcal{MT}$.*

Proof Observe that every simple algebra is critical. Let A be a critical algebra. Then A is finite. Suposse that A is not simple. Then the set $\{D_i\}_{i=1}^{n}$ of maximal monadic deductive systems of A is non-empty. Let $i : A \to \prod_{i=1}^{n} A/D_i \cong \prod_{i=1}^{n} B_{k_i}$ the subdirect representation of A. Thus by Lemma 3.3, for each i there exists $b_i \in Ant(\forall A)$ such that $[b_i) \cong B_{k_i}$. So $A \in ISP(\{[b_i)\}_{i=1}^{n})$ and the algebras $[b_i)$ are proper subalgebras of A. A contradiction, as A is critical. ■

Let $V(A)$ denote the variety generated by A.

Corollary 3.7 *Let A be a finite algebra in $\mathcal{MT}$. Then $Q(A) = V(A) = V(B_j)$, where B_j is the greatest simple homomorphic image of A.*

Proof First observe that since B_{k_n} is simple, $Q(B_{k_n}) = V(B_{k_n})$. Let $A \hookrightarrow \prod_{i=1}^{n} A/D_i \cong \prod_{i=1}^{n} B_{k_i}$. Then $A \in ISP(B_{k_n}) = Q(B_{k_n})$. So $Q(A) \subseteq Q(B_{k_n})$. On the other hand, by Proposition 3.2, $B_{k_n} \in IS(A)$, so $Q(B_{k_n}) \subseteq Q(A)$. ■

¿From this corollary and Theorem 3.6 we have the following result.

Theorem 3.8 *The subvarieties and the subquasivarieties of $\mathcal{MT}$ coincide.*

4 Monadic Boolean Quasivarietes. The aim of this section is to show that, in spite of $\Lambda(\mathcal{MB})$ is isomorphic to $\Lambda(\mathcal{MT})$, there is a great difference between $L(\mathcal{MB})$ and $L(\mathcal{MT})$. It is worth noting that the class of monadic Boolean algebras is the class of monadic Tarski algebras with a new constant "0" in the language, satisfying $0 \wedge x \approx 0$. In this section we will also give an effective axiomatization for each quasivariety in $L(\mathcal{MB})$.

As we have already pointed out, the class $\mathcal{MB}$ of monadic Boolean algebras is a variety the subvarieties of which form an $\omega + 1$ chain under inclusion:

$$T \subset M_1 \subset M_2 \subset M_3 \subset \ldots \subset \mathcal{MB}$$

such that, for each $p \in \omega$, M_p is the variety generated by the simple monadic Boolean algebra B_p.

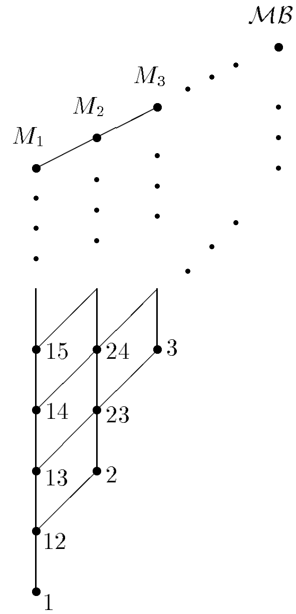
In [3], Adams and Dziobiak proved that the critical algebras in $\mathcal{MB}$ are the simple algebras B_p, and the algebras $B_p \times B_q$, $1 \leq p < q < \omega$.

Let $P \subset \omega \times \omega$ denote the set consisting of all ordered pairs (i, j) such that $1 \leq i < j < \omega$. Let $\sqsubseteq$ be defined on P by $(i, j) \sqsubseteq (k, l)$ if and only if $i \leq k$ and $j \leq l$. For $1 \leq i < \omega$, let P_i denote the principal order ideal of P determined by (i, i). Let $D(P)$ and $D(P_i)$ denote the distributive lattices of all decreasing subsets of $(P, \sqsubseteq)$ and P_i, respectively. Then, in [3], Adams and Dziobiak proved that

$$L(\mathcal{MB}) \cong D(P), \quad \text{and, for} \quad 1 \leq i < \omega, \; L(M_i) \cong D(P_i).$$

We may define a partial preorder on the set $Cr(K)$ of critical algebras of a variety K: for $A, B \in Cr(K)$, $A \leq B$ if and only if $A \in Q(B)$, so that $Q(A) \leq Q(B)$ if and only if $A \leq B$.

For each n, consider the class $M_n = \bigvee_{i>n} Q(B_n \times B_i)$. The following figure, where ij stands for $Q(B_i \times B_j)$ and i stands for $Q(B_i)$, shows the ordered set of subquasivarieties $Q(B_i \times B_j)$, $i < j$, $Q(B_i)$ and M_n.



Now we will prove that the quasivarieties shown in the figure are exactly the join-irreducible elements of the lattice $L(\mathcal{MB})$. First we have the following easy lemma.

Lemma 4.1 *If A is a critical monadic Boolean algebra, then $Q(A)$ is join-irreducible in $L(\mathcal{MB})$.*

Proof Since $L(\mathcal{MB})$ is distributive, $Cr(Q_1 \vee Q_2) = Cr(Q_1) \cup Cr(Q_2)$. ∎

Corollary 4.2 *$Q(B_i \times B_j)$ and $Q(B_r)$ are join-irreducible.*

Lemma 4.3 *$\mathcal{MB}$ and M_n are join-irreducible, for each n.*

Proof If $\mathcal{MB} = Q_1 \vee Q_2$, one of the sets $I_1 = \{j : B_j \in Q_1\}$ and $I_2 = \{j : B_j \in Q_2\}$ is infinite. Suppose that I_1 is infinite. Then $B_k \times B_l \in Q_1$ for all (k, l), $k < l$. Then $\mathcal{MB} = Q_1$.

If $M_n = Q_1 \vee Q_2$, one of the sets $I_1 = \{j : B_n \times B_j \in Q_1\}$ and $I_2 = \{j : B_n \times B_j \in Q_2\}$ is infinite, and so $M_n = Q_1$ or $M_n = Q_2$. ∎

Lemma 4.4 *$Q(B_i \times B_j)$, $Q(B_i)$, M_i and $\mathcal{MB}$ are the unique join-irreducible subquasivarieties of $\mathcal{MB}$.*

Proof Let Q be a join-irreducible subquasivariety of $\mathcal{MB}$. Suppose that $Q \neq \mathcal{MB}$ and that Q is finitely generated (by finitely many critical algebras). Then $Cr(Q)$ is finite, say $Cr(Q) = \{A_1, \ldots, A_n\}$, where $A_k = B_i \times B_j$ or $A_k = B_l$. Then $Q = \bigvee_{k=1}^{n} Q(A_k)$. Since Q is join irreducible, $Q = Q(A_k)$, for some k, $1 \leq k \leq n$.

Suppose that Q is not finitely generated. Then $Cr(Q)$ is not finite. Since $Q \neq \mathcal{MB}$, the set $\{j : B_j \in Q\}$ is finite. So there exists j_0 such that $B_{j_0} \in Q$ and $B_{j_0+1} \notin Q$. Again, since Q is not finitely generated, $\{k : B_{j_0} \times B_k \in Q\}$ is not finite, and consequently, $B_{j_0} \times B_k \in Q$ for every k. Hence, $Q = M_{j_0}$. ∎

Lemma 4.5 *Every quasivariety Q is a finite join of join-irreducible quasivarieties.*

Proof If $Q = \mathcal{MB}$ or Q is finitely generated, the Lemma is clear. Suppose that $Q \neq \mathcal{MB}$ and Q is not finitely generated. Then the sets $\{i : M_i \subseteq Q\}$ and $\{j : B_j \in Q\}$ are bounded and non-empty. If i_0 and j_0 respectively denote the greatest elements of these sets, $P(i_0, j_0) = \{(k,l) : B_k \times B_l \in Q \text{ and } i_0 < k,\ j_0 < l\}$ is finite. Then

$$Q = M_{i_0} \vee Q(B_{j_0}) \vee \bigvee_{(k,l)\in P(i_0,j_0)} Q(B_k \times B_l).$$

Observe that the set $P(i_0, j_0)$ can be empty.

∎

Consider now the following subquasivarieties of $\mathcal{MB}$:

$$(\mathcal{MB} : B_k \times B_l) = Q(\{A \in Cr(\mathcal{MB}) : B_k \times B_l \not\leq A\}),$$

$$(\mathcal{MB} : B_n) = \{A \in \mathcal{MB} : B_n \notin IS(A)\} = M_{n-1}.$$

Lemma 4.6 $Q(B_n \times B_m) = (\mathcal{MB} : B_1 \times B_{m+1}) \cap (\mathcal{MB} : B_{n+1})$.

Proof Since $B_{n+1} \notin IS(B_n \times B_m)$, $Q(B_n \times B_m) \subseteq (\mathcal{MB} : B_{n+1})$. Since $B_1 \times B_{m+1} \not\leq B_n \times B_m$, $Q(B_n \times B_m) \subseteq (\mathcal{MB} : B_1 \times B_{m+1})$. So $Q(B_n \times B_m) \subseteq (\mathcal{MB} : B_1 \times B_{m+1}) \cap (\mathcal{MB} : B_{n+1})$. For the converse, suppose that $A \in Cr\,[(\mathcal{MB} : B_1 \times B_{m+1}) \cap (\mathcal{MB} : B_{n+1})]$. If $A \cong B_r \times B_s$, $r < s$, then $s < m + 1$, that is, $s \leq m$, and $n + 1 > r$, that is, $r \leq n$. So $B_r \times B_s \in Q(B_n \times B_m)$. If $A \cong B_p$, then $p < n + 1$, that is, $p \leq n$, and consequently, $B_p \in IS(B_n \times B_m)$. So $B_p \in Q(B_n \times B_m)$. ∎

Remark. With the notation of Lemma 4.5, if $Q = M_{i_0} \vee Q(B_{j_0}) \vee \bigvee_{(k,l)\in P(i_0,j_0)} Q(B_k \times B_l)$, then

$$Q = (\mathcal{MB} : B_{j_0+1}) \cap (\bigcap_{(k,l)\in P(i_0,j_0)} (\mathcal{MB} : B_k \times B_l)) \cap (\bigcap_{m_0 < n \leq j_0} (\mathcal{MB} : B_k \times B_{j_0+1})),$$

where $m_0 = max\{k : (k,l) \in P(i_0, j_0)\}$.

Lemma 4.7 *The quasivarieties $(\mathcal{MB} : B_i \times B_j)$, $i < j$, and $(\mathcal{MB} : B_i)$ are meet-irreducible.*

Proof Suppose that $(\mathcal{MB} : B_i \times B_j) = K_1 \wedge K_2$, K_1, K_2 quasivarieties, and suppose that $K_1 \wedge K_2 \neq K_1$ and $K_1 \wedge K_2 \neq K_2$. Then there exist critical algebras $A_1 \in K_1 \setminus K_2$ and $A_2 \in K_2 \setminus K_1$. Then A_1, $A_2 \notin (\mathcal{MB} : B_i \times B_j)$. So A_1 and A_2 are of the form $B_p \times B_q$, $p < q$ or B_p. Suppose, for instance, that $A_1 = B_k \times B_l$, $i \leq k$, $j \leq l$, and $A_2 = B_r \times B_s$, $i \leq r$, $j \leq s$, the other cases being similar. Then $B_i \times B_j$ is a subalgebra of A_1 and $B_i \times B_j$ is a subalgebra of A_2. Since A_1 is not a subalgebra of A_2 and A_2 is not a subalgebra of A_1, it follows that $k \neq r$ and $l \neq s$. In addition, if $k < r$, then $l > s$, and if $k > r$, then $l < s$. Suppose, for instance, that $k < r$ and $l > s$, and consider the algebra $B_k \times B_s$. Then $B_k \times B_s$ is a subalgebra of A_1 and of A_2. Hence $B_k \times B_s \in K_1 \wedge K_2$. On the other hand, $B_k \times B_s$ contains $B_i \times B_j$ as a subalgebra, that is, $B_k \times B_s \notin (\mathcal{MB} : B_i \times B_j)$, a contradiction. ∎

Lemma 4.8 *The quasivarieties $(\mathcal{MB} : B_i \times B_j)$, $i < j$, and $(\mathcal{MB} : B_i)$ are the unique meet-irreducible subquasivarieties of $\mathcal{MB}$.*

Corollary 4.9 *Every subquasivariety of $\mathcal{MB}$ is a finite meet of subquasivarieties of the form $(\mathcal{MB} : B_i \times B_j)$ and $(\mathcal{MB} : B_i)$.*

So, in order to obtain an axiomatization for each subquasivariety of $\mathcal{MB}$, it suffices to give an axiomatization for the quasivarieties $(\mathcal{MB} : B_i \times B_j)$ and $(\mathcal{MB} : B_i)$.

We now turn our attention to quasi-identities characterizing meet-irreducibles in $L(\mathcal{MB})$. A *quasi-identity* in an algebraic language $\mathcal{L}$ is an expression of the form

$$\varphi_1 \approx \psi_1 \ \& \ \ldots \ \& \ \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n$$

where $n > 0$ and $\varphi_1 \approx \psi_1, \ldots, \varphi_{n-1} \approx \psi_{n-1}, \ \varphi_n \approx \psi_n$ are identities in $\mathcal{L}$.

An algebra A satisfies a quasi-identity $\varphi_1 \approx \psi_1 \ \& \ \ldots \ \& \ \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n$, denoted by $A \models \varphi_1 \approx \psi_1 \ \& \ \ldots \ \& \ \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n$ if and only if for every $\vec{a} \in A^m$, $\left[\varphi_1^A(\vec{a}) = \psi_1^A(\vec{a}), \ \ldots \ , \varphi_{n-1}^A(\vec{a}) = \psi_{n-1}^A(\vec{a})\right]$ implies $\varphi_n^A(\vec{a}) = \psi_n^A(\vec{a})$.

A class K of algebras is a quasivariety if and only if there exists a set Δ of quasi-identities such that K is the class of all algebras which satisfy all quasi-identities in Δ.

The following simple lemmas are the basis for constructing the quasi-identities characterizing the quasivarieties of $\mathcal{MB}$.

Lemma 4.10 *A monadic Boolean algebra A contains a subalgebra isomorphic to B_n if and only if there exist $a_1, \ldots, a_n \in A$ such that $\exists a_i = 1$ for all i, $a_i \wedge a_j = 0$ for all $i < j$, and $\bigvee_{i=1}^n a_i = 1$.*

Lemma 4.11 *A monadic Boolean algebra A contains a subalgebra isomorphic to $B_k \times B_l$ if and only if there exist $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$ different from zero such that if $a = \bigvee_{i=1}^k a_i$ and $b = \bigvee_{j=1}^l b_j$, then $b = -a$, $a_i \wedge a_j = b_i \wedge b_j = 0$, for all $i < j$, $\exists a_i = a$, $1 \leq i \leq k$, and $\exists b_j = b$, $1 \leq j \leq l$.*

Proof Let $f : B_k \times B_l \to A$ be an embedding. Let $x_1, \ldots, x_k$ be the atoms of B_k, and $y_1, \ldots, y_l$ be the atoms of B_l. Then the elements $a_i = f(x_i, 0)$, $1 \leq i \leq k$, and $b_j = f(0, y_j)$, $1 \leq j \leq l$, satisfy the required conditions. For the converse, it is enoughto consider the subalgebra generated by $a_1, \ldots a_k, b_1, \ldots b_k$. ■

By Lemma 4.10, the quasi-identity

$$\left[\left(\bigwedge_{i=1}^{n} \exists x_i \approx 1\right) \ \& \ \left(\bigvee_{i<j,i,j=1}^{n} x_i \wedge x_j \approx 0\right) \ \& \ \left(\bigvee_{i=1}^{n} x_i \approx 1\right)\right] \implies 0 \approx 1 \qquad (*)$$

holds in a monadic Boolean algebra A if and only if $A \in (\mathcal{MB} : B_n)$. Therefore $(\mathcal{MB} : B_n)$ is axiomatized by the axioms of $\mathcal{MB}$ and $(*)$.

By lemma 4.11, it is easy to see that the quasi-identity

$$\left[\left(\exists(\bigvee_{i=1}^{k} x_i) \approx \bigvee_{i=1}^{k} x_i\right) \ \& \ \left(\exists(\bigvee_{j=1}^{l} y_j) \approx \bigvee_{j=1}^{l} y_i\right) \ \& \ \left(\bigvee_{i<j,i,j=1}^{k} (x_i \wedge x_j) \approx 0\right)\right.$$

$$\& \ \left(\bigvee_{i<j,i,j=1}^{l} (y_i \wedge y_j) \approx 0\right) \ \& \ \left(\bigvee_{i=1}^{k} x_i \approx -\bigvee_{j=1}^{l} y_j\right) \ \& \ \left(\&_{s=1}^{k} (\exists x_s \approx \bigvee_{i=1}^{k} x_i)\right)$$

$$\left.\& \ \left(\&_{t=1}^{l} (\exists y_t \approx \bigvee_{j=1}^{l} y_j)\right)\right] \implies \bigvee_{j=1}^{l} y_j \approx 0 \qquad (**)$$

holds in a monadic Boolean algebra A if and only if $A \in (\mathcal{MB} : B_k \times B_l)$.

Therefore $(\mathcal{MB} : B_k \times B_l)$ is axiomatized by the axioms of $\mathcal{MB}$ and $(**)$.

Let γ_n denote the set of axioms of $\mathcal{MB} + (*)$, and let $\beta_{k,l}$ denote the set of axioms of $\mathcal{MB} + (**)$.

Corollary 4.12 *An axiomatization for $Q(B_n \times B_m)$ is given by* γ_{n+1} & $\beta_{1,m+1}$.

Corollary 4.13 *M_n is axiomatized by the axioms of $\mathcal{MB}$ and γ_{n+1}.*

Then we have given an axiomatization for all meet-irreducible quasivarieties in $L(\mathcal{MB})$. An axiomatization for an arbitrary quasivariety in $L(\mathcal{MB})$ follows from Corollary 4.9.

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Departamento de Matemática, Universidad Nacional del Sur,
8000 Bahía Blanca, Argentina
e-mail: {*imabad, usdiavar, mzander*}*@criba.edu.ar*