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NON STANDARD, NO INFORMATION SECRETARY PROBLEMS

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ABSTRACT. A version of the secretary problem is considered. Items ranked from 1 to N are randomly selected without replacement, one at a time, and to win is to stop at any item whose overall (absolute) rank belongs to a given set of ranks. Only the relative ranks of the items drawn so far are observed. The analysis is based on the existence of an embedded Markov chain and uses the technique of backward induction. The requirement of choosing an item with a prescribed value of the absolute rank can lead to more complicated strategies than threshold strategies. This approach can be used to give exact results for any set of absolute ranks. Exact results for the optimal strategy and the probability of success are given for a few sets. These examples are chosen to illustrate the variety of character of optimal stopping sets. Asymptotic behaviour is also investigated.

1. INTRODUCTION AND SUMMARY

Although a version of the secretary problem (the beauty contest problem, the dowry problem or the marriage problem) was first solved by Cayley [1] in 1875, it was not until four decades ago there was a sudden resurgence of interest in this problem. Since the articles by Gardner [6, 7] the secretary problem has been extended and generalized in many different directions. Excellent reviews of the development of this colourful problem and its extensions have been given by Rose [13], Freeman [5], Samuels [14] and Ferguson [3]. The formulation of the classical secretary problem in its simplest form can be formulated following Ferguson [3]. He defined the secretary problem in its standard form to have the following features:

1. There is only one secretarial position available.
2. The number of applicants, N, is known in advance.
3. The applicants are interviewed sequentially in a random order.
4. All the applicants can be ranked from the best to the worst without any ties. Further, the decision to accept or to reject an applicant must be based solely on the relative ranks of the interviewed applicants.
5. An applicant once rejected cannot be recalled later.
6. The employer is satisfied with nothing but the very best. The payoff is 1 if the best of the N applicants is chosen and 0 otherwise.

In our consideration we change Assumption 6 and we will be happy to accept a candidate, who has rank belonging to a fixed set A. In the literature on the original 'secretary problem', *i.e.* when $A = \{1\}$, and its extension (see *e.g.* [3] for a comprehensive bibliography), the exact optimal strategy for the more general secretary problem is not given. In this paper the derivation of the exact optimal strategy for a more general secretary problem is based on the backward induction and using the existence of an embedded Markov chain. These techniques have been used by several authors (see *e.g.* [5] for a review of papers with such

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an approach), difficulties appear in trying to extend existing approaches to derive exact results for more sophisticated sets A. In special cases, when $A = \{1, 2, \dots, s\}$, the optimal strategy for $s = 2$ was given by Gilbert and Mosteller [8]. Dynkin and Yushkevich [2] outline a proof. The paper by Quine and Law [11] was devoted to the case $s = 3$. Authors such as Gusein–Zade [9] and Frank and Samuels [4] provide asymptotic results for the optimal strategy for $s \geq 3$. In all these papers character of set A is such that it contains all ranks from 1 to some s. More complicated cases, where the sequence of ranks has 'holes', have been considered by Rose [12], Mori [10] and Szajowski [15]. In these papers the set A contains some non-extremal ranks. The set A in [15] consists of only one element s (one relative rank). Exact results have been given for $s = 1, 2$ and the asymptotic solution has been obtained for $s = 3, 4, 5$. In this paper the results of [15] are extended to some subsets of $\{1, 2, \dots, s\}$.

In Section 2 Markov chain related to the secretary problem is formulated. This section is based mainly on the suggestion from [2] and the results of [15]. In the next sections the solution of the secretary problem with A being a subset of $\{1, 2, \dots, s\}$ for $s = 3, 4$ are given. We provide exact and asymptotic solutions for all the cases. In Section 5 we present the optimal strategy for $A = \{6, 7\}$. In this special case the optimal strategy is not a threshold strategy.

In the last section we give a comparison of the results obtained.

2. The embedded Markovian model of the secretary problem

Let $\mathbb{S} = \{1, 2, \dots, N\}$ be the set of ranks of items and $\{x_1, x_2, \dots, x_N\}$ a permutation of these ranks. We assume that all of these permutations are equally likely. Let X_k be the rank of the k-th candidate. We define

$$Y_k = \#\{1 \leq i \leq k : X_i \leq X_k\}.$$

The random variable Y_k is called *the relative rank* of the k-th candidate with respect to the items investigated up to the moment k.

We observe sequentially a permutation of items from the set $\mathbb{S}$. The mathematical model of such an experiment is the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The elementary events are permutations of the elements from $\mathbb{S}$ and the probability measure $\mathbf{P}$ is the uniform distribution on Ω. The observation of the random variables Y_k, $k = 1, 2, \dots, N$, generate a sequence of σ-fields, $\mathcal{F}_k = \sigma\{Y_1, Y_2, \dots, Y_k\}$, $k = 1, 2, \dots, N$. The random variables Y_k are independent and $\mathbf{P}\{Y_k = i\} = \frac{1}{k}$, $i = 1, 2, \dots, k$.

Denote by $\mathfrak{M}^N$ the set of all Markov moments τ with respect to σ-fields $\{\mathcal{F}_k\}_{k=1}^N$. Let $q : \mathbb{S} \to \Re^+$ be the gain function. Define

$$v_N = \sup_{\tau \in \mathfrak{M}^N} \mathbf{E} q(X_\tau). \tag{1}$$

We are looking for $\tau^* \in \mathfrak{M}^N$, such that $\mathbf{E} q(X_{\tau^*}) = v_N$. Since $\mathfrak{M}^N$ is finite, then such a τ^* exists and v_N is finite. In this paper we consider a gain function of the form

$$q(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $A \subset \mathbb{S}$. From (1) we have $v_N = \mathbf{P}\{X_{\tau^*} \in A\} = \sup_{\tau \in \mathfrak{M}^N} \mathbf{P}\{X_\tau \in A\}$.

Such problems have been investigated, as stated in Section 1, by Gilbert and Mosteler [8] and others. They constructed the optimal strategy for $A = \{1\}$, $A = \{1, 2\}$. Frank and Samuels [4] and Gusein–Zade [9] have considered $A = \{1, 2, \dots, s\}$. In these papers the gain functions are monotone. We consider problems with gain functions which are not monotone.

2.1. **The probability of success.** Let $q(\cdot)$ be defined by (2). We have

$$\begin{aligned}\mathbf{P}\{X_\tau \in A\} &= \mathbf{E}q(X_\tau) = \sum_{r=1}^{N} \int_{\{\tau=r\}} q(X_\tau) d\mathbf{P} = \sum_{r=1}^{N} \int_{\{\tau=r\}} \sum_{a\in A} \mathbf{P}\{X_r = a|Y_r\} d\mathbf{P} \\ &= \mathbf{E} \sum_{a\in A} g_a(\tau, Y_\tau) = \mathbf{E} g_A(\tau, Y_\tau),\end{aligned}$$

where

$$g_a(r,l) = \mathbf{P}\{X_r = a|Y_r = l\} = \frac{\binom{a-1}{l-1}\binom{N-a}{r-l}}{\binom{N}{r}} \tag{3}$$

and

$$g_A(r,l) = \sum_{a\in A} g_a(r,l)$$

for $a = 1, 2, \ldots, N$, $l = 1, 2, \ldots, \min(a, r)$, $r = 1, 2, \ldots, N$ (see [8]).

2.2. **Solution by recursive algorithm.** Let $\mathfrak{M}_r^N = \{\tau \in \mathfrak{M}^N : r \le \tau \le N\}$ and $\tilde{v}_N(r) = \sup_{\tau\in\mathfrak{M}_r^N} \mathbf{E}q(X_\tau)$. The following algorithm permits the construction of the value of the problem v_N.

$$\tilde{v}_N(N) = \mathbf{E}q(X_N) = \frac{\operatorname{card}(A)}{N}. \tag{4}$$

Let

$$w_N(N,l) = q(l) = \begin{cases} 1, & \text{if } l \in A, \\ 0, & \text{otherwise,}\end{cases} \tag{5}$$

$$w_N(r,l) = \max\{g_A(r,l), \mathbf{E}w_N(r+1, Y_{r+1})\}, \tag{6}$$

$$\tilde{v}_N(r) = \mathbf{E}w_N(r, Y_r) = \frac{1}{r}\sum_{l=1}^{r} w_N(r,l). \tag{7}$$

We thus have $v_N = \tilde{v}_N(1)$. The optimal stopping time τ^* is defined as follows: one has to stop at the first moment r at which $Y_r = l$, unless $w_N(r,l) > g_A(r,l)$. We can define the stopping set $\Gamma = \{(r,l) : g_A(r,l) \ge w_N(r+1)\}$.

2.3. **The embedded Markov chain.** Let $a = \max(A)$. The function $g_a(r,l)$ defined in (3) is equal to 0 for $l > \min(a,r)$ and is non-negative for $l \le \min(a,r)$. This means that we should only choose the required item at moments r with state (r,l) such that $l \le \min(a,r)$.

Define $W_0 = (1, Y_1) = (1,1)$, $\gamma_t = \inf\{r > \gamma_{t-1} : Y_r \le \min(a,r)\}$ ($\inf\emptyset = \infty$) and $W_t = (\gamma_t, Y_{\gamma_t})$. If $\gamma_t = \infty$, then W_t is defined to be (∞,∞). W_t is a Markov chain with the following one step transition probabilities (see [15])

$$p(r,s) = \mathbf{P}\{W_{t+1} = (s, l_s)|W_t = (r, l_r)\} = \begin{cases} \frac{1}{s}, & \text{if } r < a,\ s = r+1, \\ \frac{(r)_a}{(s)_{a+1}}, & \text{if } a \le r < s, \\ 0, & \text{if } r \ge s \text{ or } r < a,\ s \ne r+1,\end{cases} \tag{8}$$

with $p(\infty,\infty) = 1$, $p(r,\infty) = 1 - a\sum_{s=r+1}^{N} p(r,s)$, where $(s)_a = s(s-1)(s-2)\ldots(s-a+1)$, $(s)_0 = 1$. Let $\mathcal{G}_t = \sigma\{W_1, W_2, \ldots, W_t\}$ and $\tilde{\mathfrak{M}}^N$ be the set of stopping times with respect to $\{\mathcal{G}_t\}_{t=1}^N$. Since γ_t is increasing, then we can define $\tilde{\mathfrak{M}}_{r+1}^N = \{\sigma \in \tilde{\mathfrak{M}}^N : \gamma_\sigma > r\}$.

Let $\mathbf{P}_{(r,l)}(\cdot)$ be the probability measure related to the Markov chain W_t, with trajectory starting in state (r,l) and $\mathbf{E}_{(r,l)}(\cdot)$ the expected value with respect to $\mathbf{P}_{(r,l)}(\cdot)$. From (8) we can see that the transition probabilities do not depend on relative ranks, but only on moments r where items with relative rank $l \le \min(a,r)$ appear. Based on the following lemma we can solve problem (1) with gain function (2) using the embedded Markov chain $\{W_t\}$ and the gain function given by (3).

Lemma 2.1. *(see* [15]*)*

$$\mathbf{E}w_N(s+1,Y_{s+1}) = \mathbf{E}_{(s,l)}w_N(W_1) \text{ for every } l \le \min(a,r). \tag{9}$$

2.4. **Distribution of stopped Markov process.** Let $\Gamma_r = \{(s,l) : s > r, g_A(s,l) \ge \mathbf{E}_{(s,l)}w_N(W_1)\}$ and $\sigma_r = \inf\{t : W_t \in \Gamma_r\}$. For $\sigma_r \in \tilde{\mathfrak{M}}_{r+1}^N$ we have $\tau_{r+1}^* = \inf\{s > r : (s,Y_s) \in \Gamma_r\}$. The moment τ_{r+1}^* is optimal Markov time in $\mathfrak{M}_{r+1}^N$ from the definition of Γ_r. From (7) and (9) we have $\tilde{v}_N(r+1) = \mathbf{E}_{(r,l)}w_N(W_1)$, and by (6) and the optimality of τ_{r+1}^* in $\mathfrak{M}_{r+1}^N$ we get $\tilde{v}_N(r+1) = \mathbf{E}_{(r,l)}g_A(\tau_{r+1}^*, Y_{\tau_{r+1}^*}) = \mathbf{E}_{(r,l)}g_A(W_{\sigma_r})$. We have $\mathbf{E}_{(r,l)}w_N(W_1) = \mathbf{E}_{(r,l)}g_A(W_{\sigma_r})$. We need the distribution of the random variables W_ν, where $\nu = \inf\{t : W_t \in \Gamma\}$ and Γ is a subset of $\mathbb{E} = \{(r,l) : 1 \le r \le N,\ l = 1,2,\dots,a\}$.

Let $A = \{m_1, m_2, \dots, m_k\}$. Denote $\Gamma_{r,s}(A) = \{(u,l) : r < u \le s,\ l \in A\}$ for $k \le a$, $m_i \le a$, $i = 1,2,\dots,k$, $1 \le r < s \le N$, $\Gamma = \bigcup_{i=1}^N \Gamma_{r_{i-1},r_i}(A_i)$ and $\Gamma^i = \bigcup_{j=i}^N \Gamma_{r_{j-1},r_j}(A_j)$, where A_i is some set of relative ranks. Let $\nu_{r,s}(A) = \inf\{t : W_t \in \Gamma_{r,s}(A)\}$ and $\nu^i = \inf\{t : W_t \in \Gamma^i\}$. For $r' \le r$, $1 \le l' \le a$ we have

$$\begin{aligned} \mathbf{P}_{(r',l')}&\{W_{\nu_{r,s}(A)} = (u,m_i), m_i \in A\} \\ &= \frac{(r)_k}{(u)_{k+1}} \text{ for } r < u \le s,\ i = 1,2,\dots,k,\ k = \mathrm{card}(A), \end{aligned} \tag{10}$$

$$\begin{aligned} \mathbf{P}_{(r',l')}&\{W_{\nu^i} = (u,l)\} \\ &= \begin{cases} \frac{(r_{i-1})_{k_i}}{(r_i)_{k_i}}\mathbf{P}_{(r',l')}\{W_{\nu^{i+1}} = (u,l)\} & \text{for } (u,l) \in \Gamma^{i+1}, \\ \frac{(r_{i-1})_{k_i}}{(u)_{k_i+1}} & \text{for } (u,l) \in \Gamma_{r_{i-1},r_i}(A_i),\ k_i = \mathrm{card}(A_i). \end{cases} \end{aligned} \tag{11}$$

Using the formulae (10) and (11) we can calculate the distribution of W_ν, $\nu = \inf\{t : W_t \in \Gamma\}$ recursively.

2.5. **Construction of the optimal stopping set.** We derive the recursive algorithm for determining the optimal strategy and the value of the problem for the optimal choice of an item with absolute rank in A. This is justified by backward induction.

(i) For each l we assume $(N,l) \in \Gamma$. Let $S_0 = \{l : g_A(N-1,l) \ge \mathbf{E}_{(N-1,l)}g_A(W_1)\}$. The set $\{(N-1,l) : l \in S_0\} \subset \Gamma$.

(ii) Let $(s,l) \in \Gamma$ for $l \in S_0$ and $s > r$. Denote $\Gamma_r = \{(s,l) : l \in S_0, s > r\}$ and $\sigma_r = \inf\{t : W_t \in \Gamma_r\}$. It follows that

$$g_A(s,l) \ge \mathbf{E}_{(s,l)}g_A(W_{\sigma_r}) \text{ for } s > r,\ l \in S_0 \tag{12}$$

and

$$g_A(s,l) < \mathbf{E}_{(s,l)}g_A(W_{\sigma_r}) \text{ for } s > r,\ l \notin S_0. \tag{13}$$

(iii) Let r_1 be the greatest r such that condition (12) or (13) is not valid.

(a) Let (12) be invalid at $r = r_1$ and $l \in S' = \{m_1', \dots, m_k'\}$. The subset of the stopping set Γ, for the induction assumption, will be $\Gamma_{r_1-1} = \Gamma_{r_1-1,r_1}(S_1) \cup \Gamma_{r_1,N}(S_0)$, where $S_1 = S_0 \setminus S'$.

(b) Let (13) be invalid at $r = r_1$ and $l \in S' = \{m_1', \dots, m_k'\}$. The subset of the stopping set Γ, for the induction assumption, will be $\Gamma_{r_1-1} = \Gamma_{r_1-1,r_1}(S_1) \cup \Gamma_{r_1,N}(S_0)$, where $S_1 = S_0 \cup S'$.

(c) If both conditions are broken at r_1, (12) for $l \in S_1^{'} = \{m_1^{'}, \ldots, m_{k'}^{'}\}$ and (13) for $l \in S_2^{'} = \{m_1^{''}, \ldots, m_{k''}^{''}\}$, then the subset of the stopping set Γ, for the induction assumption, will be $\Gamma_{r_1-1} = \Gamma_{r_1-1,r_1}(S_1) \cup \Gamma_{r_1,N}(S_0)$, where $S_1 = (S_0 \setminus S_1^{'}) \cup S_2^{'}$.

2.6. **Asymptotic solution.** Let the number of candidates go to infinity. In this case we can find the optimal solution from the following argument. As $N \to \infty$ such that $\frac{r}{N} \to x \in (0,1]$, the embedded Markov chain $(W_t, \mathcal{F}_t, \mathbf{P}_{(1,1)})$ with state space $\mathbb{E} = \{1,2,\ldots,N\} \times \{1,2,\ldots,\max(A)\}$ can be treated as a Markov chain $(W_t^{'}, \mathcal{F}_t, \mathbf{P}_{(\frac{1}{N},1)})$ on $\{\frac{1}{N}, \frac{2}{N}, \ldots, 1\} \times \{1,2,\ldots,\max(A)\}$. The gain function $g_A([Nx], l)$ has limit

$$g_A(x,l) = \sum_{a \in A} \binom{a-1}{l-1} x^l (1-x)^{a-l},\ l = 1,2,\ldots,\max(A).$$

We get $\lim_{N\to\infty} \mathbf{E}_{(\frac{r}{N},l)} g_A(W_1) = \mathbf{E}_{(x,l)} g_A(W_1^{''})$, where $(W_t^{''}, \mathcal{F}_t, \mathbf{P}_{(x,1)})$ is a Markov chain with state space $(0,1] \times \{1,2,\ldots,\max(A)\}$ and transition density function

$$p(x,y) = \begin{cases} \frac{x^a}{y^{a+1}}, & 0 < x < y \le 1, \\ 0, & x \ge y. \end{cases} \tag{14}$$

The expected value with respect to the conditional distribution given in (14) is as follows

$$\mathbf{E}_{(x,l')} g_A(W_1^{''}) = \sum_{l=1}^{\max(A)} \int_x^1 p(x,y) g_A(y,l) dy. \tag{15}$$

In this asymptotic case the recursive formulae (4)–(7) are of the form

$$v(1) = 0 \tag{16}$$

$$w(x,l) = \max\{g_A(x,l), \mathbf{E}_{(x,l)} w(W_1^{''})\}, \tag{17}$$

$$v(x) = \mathbf{E}_{(x,l)} w(W_1^{''}). \tag{18}$$

Where $w(x,l)$ is the limit of $w_N(r,l)$, when $\frac{r}{N} \to x \in (0,1]$, *i.e.* $\lim_{N\to\infty} w_N([Nx], l) = w(x,l)$. The asymptotic solution is obtained by a recursive method based on (16)–(18). The distribution of stopped Markov process, given by (10)–(11) in the finite case, in the asymptotic case is of the form

$$f_{(x',l')}((y,m_i)) = \frac{x^k}{y^{k+1}},\ x^{'} \le x < y \le 1 \tag{19}$$

$$f_{(x',l')}((y,m_i)) = \begin{cases} \frac{x_{i-1}^{k_i}}{x_i^{k_i}} f_{(x',l')}((y,l)), & (y,l) \in \Gamma^{'}(i+1) \\ \frac{x_{i-1}^{k_i}}{y^{k_i+1}}, & (y,l) \in \Gamma_{x_{i-1},x_i}(k_i), \end{cases} \tag{20}$$

where $\Gamma_{x,z}(k) = \{(y,l) : x < y \le z, l = m_1, m_2, \ldots, m_k\}$, $\Gamma^{'}(i+1) = \bigcup_{j=i}^{n} \Gamma_{x_{j-1},x_j}(k_j)$.

The algorithm for constructing the optimal stopping set (analogous to the one introduced in Section 2.5) will be presented using examples in Section 3.1.

3. The optimal strategy for choosing items with one of two ranks ≤ 3

We construct the optimal strategy for choosing an item with absolute rank belonging to a two element subset of ranks less than or equal to 3.

3.1. **Optimal stopping at the best or the third absolute ranked item.** Based on the results from Section 2, we construct the optimal strategy for choosing an item with absolute rank 1 or 3. For finite horizon N we can give the numerical solution. The results of the calculation for various N are given in Table 1.

Let the goal of the decision maker be to choose the first or the third ranked applicant. Taking into account the arguments in Section 2, we solve the optimal stopping problem for a Markov chain with transition probability function given by (8), with $a = \max(A)$, where $A = \{1, 3\}$, and the gain function

$$g_A(r,l) = \begin{cases} g_1(r,1) + g_3(r,1), & l = 1, \\ g_3(r,2), & l = 2, \\ g_3(r,3), & l = 3. \end{cases}$$

Based on the algorithm given in Section 2.5, we get the results given in Table 1. For $N = 10$ we show how to get the strategy from Table 1. The stopping set for this horizon is

$$\Gamma = \{(r,l) : 4 \le r \le 10, l = 1\} \cup \{(r,l) : 9 \le r \le 10, l = 2\} \cup \{(r,l) : 8 \le r \le 10, l = 3\},$$

and the maximal probability of the realization of the goal is $v \cong 0.5379$.

TABLE 1. Optimal choice of an item from $A = \{1, 3\}$.

N	Strategies - relative ranks						Probability
	1		2		3		
4	2	4	3	4	4	4	0.6250
5	2	5	5	5	5	5	0.5833
6	3	6	5	6	5	6	0.5722
7	3	7	6	7	6	7	0.5619
8	3	8	7	8	7	8	0.5464
9	4	9	8	9	7	9	0.5421
10	4	10	9	10	8	10	0.5379
20	8	20	19	20	15	20	0.5107
100	35	100	99	100	72	100	0.4917
200	69	200	199	200	143	200	0.4894
∞	[0.339N]	N	N	N	[0.710N]	N	0.4870

For the asymptotic solution we use the gain function

$$g_A(x,l) = \begin{cases} x + x(1-x)^2, & l = 1, \\ 2x^2(1-x), & l = 2, \\ x^3, & l = 3. \end{cases} \tag{21}$$

We get the asymptotically optimal stopping time by constructing the asymptotic stopping set. Let us assume that $\Gamma_x(1) = \Gamma_{x,1}(\{1,2,3\}) = \{(x,l) : x \in (1-\epsilon, 1], l = 1,2,3\} \subset \Gamma$ for small enough $\epsilon > 0$, where Γ is the asymptotically optimal stopping set. From (21), (14) and (15) we get

$$w_0(x) = \sum_{l=1}^{3} \int_x^1 p(x,y) g_A(y,l) dy = x(1-x^2).$$

Solving inequality $w_0(x) - g_A(x,l) \le 0$ for $l = 1,2,3$ and $x \in (1-\epsilon, 1]$ we get that there are no $\epsilon > 0$ for which this inequality holds when $l = 2$. Thus our assumption is false. Suppose now, that we change the definition of the stopping set in the neighbourhood of 1 and set

$\Gamma_x(1) = \Gamma_{x,1}(\{1,3\}) = \{(x,l) : x \in (1-\epsilon, 1], l = 1,3\}$ and $\nu_1 = \inf\{t : W_t'' \in \Gamma^2_{x,1}, x \in (1-\epsilon, 1]\}$. We have

$$w_1(x) = \mathbf{E}_{(x,l)} g_A(W''_{\nu_1}) = \int_x^1 \frac{x^2}{y^3}(x + x(1-x)^2 + x^3)dy = 2x(1 + x\log(x) - x^2).$$

For $x \in (1-\epsilon, 1]$ and $\epsilon > 0$ small enough, the inequality

$$w_1(x) - g_A(x,l) \leq 0 \tag{22}$$

holds for $l = 1,3$ and $w_0(x) \leq w_1(x)$. The nearest point on the left hand side of 1, at which (22) does not hold is $\alpha \cong 0.7105$. This point satisfies the equation $w_1(x) = g_A(x,3)$. Define $v(x) = w_1(x)$ for $x \in (\alpha, 1]$.

Let $\Gamma_x(2) = \Gamma_{x,\alpha}(\{1\}) \cup \Gamma_{\alpha,1}(\{1,3\})$ and $\nu_2 = \inf\{t : W_t'' \in \Gamma_x(2)\}$. We have from (19) and (20)

$$\begin{aligned} w_2(x) &= \mathbf{E}_{(x,l)} g_A(W''_{\nu_2}) = \int_x^\alpha \frac{x}{y^2}(x + x(1-x)^2)dy + \frac{x}{\alpha}w_1(\alpha) \\ &= x(2\log(\frac{\alpha}{x}) - 2(\alpha - x) + \frac{1}{2}(\alpha^2 - x^2)) + \frac{x}{\alpha}w_1(\alpha). \end{aligned}$$

The recursive procedure gives $w_2(x) \leq g_A(x,1)$ and the next change in the stopping set is at the point $\beta \cong 0.3389$ which is the solution of the equation $w_2(x) = g_A(x,1)$ in $(0,\alpha]$. Define $v(x) = w_2(x)$ for $x \in (\beta, \alpha]$ and $v(x) = w_2(\beta)$ for $x \in (0, \beta]$.

We have derived an optimal stopping set of the form

$$\Gamma = \Gamma_{\beta,\alpha}(\{1\}) \cup \Gamma_{\alpha,1}(\{1,3\})$$

and

$$v(x) = w_2(\beta)\mathbb{I}_{\{x \leq \beta\}} + w_2(x)\mathbb{I}_{\{\beta < x \leq \alpha\}} + w_1(x)\mathbb{I}_{\{\alpha < x \leq 1\}}.$$

The value of the problem is thus $v = v(\beta) \cong 0.4870$.
The last row of Table 1 contains the form of the asymptotically optimal strategy when there are a large number of candidates N.

This method of determining the asymptotically optimal stopping set is a consequence of dynamic programming.

3.2. **Optimal stopping at the second or the third best.** Analogous to the solution presented in detail in Section 3.1, we present the solution of the problem when $A = \{2,3\}$, with $a = \max(A)$. We have the gain function

$$g_A(r,l) = \begin{cases} g_2(r,1) + g_3(r,1), & l = 1, \\ g_2(r,2) + g_3(r,2), & l = 2, \\ g_3(r,3), & l = 3. \end{cases}$$

For the asymptotic solution we use the gain function

$$g_A(x,l) = \begin{cases} x(1-x) + x(1-x)^2, & l = 1, \\ x^2 + 2x^2(1-x), & l = 2, \\ x^3, & l = 3. \end{cases} \tag{23}$$

By arguing as in Section 3.1, we get the form of the value function and the optimal stopping set. We have

$$v(x) = w_2(\beta)\mathbb{I}_{\{x \leq \beta\}} + w_2(x)\mathbb{I}_{\{\beta < x \leq \alpha\}} + w_1(x)\mathbb{I}_{\{\alpha < x \leq 1\}},$$

TABLE 2. Optimal choice of an item from $A = \{2,3\}$.

N	Strategies - relative ranks						Probability
	1		2		3		
4	4	4	3	4	4	4	0.6667
5	5	5	3	5	5	5	0.6000
6	6	6	4	6	5	6	0.5750
7	7	7	4	7	6	7	0.5571
8	8	8	5	8	7	8	0.5357
9	9	9	5	9	7	9	0.5278
10	10	10	6	10	8	10	0.5179
20	20	20	10	20	15	20	0.4830
100	100	100	48	100	73	100	0.4575
200	200	200	94	200	144	200	0.4544
∞	N	N	[0.468N]	N	[0.716N]	N	0.4514

where

$$\begin{aligned} w_1(x) &= x^2(x - 3\log(x) - 1) \\ w_2(x) &= x^3 - 3x^2 + 3xe^{-\frac{1}{3}} + \frac{x}{\alpha} w_1(\alpha), \end{aligned}$$

and the constants α and β are determined as in Section 3.1. The constant $\alpha = e^{-\frac{1}{3}} \cong 0.7165$ is the nearest to 1 on the left hand side solution of the equation $w_1(x) - g_A(x,3) = 0$. The constant $\beta = 1 - \sqrt{1 - e^{-\frac{1}{3}}} \cong 0.4676$ is the nearest to α on the left hand side solution of the equation $w_2(x) - g_A(x,2) = 0$.
The optimal stopping set is of the form

$$\Gamma = \Gamma_{\beta,\alpha}(\{2\}) \cup \Gamma_{\alpha,1}(\{2,3\})$$

The value of the problem is $v = v(\beta) \cong 0.4514$.

4. Optimal strategy for choosing items with one of two ranks ≤ 4

We construct the optimal strategy for choosing an item with absolute rank belonging to a two element subset of the ranks less than or equal to 4.

4.1. Optimal stopping at the best or the fourth best. Analogous to the solution presented in detail in Section 3.1, we present the solution of the problem when $A = \{1,4\}$, with $a = \max(A)$. We have the gain function

$$g_A(r,l) = \begin{cases} g_1(r,1) + g_4(r,1), & l = 1, \\ g_4(r,2), & l = 2, \\ g_4(r,3), & l = 3, \\ g_4(r,4), & l = 4. \end{cases}$$

For the asymptotic solution we use the gain function

$$g_A(x,l) = \begin{cases} x + x(1-x)^3, & l = 1, \\ 3x^2(1-x)^2, & l = 2, \\ 3x^3(1-x), & l = 3, \\ x^4, & l = 4. \end{cases} \tag{24}$$

TABLE 3. The solution for $A = \{1, 4\}$.

N	Strategies - relative ranks								Probability
	1		2		3		4		
4	3	4	4	4	3	4	4	4	0.5667
5	3	5	5	5	4	5	5	5	0.5278
6	3	6	6	6	5	6	5	6	0.5278
7	4	7	7	7	5	7	6	7	0.5147
8	4	8	8	8	6	8	7	8	0.5107
9	4	9	9	9	7	9	8	9	0.5024
10	4	10	10	10	8	10	8	10	0.4926
20	8	20	20	20	15	20	16	20	0.4716
100	36	100	100	100	75	100	76	100	0.4531
200	71	200	200	200	151	200	151	200	0.4509
∞	[0.353N]	N	N	N	[0.751N]	N	[0.753N]	N	0.4487

By arguing as in Section 3.1, we get the form of the value function and the optimal stopping set. We have

$$v(x) = w_3(\gamma)\mathbb{I}_{\{x\leq\gamma\}} + w_3(x)\mathbb{I}_{\{\gamma<x\leq\beta\}} + w_2(x)\mathbb{I}_{\{\beta<x\leq\alpha\}} + w_1(x)\mathbb{I}_{\{\alpha<x\leq 1\}},$$

where

$$\begin{aligned}
w_1(x) &= 3x^4 - x^3 - 3x^2 + x - 6x^3\log(x) \\
w_2(x) &= -3x^3 + x^2(3\alpha - 3\log(\alpha) - \frac{2}{\alpha}) + 2x + 3x^2\log(x) + \frac{x^2}{\alpha^2}w_1(\alpha) \\
w_3(x) &= \frac{1}{3}x^4 - \frac{3}{2}x^3 + 3x^2 + x(-\frac{1}{3}\beta^3 + \frac{3}{2}\beta^2 - 3\beta + 2\log(\beta)) - 2x\log(x) + \frac{x}{\beta}w_2(\beta),
\end{aligned}$$

and the constants α and β are determined as in Section 3.1. Then $\alpha \cong 0.7528$ satisfies $w_1(\alpha) - g_A(\alpha, 3) = 0$, $\beta \cong 0.7507$ satisfies $w_2(\beta) - g_A(\beta, 4) = 0$ and $\gamma \cong 0.3531$ satisfies $w_3(\gamma) - g_A(\gamma, 1) = 0$.
The optimal stopping set is of the form

$$\Gamma = \Gamma_{\gamma,\beta}(\{1\}) \cup \Gamma_{\beta,\alpha}(\{1,4\}) \cup \Gamma_{\alpha,1}(\{1,3,4\})$$

The value of the problem is $v = v(\gamma) \cong 0.4487$.

4.2. **Optimal stopping at the second or the fourth best.** Analogous to the solution presented in detail in Section 3.1, we present the solution of the problem when $A = \{2, 4\}$, with $a = \max(A)$. We have the gain function

$$g_A(r,l) = \begin{cases} g_2(r,1) + g_4(r,1), & l = 1, \\ g_2(r,2) + g_4(r,2), & l = 2, \\ g_4(r,3), & l = 3, \\ g_4(r,4), & l = 4. \end{cases}$$

For the asymptotic solution we use the gain function

$$g_A(x,l) = \begin{cases} x(1-x) + x(1-x)^3, & l = 1, \\ x^2 + 3x^2(1-x)^2, & l = 2, \\ 3x^3(1-x), & l = 3, \\ x^4, & l = 4. \end{cases}$$

TABLE 4. The solution for $A = \{2, 4\}$.

N	Strategies - relative ranks								Probability
	1		2		3		4		
4	4	4	2	4	3	4	4	4	0.6250
5	5	5	3	5	4	5	5	5	0.5333
6	6	6	3	6	5	6	5	6	0.4833
7	7	7	4	7	5	7	6	7	0.4750
8	8	8	4	8	6	8	7	8	0.4607
9	9	9	5	9	7	9	8	9	0.4511
10	10	10	5	10	8	10	8	10	0.4405
20	20	20	10	20	15	20	16	20	0.4123
100	100	100	46	100	73	100	75	100	0.3904
200	200	200	91	200	146	200	150	200	0.3878
∞	N	N	[0.449N]	N	[0.727N]	N	[0.744N]	N	0.3853

By arguing as in Section 3.1, we get the form of the value function and the optimal stopping set. We have

$$v(x) = w_3(\gamma)\mathbb{I}_{\{x\leq\gamma\}} + w_3(x)\mathbb{I}_{\{\gamma<x\leq\beta\}} + w_2(x)\mathbb{I}_{\{\beta<x\leq\alpha\}} + w_1(x)\mathbb{I}_{\{\alpha<x\leq 1\}},$$

where

$$\begin{aligned} w_1(x) &= -x^4 - 3x^3 + 4x^2 + 3x^2\log(x) \\ w_2(x) &= 3x^3 + x^2(4\log(\alpha) - 3\alpha) - 4x^2\log(x) + \frac{x^2}{\alpha^2}w_1(\alpha) \\ w_3(x) &= -x^4 + 3x^3 - 4x^2 + x(\beta^3 - 3\beta^2 + 4\beta) + \frac{x}{\beta}w_2(\beta), \end{aligned}$$

and $\alpha \cong 0.7442$ satisfies $w_1(\alpha) - g_A(\alpha, 4) = 0$, $\beta \cong 0.7274$ satisfies $w_2(\beta) - g_A(\beta, 3) = 0$ and $\gamma \cong 0.4491$ satisfies $w_3(\gamma) - g_A(\gamma, 2) = 0$. For details of the method see Sections 3.1 and 3.2.

The optimal stopping set is of the form

$$\Gamma = \Gamma_{\gamma,\beta}(\{2\}) \cup \Gamma_{\beta,\alpha}(\{2,3\}) \cup \Gamma_{\alpha,1}(\{2,3,4\})$$

The value of the problem is $v = v(\gamma) \cong 0.3853$.

4.3. **Optimal stopping at the third or the fourth best.** Analogous to the solution presented in detail in Section 3.1, we present the solution of the problem when $A = \{3, 4\}$, with $a = \max(A)$. We have the gain function

$$g_A(r,l) = \begin{cases} g_3(r,1) + g_4(r,1), & l = 1, \\ g_3(r,2) + g_4(r,2), & l = 2, \\ g_3(r,3) + g_4(r,3), & l = 3, \\ g_4(r,4), & l = 4. \end{cases}$$

For the asymptotic solution we use the gain function

$$g_A(x,l) = \begin{cases} x(1-x)^2 + x(1-x)^3, & l = 1, \\ 2x^2(1-x) + 3x^2(1-x)^2, & l = 2, \\ x^3 + 3x^3(1-x), & l = 3, \\ x^4, & l = 4. \end{cases}$$

TABLE 5. The solution for $A = \{3,4\}$.

N	Strategies - relative ranks								Probability
	1		2		3		4		
4	4	4	2	4	3	4	4	4	0.7500
5	5	5	3	5	4	5	5	5	0.6000
6	6	6	3	6	4	6	5	6	0.5556
7	7	7	4	7	5	7	6	7	0.5286
8	8	8	4	8	5	8	7	8	0.5086
9	9	9	5	9	6	9	8	9	0.4947
10	10	10	5	10	7	10	8	10	0.4804
20	20	20	10	20	12	20	16	20	0.4421
100	100	100	46	100	57	100	76	100	0.4134
200	200	200	91	200	112	200	152	200	0.4101
∞	N	N	[0.450N]	N	[0.556N]	N	[0.753N]	N	0.4069

By arguing as in Section 3.1, we get the form of the value function and the optimal stopping set. We have

$$v(x) = w_3(\gamma)\mathbb{I}_{\{x\le\gamma\}} + w_3(x)\mathbb{I}_{\{\gamma<x\le\beta\}} + w_2(x)\mathbb{I}_{\{\beta<x\le\alpha\}} + w_1(x)\mathbb{I}_{\{\alpha<x\le 1\}},$$

where

$$\begin{aligned}
w_1(x) &= -x^4 - 4x^3 + 5x^2 + 4x^3\log(x)\\
w_2(x) &= 4x^3 + x^2(-4\alpha + 5\log(\alpha)) - 5x^2\log(x) + \frac{x^2}{\alpha^2}w_1(\alpha)\\
w_3(x) &= -x^4 + 4x^3 - 5x^2 + x(\beta^3 - 4\beta^2 + 5\beta) + \frac{x}{\beta}w_2(\beta),
\end{aligned}$$

and $\alpha \cong 0.7529$ satisfies $w_1(\alpha) - g_A(\alpha,4) = 0$, $\beta \cong 0.5557$ satisfies $w_2(\beta) - g_A(\beta,3) = 0$ and $\gamma \cong 0.4505$ satisfies $w_3(\gamma) - g_A(\gamma,2) = 0$. For details of the method see Sections 3.1 and 3.2.
The optimal stopping set is of the form

$$\Gamma = \Gamma_{\gamma,\beta}(\{2\}) \cup \Gamma_{\beta,\alpha}(\{2,3\}) \cup \Gamma_{\alpha,1}(\{2,3,4\})$$

The value of the problem is $v = v(\gamma) \cong 0.4069$.

5. Choosing an item with rank 6 or 7

Analogous to the solution presented in detail in Section 3.1, we present the solution of the problem when $A = \{6,7\}$, with $a = \max(A)$. We have the gain function

$$g_A(r,l) = \begin{cases} g_6(r,1) + g_7(r,1), & l = 1,\\ g_6(r,2) + g_7(r,2), & l = 2,\\ g_6(r,3) + g_7(r,3), & l = 3,\\ g_6(r,4) + g_7(r,4), & l = 4,\\ g_6(r,5) + g_7(r,5), & l = 5,\\ g_6(r,6) + g_7(r,6), & l = 6,\\ g_7(r,1), & l = 7. \end{cases}$$

The results in Table 6 are presented in a slightly different form because of the occurrence of an 'island' strategy. We show for $N = 20$ how to find the optimal strategy. The stopping set for this horizon is

$$\begin{aligned}\Gamma &= \{(r,l) : r = 20, l = 1,2,3\} \cup \{(r,l) : 11 \leq r \leq 15, r = 20, l = 4\}\\ &\cup \{(r,l) : 13 \leq r \leq 20, l = 5\} \cup \{(r,l) : 15 \leq r \leq 20, l = 6\}\\ &\cup \{(r,l) : 17 \leq r \leq 20, l = 7\},\end{aligned}$$

and the maximal probability of the realization of the goal is $v \cong 0.3904$. Because in all cases we stop at ranks ≤ 3 only at the last moment (the only exception is in the case $N = 7$, in which we stop at the relatively third best item at moment $r = 3$), we have omitted these ranks in Table 6 to simplify the notation.

TABLE 6. The solution for $A = \{6, 7\}$.

N	Strategies - relative ranks										Probability
	4		4		5		6		7		
7	4	5	7	7	5	7	6	7	7	7	0.6667
8	5	5	8	8	6	8	7	8	8	8	0.5357
9	5	6	9	9	6	9	7	9	8	9	0.4947
10	6	7	10	10	7	10	8	10	9	10	0.4755
20	11	15	20	20	13	20	15	20	17	20	0.3904
100	54	74	100	100	61	100	71	100	83	100	0.3478
200	107	149	200	200	120	200	140	200	165	200	0.3433
∞	[0.531N]	[0.748N]	N	N	[0.596N]	N	[0.695N]	N	[0.821N]	N	0.3389

For the asymptotic solution we use the gain function

$$g_A(x,l) = \begin{cases} x(1-x)^5 + x(1-x)^6, & l = 1,\\ 5x^2(1-x)^4 + 6x^2(1-x)^5, & l = 2,\\ 10x^3(1-x)^3 + 15x^3(1-x)^4, & l = 3,\\ 10x^4(1-x)^2 + 20x^4(1-x)^3, & l = 4,\\ 5x^5(1-x) + 15x^5(1-x)^2, & l = 5,\\ x^6 + 6x^6(1-x), & l = 6,\\ x^7, & l = 7.\end{cases}$$

By arguing as in Section 3.1, we get the form of the value function and the optimal stopping set. We have

$$v(x) = w_5(\epsilon)\mathbb{I}_{\{x\leq\epsilon\}} + w_5(x)\mathbb{I}_{\{\epsilon<x\leq\delta\}} + w_4(x)\mathbb{I}_{\{\delta<x\leq\gamma\}} + w_3(x)\mathbb{I}_{\{\gamma<x\leq\beta\}} + w_2(x)\mathbb{I}_{\{\beta<x\leq\alpha\}} + w_1(x)\mathbb{I}_{\{\alpha<x\leq 1\}},$$

where

$$\begin{aligned} w_1(x) &= \int_x^1 \frac{x^3}{y^4}(g_A(y,5) + g_A(y,6) + g_A(y,7))dy\\ w_2(x) &= \int_x^\alpha \frac{x^2}{y^3}(g_A(y,5) + g_A(y,6))dy + \frac{x^2}{\alpha^2}w_1(\alpha)\\ w_3(x) &= \int_x^\beta \frac{x^3}{y^4}(g_A(y,4) + g_A(y,5) + g_A(y,6))dy + \frac{x^3}{\beta^3}w_2(\beta)\\ w_4(x) &= \int_x^\gamma \frac{x^2}{y^3}(g_A(y,4) + g_A(y,5))dy + \frac{x^2}{\gamma^2}w_3(\gamma)\\ w_5(x) &= \int_x^\delta \frac{x}{y^2}g_A(y,4)dy + \frac{x}{\delta}w_4(\delta),\end{aligned}$$

and $\alpha \cong 0.8212$ satisfies $w_1(\alpha) - g_A(\alpha, 7) = 0$, $\beta \cong 0.7483$ satisfies $w_2(\beta) - g_A(\beta, 4) = 0$, $\gamma \cong 0.6950$ satisfies $w_3(\gamma) - g_A(\gamma, 6) = 0$, $\delta \cong 0.5963$ satisfies $w_4(\delta) - g_A(\delta, 5) = 0$ and $\epsilon \cong 0.5310$ satisfies $w_5(\epsilon) - g_A(\epsilon, 4) = 0$. For the details of the method see Sections 3.1 and 3.2.
The optimal stopping set is of the form

$$\Gamma = \Gamma_{\epsilon,\delta}(\{4\}) \cup \Gamma_{\delta,\gamma}(\{4,5\}) \cup \Gamma_{\gamma,\beta}(\{4,5,6\}) \cup \Gamma_{\beta,\alpha}(\{5,6\}) \cup \Gamma_{\alpha,1}(\{5,6,7\})$$

The value of the problem is $v = v(\epsilon) \cong 0.3389$.

6. Conclusion.

We have investigated optimal strategies for problems with non-monotone gain functions. The recursive algorithm introduced for determining the optimal strategy (equivalent to defining the optimal stopping set), is based on the distribution of the stopped process (see Section 2.4).

We have given examples of a wide variety of optimal stopping sets. We have presented the complete analysis for some of the simplest sets A (see Sections 3 and 4). We have got interesting results. For example in the problem with $A = \{1, 3\}$ we never stop at the second relative rank. Similarly, if we consider the problem with $A = \{2, 3\}$, there is no stop at the relative first rank. In the problem with $A = \{1, 4\}$ we can observe that stopping at the second rank is not optimal. Moreover, the optimal thresholds for stopping at the third and at the fourth relative ranks are almost equal.

Let us compare the values of the problems. Unexpectedly, it is easier to choose a candidate with absolute rank from $A = \{1, 3\}$ than from $A = \{2, 3\}$. Similar situation occurs in the cases $A = \{1, 4\}$ and $A = \{3, 4\}$. Moreover, the probability of winning according to the optimal strategy does not decrease with $1 \leq a \leq 3$ for sets $A = \{a, 4\}$. The strategy can be given in closed form only in case $A = \{2, 3\}$.

In the problem with $A = \{5\}$ we have an island asymptotic optimal strategy (for details see [15]). If we add one more absolute rank < 5, the optimal strategy is threshold. We have tried to get an answer to the question 'what sets A generate island optimal strategies?'.

Island strategies can arise in two simple cases: if we try to choose from small group of big absolute ranks or if the ranks in A are very different. The problems from the second class (remote groups of ranks) were inspired by result from [15]. We had to study, how distant must the ranks be, to get an island strategy. Thus, we investigated sets of the form $\{1, a\}$. The first class consists of sets of the type $\{a, a+1\}$ (big ranks, but close each other). We provided an numerical analysis of the optimal strategies for the sets A belonging to each of the classes with $a = \{2, 3, \dots\}$ and with a large number ($N = 300$) of candidates. If we found the island strategy, we proved the related asymptotic result. Island strategies appear in the cases $A = \{6, 7\}$ and $A = \{1, 10\}$. To get these results in the manner introduced is arduous, but using algorithm presented we can obtain the results numerically for any number of applicants. In Section 5 we presented results for $A = \{6, 7\}$, but the result for $A = \{1, 10\}$ can be obtained in such a way too.

Let us consider the asymptotic results when $A = \{6, 7\}$. We can see, that if we didn't encounter a relatively fourth best item between the moments $[0.531N]$ and $[0.748N]$, stopping later at the relatively fourth best is no longer optimal.

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