A WEIGHTED VERSION OF OZEKI'S INEQUALITY

Saichi Izumino* and Josip E. Pečarić**

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Abstract. As an extension of Ozeki's inequality we give an inequality which estimates the difference

$$\sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 - (\sum_{k=1}^{n} p_k a_k b_k)^2$$

derived from the weighted Cauchy-Schwartz inequality for n-tuples $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and $p = (p_1, ..., p_n)$ of positive numbers under certain conditions. We discuss the upper bound of the difference not only in the general case but also in the special cases that a and b are monotonic in the opposite sense and in the same sense.

1 Introduction As a complement of Cauchy-Schwartz inequality, the following inequality was given in [4] (cf. [7, p. 121]) which was originally presented by Ozeki [8]: If $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ are n-tuples of positive numbers satisfying

$$m_1 \le a_k \le M_1, \quad m_2 \le b_k \le M_2 \quad (k = 1, 2, \dots, n), \qquad 0 < m_1 < M_1 \quad \text{and} \quad 0 < m_2 < M_2, \tag{1}$$

then

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - (\sum_{k=1}^{n} a_k b_k)^2 \le \frac{n^2}{3}(M_1 M_2 - m_1 m_2)^2. \tag{2}$$

Put $T(a,b)$ the left-hand side of the above inequality, then $T(a,b)$ is considered as a function on the product $[m_1, M_1]^n \times [m_2, M_2]^n$ of n-dimensional cubes $[m_1, M_1]^n$ and $[m_2, M_2]^n$. Then it is Ozeki's idea to make use of the following two facts in order to prove the inequality (2) (and the technique was also useful for further results in [3], [5]):

(i) $T(a,b)$ is a separately convex function with respect to a and b, so that its maximum is attained at an extreme point, namely, vertex of $2n$-dimensional rectangle $[m_1, M_1]^n \times [m_2, M_2]^n$.

(ii) Denote by $\underline{c} = (\underline{c}_1, \dots, \underline{c}_n)$ and $\overline{c} = (\overline{c}_1, \dots, \overline{c}_n)$ the rearrangements of a nonnegative n-tuple $c = (c_1, \dots, c_n)$ in nonincreasing order and in nondecreasing order, respectively. Then for a and b, $\sum \underline{a}_k \overline{b}_k = \sum \overline{a}_k \underline{b}_k \le \sum a_k b_k$ [2, p. 261], so that

$$T(\underline{a}, \overline{b}) = T(\overline{a}, \underline{b}) \ge T(a,b). \tag{3}$$

As a result, from (3) the inequality (2) was obtained by considering $T(a,b)$ for a and b such that they are monotonic in the *opposite* sense.

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Now let $D(a,b) = n\sum_{k=1}^n a_k b_k - \sum_{k=1}^n a_k \sum_{k=1}^n b_k$, which is n^2 times of the covarience between a and b. As an estimation of $D(a,b)$, Biernacki, Pidek and Ryll-Nardzewski [1] (cf. [7, p. 299]) presented the following result:

$$|D(a,b)| \le \left[\frac{n}{2}\right]\left(n - \left[\frac{n}{2}\right]\right)(M_1 - m_1)(M_2 - m_2) \quad \text{(for } (a,b) \text{ satisfying (1))}.$$

In particular, taking $D(a,b)$ for a and b such that they are monotonic in the *same* sense, (say, $a = \overline{a}$ and $b = \overline{b}$), we obtain an inequality, which is nothing but a complement of the well-known Čebyšev's inequality, a kind of Grüss type inequalities.

It is a problem to estimate $T(a,b)$ with the restriction that a and b are monotonic in the *same* sense, likely to the above consideration and several works [6], [9], [10], etc. related to Grüss' inequality.

Now to consider the problem more generally, define by

$$T(a,b;p) = \sum_{k=1}^n p_k a_k^2 \sum_{k=1}^n p_k b_k^2 - (\sum_{k=1}^n p_k a_k b_k)^2 \tag{4}$$

the difference derived from the weighted Cauchy-Schwartz inequality with a positive n-weight (n-tuple) $p = (p_1, ..., p_n)$, $\sum_{k=1}^n p_k = 1$. Then unlike $T(a,b)$ the equality-inequality $T(\underline{a}, \overline{b}; p) = T(\overline{a}, \underline{b}; p) \ge T(a,b;p)$ corresponding to (3) are false in general. (For example, if $a = (1,1,1)$, $b = (2,1,2)$ and $p = (\frac{3}{15}, \frac{7}{15}, \frac{5}{15})$ then $T(\underline{a}, \overline{b}; p) = \frac{36}{15}$, $T(\overline{a}, \underline{b}; p) = \frac{50}{15}$ and $T(a,b;p) = \frac{56}{15}$.) This means that rearrangements of a and b to be monotonic in the opposite sense are not effective to obtain the maximum of $T_p(a,b) = T(a,b;p)$. However, the calculation of the maximum for such a and b yields, in a sense, an extension of (2).

In this paper, using Ozeki's technique on convex functions, we give upper bounds of (4) not only in the general case for a and b, but also in the special cases that a and b are monotonic in the opposite sense and in the same sense.

2 Preliminaries

We prepare some useful facts for our discussion. Let $I_n = \{1, ..., n\}$ and define an index set Δ in $I_n^2 = I_n \times I_n$ by

$$\Delta = \left\{(i,j) \in I_n^2; i < j\right\}. \tag{5}$$

Now we state a weighted version of Lagrange's formula (cf. [7, p. 84]), which we can prove easily.

Lemma 2.1

$$T(a,b;p) = \sum_{(i,j)\in\Delta} p_i p_j (a_i b_j - a_j b_i)^2. \tag{6}$$

¿From this lemma we can see the following:

Lemma 2.2 *$T_p(a,b) = T(a,b;p)$ is a separately convex function on $[m_1, M_1]^n \times [m_2, M_2]^n$ with respect to a and b, that is,*

$$T_p(\lambda a + (1-\lambda)a', b) \le \lambda T_p(a,b) + (1-\lambda)T_p(a',b), \quad \lambda \in [0,1]$$

and

$$T_p(a, \mu b + (1-\mu)b') \le \mu T_p(a,b) + (1-\mu)T_p(a,b'), \quad \mu \in [0,1].$$

Consequently, we see that $T_p(a,b)$ attains its maximum at a point (a,b) of $[m_1,M_1]^n \times [m_2,M_2]^n$, with both a and b being vertices of $[m_1,M_1]^n$ and $[m_2,M_2]^n$, respectively. (Note that a point $v=(v_1,...,v_n)\in[m,M]^n$ is a vertex if (and only if) each v_k is equal to m or M.)

For two real numbers $m,\ M,\ \ m<M$, let

$$K=\{(x_1,\dots,x_n)\in[m,M]^n; x_1\le\cdots\le x_n\}$$

and

$$L=\{(x_1,\dots,x_n)\in[m,M]^n; x_1\ge\cdots\ge x_n\}.$$

Then K and L are convex subsets in $[m,M]^n$. The following fact related to their extreme points is easily seen, say, by the induction method.

Lemma 2.3 *Every extreme point of K (L) is a vertex of $[m,M]^n$.*

Now assume that $A,B,C>0$, and put

$$\begin{aligned}&\tilde{A}=B+C-A,\quad \tilde{B}=C+A-B,\quad \tilde{C}=A+B-C\quad\text{and}\\ &D=A\tilde{A}+B\tilde{B}+C\tilde{C}\ (=2AB+2BC+2CA-A^2-B^2-C^2).\end{aligned}\tag{7}$$

Then it is not difficult to see that

(i) at least two of $\tilde{A},\tilde{B}$ and $\tilde{C}$ are positive, and
(ii) if all of $\tilde{A},\tilde{B}$ and $\tilde{C}$ are positive then $D>0$.

The following general fact (cf. [4]) is very useful for our discussion.

Lemma 2.4 *With the same notations as above, consider the function*

$$u=f(x,y,z)=Axy+Bxz+Cyz\tag{8}$$

under the condition

$$x,y,z\ge0,\quad x+y+z=k>0\quad (k\ \textit{is a constant}).\tag{9}$$

(i) *If $\tilde{A},\tilde{B},\tilde{C}>0$, then $D>0$ and*

$$u=-C\left\{\left(y-\frac{B\tilde{B}}{D}k\right)+\frac{\tilde{A}}{2C}\left(x-\frac{C\tilde{C}}{D}k\right)\right\}^2-\frac{D}{4C}\left(x-\frac{C\tilde{C}}{D}k\right)^2+\frac{ABC}{D}k^2,\tag{10}$$

so that

$$u\le u_{max}(=\textit{the maximum of } u)=\frac{ABC}{D}k^2,$$

and u_{max} is attained at a point

$$(x,y,z)=\left(\frac{C\tilde{C}}{D}k,\frac{B\tilde{B}}{D}k,\frac{A\tilde{A}}{D}k\right).$$

(ii) *If one of $\tilde{A},\tilde{B},\tilde{C}$ is nonpositive, say, $\tilde{B}\le0$, (hence $\tilde{A},\tilde{C}>0$), then*

$$u=-\tilde{B}xz+Ax(k-x)+Cz(k-z)\tag{11}$$

and

$$u \le u_{max} = \frac{B}{4}k^2.$$

The value u_{max} is attained at

$$(x, y, z) = (k/2, 0, k/2).$$

Proof. (i) Putting $z = k - x - y$, we have, from (8),

$$u = -Cy^2 - \left(\tilde{A}x - Ck\right) y - Bx^2 + Bkx.$$

Taking the $4C$ times of the both sides, we have

$$\begin{aligned} 4Cu &= -4C^2y^2 - 4C\left(\tilde{A}x - Ck\right) y - 4BCx^2 + 4BCkx \\ &= -\left(2Cy + \tilde{A}x - Ck\right)^2 - D\left(x - \frac{C\tilde{C}}{D}k\right)^2 + \frac{4ABC^2}{D}k^2. \end{aligned}$$

Hence we have

$$\begin{aligned} u &= -C\left(y + \frac{\tilde{A}x - Ck}{2C}\right)^2 - \frac{D}{4C}\left(x - \frac{C\tilde{C}}{D}k\right)^2 + \frac{ABC}{D}k^2 \\ &= -C\left\{\left(y - \frac{B\tilde{B}}{D}k\right) + \frac{\tilde{A}}{2C}\left(x - \frac{C\tilde{C}}{D}k\right)\right\}^2 - \frac{D}{4C}\left(x - \frac{C\tilde{C}}{D}k\right)^2 + \frac{ABC}{D}k^2. \end{aligned}$$

Now, if $x = \frac{C\tilde{C}}{D}\,k,\ y = \frac{B\tilde{B}}{D}\,k$, (so that $z = k - x - y = \frac{A\tilde{A}}{D}\,k$), then $u = u_{max} = \frac{ABC}{D}\,k^2$.

(ii) Putting $y = k - x - z$, we have, from (8),

$$u = -\tilde{B}xz + Ax(k - x) + Cz(k - z).$$

Since $xz \le \left(\frac{x+z}{2}\right)^2 \le \frac{k^2}{4}$, $x(k - x) \le \frac{k^2}{4}$ and $z(k - z) \le \frac{k^2}{4}$, we have

$$u \le -\tilde{B}\cdot\frac{1}{4}k^2 + A\cdot\frac{1}{4}k^2 + C\cdot\frac{1}{4}k^2 = \frac{1}{4}Bk^2.$$

Hence $u_{max} = \frac{1}{4}Bk^2$, which is attained at $(x, y, z) = (k/2, 0, k/2)$. □

3 Weighted Ozeki's inequality

In this section we give an upper bound of $T(a, b; p)$ without any assumption of monotony on positive n-tuples a and b. Let us define, for a positive n-weight $p = (p_1, \dots, p_n)$ with $\sum_{k=1}^{n} p_k = 1$,

$$P(X) = \sum_{k\in X} p_k \quad \text{for } X \subset I_n.$$

say, as in [11]. Then we have:

Lemma 3.1 *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be n-tuples such that $a_k = 1$ or α and $b_k = 1$ or β $(k = 1, \dots, n)$, and let $p = (p_1, \dots, p_n)$ be a positive n-weight with $\sum_{k=1}^n p_k = 1$. Put*

$$J_a = \{k \in I_n; a_k = 1\} \quad \text{and} \quad J_b = \{k \in I_n; b_k = 1\}.$$

Then

$$\begin{aligned} &T(a, b; p) = P(J_a \cap J_b)P(J_a \cap J_b^c)(1-\beta)^2 + P(J_a \cap J_b)P(J_a^c \cap J_b)(1-\alpha)^2 \\ &+ P(J_a \cap J_b)P(J_a^c \cap J_b^c)(\alpha - \beta)^2 + P(J_a \cap J_b^c)P(J_a^c \cap J_b)(1-\alpha\beta)^2 \\ &+ P(J_a \cap J_b^c)P(J_a^c \cap J_b^c)\beta^2(1-\alpha)^2 + P(J_a^c \cap J_b)P(J_a^c \cap J_b^c)\alpha^2(1-\beta)^2. \end{aligned} \tag{12}$$

Proof. First note that I_n is devided into the four subsets

$$J_1 = J_a \cap J_b, \ J_2 = J_a \cap J_b^c, \ J_3 = J_a^c \cap J_b \text{ and } J_4 = J_a^c \cap J_b^c,$$

so that $\Delta = \{(i, j) \in I_n^2; i < j\}$ is devided into the ten subsets

$$\Delta_{k,l} = J_k \times J_l, \ 1 \le k \le l \le 4.$$

Let $\sum_{\Delta_{k,l}} = \sum_{(i,j)\in\Delta_{k,l}} p_i p_j (a_i b_j - a_j b_i)^2$. Then we see that $T(a, b; p)$ is the totality of sums $\sum_{\Delta_{k,l}}, 1 \le k \le l \le 4$ by Lemma 2.1. We can easily see that $\sum_{\Delta_{k,k}} = 0$. It is also easy to compute $\sum_{\Delta_{k,l}}$, for $k < l$: say, for $k = 1, l = 2$ we have

$$\sum_{\Delta_{1,2}} = \sum_{(i,j)\in J_1 \times J_2} p_i p_j (a_i b_j - a_j b_i)^2 = P(J_1)P(J_2)(1-\beta)^2.$$

Consequently, we have

$$\begin{aligned} T(a, b; p) &= \sum_{\Delta_{1,2}} + \sum_{\Delta_{1,3}} + \sum_{\Delta_{1,4}} + \sum_{\Delta_{2,3}} + \sum_{\Delta_{2,4}} + \sum_{\Delta_{3,4}} \\ &= P(J_1)P(J_2)(1-\beta)^2 + P(J_1)P(J_3)(1-\alpha)^2 + P(J_1)P(J_4)(\alpha-\beta)^2 \\ &\quad + P(J_2)P(J_3)(1-\alpha\beta)^2 + P(J_2)P(J_4)\beta^2(1-\alpha)^2 + P(J_3)P(J_4)\alpha^2(1-\beta)^2. \end{aligned}$$

□

Now we have the following extension of Ozeki's inequality (cf. [4, Theorem 2.1]).

Theorem 3.2 *Let a and b be positive n-tuples satisfying* (1) *and let p be a positive n-weight with $\sum_{k=1}^n p_k = 1$. Assume that $\alpha = m_1/M_1 \ge m_2/M_2 = \beta$. Then*

$$\begin{aligned} &T(a, b; p) \\ &\le M_1^2 M_2^2 \max_{X \subset I_n} \left\{ \frac{(1-\alpha\beta)^2}{4}(1 - P(X))^2 + (1-\beta)^2 P(X)(1 - P(X)) \right\}. \end{aligned} \tag{13}$$

Proof. We may assume that $M_1 = M_2 = 1$ (and then write $\alpha = m_1, \ \beta = m_2$) for convenience. In order to obtain the maximum or the best upper bound of $T_p(a, b) = T(a, b; p)$, we have to calculate, by convexity of $T(a, b; p)$, its value for a and b such that $a_i = 1$ or α, $b_i = 1$ or β $(i = 1, ..., n)$. Hence we may apply the preceding lemma. Put

$$A = \beta^2(1-\alpha)^2, \ B = (1-\alpha\beta)^2, \ C = \alpha^2(1-\beta)^2,$$

$$E = (1-\beta)^2,\ F = (\alpha-\beta)^2,\ G = (1-\alpha)^2,$$

and furthermore put

$$x = P(J_a \cap J_b^c),\ y = P(J_a^c \cap J_b^c),\ z = P(J_a^c \cap J_b) \quad \text{and} \quad w = P(J_a \cap J_b).$$

Then we have

$$x + y + z + w = 1 \quad (x, y, z, w \geq 0)$$

and from (12)

$$u := T(a, b; p) = Axy + Bxz + Cyz + Exw + Fyw + Gzw.$$

First note that for positive numbers A, B, C we have

$$\begin{aligned}\tilde{B} = C + A - B &= \alpha^2(1-\beta)^2 + \beta^2(1-\alpha)^2 - (1-\alpha\beta)^2 \\ &= -(1-\alpha)(1-\beta)(1+\alpha+\beta-\alpha\beta) < 0,\end{aligned}$$

because $0 < \alpha < 1$ and $0 < \beta < 1$. Hence since $x + y + z = 1 - w$, we have, by Lemma 2.4 (ii),

$$Axy + Bxz + Cyz \leq \frac{B}{4}(1-w)^2.$$

Next from the assumption $\alpha \geq \beta$, we see $E \geq F, G$, so that

$$Exw + Fyw + Gzw \leq Ew(x + y + z) = Ew(1-w).$$

Hence we have

$$T(a, b; p) \leq \frac{B}{4}(1-w)^2 + Ew(1-w), \tag{14}$$

from which we obtain the desired inequality (13). □

Now we obtain the following result [4, Theorem 4.1] from the preceding theorem.

Theorem 3.3 *With the same notations and the same assumptions as in Theorem* 3.2,

$$T(a, b; p) \leq \frac{1}{3}M_1^2 M_2^2 (1-\alpha\beta)^2 = \frac{1}{3}(M_1 M_2 - m_1 m_2)^2.$$

Proof. As before we may assume $M_1 = M_2 = 1$. Write $g(w)$ the right-hand side of (14). Then it suffices to show that

$$g(w) \leq \frac{1}{3}B \quad (0 \leq w \leq 1).$$

Since $E \leq B \leq 4E$ and

$$g(w) = -\frac{4E - B}{4}w^2 + \frac{2E - B}{2}w + \frac{B}{4},$$

we have, by an elementary computation,

$$\max_{0 \leq w \leq 1} g(w) = \begin{cases} \frac{E^2}{4E-B} & \text{if } (E \leq) B \leq 2E, \\ \frac{B}{4} & \text{if } 2E \leq B \leq 4E. \end{cases}$$

Furthermore, it is not difficult to see that

$$\frac{E^2}{4E - B} \leq \frac{1}{3}B \quad (\text{if } E \leq B \leq 2E).$$

Hence we have the desired inequality. □

4 The difference $T(a,b;p)$ for oppositely ordered a and b In this section we give an upper bound of $T_p(a,b)=T(a,b;p)$ for a and b ordered oppositely. We confine ourselves to the case that a is ordered nonincreasingly and b is ordered nondecreasingly. Recall that from Lemmas 2.2 and 2.3 the function $T_p(a,b)$ is separately convex with respect to a and b, and attains its maximum at a point (a,b) such that

$$(15)\qquad a=(\overbrace{M_1,\dots,M_1}^{s},\overbrace{m_1,\dots,m_1}^{n-s}) \quad\text{and}\quad b=(\overbrace{m_2,\dots,m_2}^{t},\overbrace{M_2,\dots,M_2}^{n-t}),\quad s,t\in I_n^*=I_n\cup\{0\}.$$

Now we have

Lemma 4.1 *Let $a^{(s)}$ and $b^{(t)}$ be n-tuples of real numbers such that*

$$(16)\qquad a^{(s)}=(\overbrace{1,\dots,1}^{s},\overbrace{\alpha,\dots,\alpha}^{n-s}) \quad\textit{and}\quad b^{(t)}=(\overbrace{\beta,\dots,\beta}^{t},\overbrace{1,\dots,1}^{n-t}),\quad s,t\in I_n^*=I_n\cup\{0\},$$

and let $p=(p_1,\cdots,p_n)$ be a positive n-weight with $\sum_{i=1}^n p_i=1$. Write $P_k=\sum_{i=1}^k p_i$, for $k\in I_n^*$ $(P_0=0)$. Then

$$(17)\qquad T(a^{(s)},b^{(t)};p)=\begin{cases} P_t(P_s-P_t)(1-\beta)^2+P_t(1-P_s)(1-\alpha\beta)^2 \\ \qquad +(P_s-P_t)(1-P_s)(1-\alpha)^2 \\ \qquad \textit{if } 0\le t\le s\le n, \\ P_s(P_t-P_s)\beta(1-\alpha)^2+P_s(1-P_t)(1-\alpha\beta)^2 \\ \qquad +(P_t-P_s)(1-P_t)\alpha(1-\beta)^2 \\ \qquad \textit{if } 0\le s\le t\le n. \end{cases}$$

Proof.

Case I : $0\le t\le s\le n$. Rewriting $a=a^{(s)}$ and $b=b^{(t)}$ more precisely, we have

$$a=(\overbrace{1,\dots,1}^{t},\overbrace{1,\dots,1}^{s-t},\overbrace{\alpha,\dots,\alpha}^{n-s}),\quad\text{and}\quad b=(\overbrace{\beta,\dots,\beta}^{t},\overbrace{1,\dots,1}^{s-t},\overbrace{1,\dots,1}^{n-s}).$$

Then with the same notations as in Section 3 we have

$$J_a=\{1,\dots,s\}\quad\text{and}\quad J_b=\{t+1,\dots,n\},$$

and $\Delta=\{(i,j)\in I^2; i<j\}$ is devided into the three subsets

$$J_a\cap J_b^c\ (=J_2),\ J_a\cap J_b\ (=J_1)\text{ and } J_a^c\cap J_b\ (=J_3).$$

Hence similarly as in Lemma 3.1 of Section 3, $T(a,b;p)$ is the sum of $\sum_{J_{1,2}}$, $\sum_{J_{1,3}}$ and $\sum_{J_{2,3}}$. Note that $P(J_2)=P_t$, $P(J_1)=P_s-P_t$ and $P(J_3)=1-P_s$. Hence we have

$$\begin{aligned} T(a,b;p)&=P(J_1)P(J_2)(1-\beta)^2+P(J_1)P(J_3)(1-\alpha)^2+P(J_2)P(J_3)(1-\alpha\beta)^2\\ &=P_t(P_s-P_t)(1-\beta)^2+P_t(1-P_s)(1-\alpha\beta)^2+(P_s-P_t)(1-P_s)(1-\alpha)^2. \end{aligned}$$

Case II: $0\le s\le t\le n$. By the similar argument as in Case I, we have

$$\begin{aligned} T(a^{(s)},b^{(t)};p)=\beta^2(1-\alpha)^2P_t(P_s-P_t)+(1-\alpha\beta)^2P_t(1-P_s)\\ +\alpha^2(1-\beta)^2(P_s-P_t)(1-P_s). \end{aligned}$$

Summarizing Cases I and II, we obtain (17). □

Now we show the following result stronger than Theorem 3.2 with the restriction that a and b are oppositely ordered.

Theorem 4.2 *Let a and b be positive n-tuples satisfying*

$$M_1 \geq a_1 \geq \cdots \geq a_n \geq m_1 \quad \textit{and} \quad m_2 \leq b_1 \leq \cdots b_n \leq M_2,$$

and let $p = (p_1, \ldots, p_n)$ be an n-weight with $\sum_{k=1}^n p_k = 1$. Put $\alpha = m_1/M_1,\ \beta = m_2/M_2$,

$$A = (1-\beta)^2,\ B = (1-\alpha\beta)^2,\ C = (1-\alpha)^2,$$

$$A_1 = \beta^2(1-\alpha)^2,\ B_1 = B,\ C_1 = \alpha^2(1-\beta)^2,$$

and define $\tilde{A}, \tilde{B}, \tilde{C}$ and D similarly as (7). (Furthermore, correspondingly define $\tilde{A}_1, \tilde{B}_1$ and $\tilde{C}_1$.) Then

$$D = \{4-(1+\alpha)(1+\beta)\}(1+\alpha)(1+\beta)(1-\alpha)^2(1-\beta)^2 \tag{18}$$
$$\textit{and} \quad \frac{ABC}{D} = \frac{(1-\alpha\beta)^2}{\{4-(1+\alpha)(1+\beta)\}(1+\alpha)(1+\beta)},$$

and the following results hold.

(i) *If $(1+\alpha)(1+\beta) < 2$, then*

$$T(a,b;p) \leq M_1^2 M_2^2 \max\left\{\frac{ABC}{D} - C\mu^2 - \frac{D}{4C}\lambda^2, \quad B\left(\frac{1}{4} - \nu^2\right)\right\}. \tag{19}$$

(ii) *If $(1+\alpha)(1+\beta) \geq 2$, then*

$$T(a,b;p) \leq M_1^2 M_2^2 B\left(\frac{1}{4} - \nu^2\right). \tag{20}$$

Here, λ, μ and ν are defined as follows:

$$\begin{cases} \lambda = \min_{1\leq t\leq n-1}\left|P_t - \frac{C\tilde{C}}{D}\right|, \\ \mu = \min_{1\leq t<s\leq n-1}\left|(P_s - P_t) - \frac{B\tilde{B}}{D} + \frac{\tilde{A}}{2C}\left(P_t - \frac{C\tilde{C}}{D}\right)\right| \quad \textit{and} \\ \nu = \min_{1\leq t\leq n-1}\left|\frac{1}{2} - P_t\right|. \end{cases} \tag{21}$$

Proof. We may assume that $M_1 = M_2 = 1$, and write $m_1 = \alpha$ and $m_2 = \beta$ as in Theorem 3.2. Then by convexity of $T(a,b;p) = T_p(a,b)$ and Lemma 2.3 we may compute the maximum of $T_p(a,b)$ for $(a,b) = (a^{(s)}, b^{(t)}),\ s,t \in I_n^*$, where $a^{(s)}$ and $b^{(t)}$ are positive n-tuples defined as (16). First we consider

Case I: $0 \leq t \leq s \leq n$. Put

$$x = P_t, \quad y = P_s - P_t \quad \text{and} \quad z = 1 - P_s.$$

Then from (17) of Lemma 4.1

$$(u =)\ T(a^{(s)}, b^{(t)}; p) = Axy + Bxz + Cyz.$$

Now consider the two subcases I-(1) and I-(2) as follows.

I-(1): Assume $(1+\alpha)(1+\beta)<2$. Then

$$\tilde{B}=C+A-B=(1-\alpha)^2+(1-\beta)^2-(1-\alpha\beta)^2=2-(1+\alpha)(1+\beta)>0.$$

(Note that $(1+\alpha)(1+\beta)<2$ is equivalent to $\tilde{B}>0$.) For $\tilde{A}$ and $\tilde{C}$, since $B=(1-\alpha\beta)^2>(1-\beta)^2=A$, we have $\tilde{A}=B+C-A>0$, and similarly $\tilde{C}>0$. By Lemma 2.4 (cf. (10)) we can write

$$u=-C\left\{\left(y-\frac{B\tilde{B}}{D}\right)+\frac{\tilde{A}}{2C}\left(x-\frac{C\tilde{C}}{D}\right)\right\}^2-\frac{D}{4C}\left(x-\frac{C\tilde{C}}{D}\right)^2+\frac{ABC}{D}.$$

Hence from the above defintion of λ and μ, we have

$$u\le -C\mu^2-\frac{D}{4C}\lambda^2+\frac{ABC}{D}.$$

Here, it is an elementary computation to show that D and ABC/D are expressed as (18) in α and β.

I-(2): Assume $(1+\alpha)(1+\beta)\ge 2$. Then $\tilde{B}\le 0$, so that $\tilde{A},\tilde{C}>0$. By Lemma 2.4 (cf. (11)) we can write

$$u=-\tilde{B}xz+Ax(1-x)+Cz(1-z),$$

and since

$$xz=x(1-x-y)\le x(1-x)=\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\le\frac{1}{4}-\nu^2,$$

$$z(1-z)\le\frac{1}{4}-\nu^2 \qquad \text{(cf. } \nu \text{ is defined in (21))},$$

we then have

$$u\le(-\tilde{B}+A+C)\left(\frac{1}{4}-\nu^2\right)=B\left(\frac{1}{4}-\nu^2\right).$$

Case II: $0\le s\le t\le n$. Put

$$x=P_s,\quad y=P_t-P_s\quad\text{and}\quad z=1-P_t.$$

Then similarly as Case I, from Lemma 4.1

$$u=T(a^{(s)},b^{(t)};p)=A_1xy+B_1xz+C_1yz,$$

and furthemore

$$\begin{aligned}\tilde{A}_1=B_1+C_1-A_1&=(1-\alpha\beta)^2+\alpha^2(1-\beta)^2-\beta^2(1-\alpha)^2\\&=(1-\beta)\left\{(1+\alpha^2)(1-\beta)+2\beta(1-\alpha)\right\}>0,\\ \tilde{B}_1=C_1+A_1-B_1&=-(1-\alpha)(1-\beta)(1+\alpha+\beta-\alpha\beta)\le 0,\\ \tilde{C}_1=A_1+B_1-C_1&=(1-\alpha)\left\{(1+\beta^2)(1-\alpha)+2\alpha(1-\beta)\right\}>0.\end{aligned}$$

Hence by Lemma 2.4 (ii)

$$u\le B_1\left(\frac{1}{4}-\nu^2\right)=B\left(\frac{1}{4}-\nu^2\right),$$

so that

$$T(a,b;p) \leq M_1^2 M_2^2 B \left(\frac{1}{4} - \nu^2 \right).$$

We notice that the constant ν is independent from $A, B, \ldots$, so that it is identical in Cases I and II. Summarizing the two cases, we obtain the desired facts (i) and (ii). □

Considering the special cases $\lambda = \mu = 0$ and $\nu = 0$ in the preceding theorem, we have:

Theorem 4.3 *With the same notations and the same assumptions as in Theorem* 4.2, *the following results hold.*

(i) *If* $(1+\alpha)(1+\beta) < 2$, *then*

$$T(a,b;p) \leq \frac{M_1^2 M_2^2 ABC}{D} = \frac{M_1^2 M_2^2 (1-\alpha\beta)^2}{\{4-(1+\alpha)(1+\beta)\}(1+\alpha)(1+\beta)}.$$

If there are integers $s = s_0,\ t = t_0\ \ (s_0 > t_0)$ *such that*

$$P_{t_0} = \frac{C\tilde{C}}{D} \quad \textit{and} \quad P_{s_0} - P_{t_0} = \frac{B\tilde{B}}{D},$$

then

$$T_{max}\ (= \textit{ the maximum of } T_p(a,b) = T(a,b;p)) = \frac{M_1^2 M_2^2 ABC}{D},$$

which is attained at (a,b) *such that*

$$a = (\overbrace{M_1, \ldots, M_1}^{s_0}, \overbrace{m_1, \ldots, m_1}^{n-s_0}) \quad \textit{and} \quad b = (\overbrace{m_2, \ldots, m_2}^{t_0}, \overbrace{M_2, \ldots, M_2}^{n-t_0}).$$

(ii) *If* $(1+\alpha)(1+\beta) \geq 2$ *then*

$$T(a,b;p) \leq \frac{M_1^2 M_2^2 B}{4} = \frac{M_1^2 M_2^2 (1-\alpha\beta)^2}{4}.$$

If there is an integer $t = t_0$ *such that* $P_{t_0} = 1/2$, *then*

$$T_{max} = \frac{M_1^2 M_2^2 B}{4},$$

which is attained at (a,b) *such that*

$$a = (\overbrace{M_1, \ldots, M_1}^{t_0}, \overbrace{m_1, \ldots, m_1}^{n-t_0}) \quad \textit{and} \quad b = (\overbrace{m_2, \ldots, m_2}^{t_0}, \overbrace{M_2, \ldots, M_2}^{n-t_0}).$$

Proof. By Theorem 4.2 it suffices to see that

$$\frac{ABC}{D} \geq \frac{B}{4},$$

which is easily obtained, say, from (18). □

5 The difference $T(a,b;p)$ for similarly ordered a and b We here give an upper bound of $T_p(a,b)=T(a,b;p)$ under the condition that a and b are similarly ordered. We may confine ourselves for the case that both a and b are nondecreasingly ordered.

Theorem 5.1 *Let a and b be positive n-tuples satisfying*

$$m_1 \le a_1 \le \cdots \le a_n \le M_1 \quad \textit{and} \quad m_2 \le b_1 \le \cdots \le b_n \le M_2,$$

and let $p=(p_1,\dots,p_n)$ be an n-weight with $\sum_{k=1}^n p_k = 1$. Put, for $\alpha = m_1/M_1,\ \beta = m_2/M_2$,

$$A=\alpha^2(1-\beta)^2,\ B=(\alpha-\beta)^2,\ C=(1-\alpha)^2,$$
$$A_1=\beta^2(1-\alpha)^2,\ B_1=B,\ C_1=(1-\beta)^2,$$

and define $\tilde{A},\ \tilde{B},\ \tilde{C}$ and D, similarly as (7). (Furthermore, correspondingly define $\tilde{A}_1,\ \tilde{B}_1$ and $\tilde{C}_1$). Then

$$\begin{aligned} &D=(1+\alpha)(1+\beta))(1-\alpha)^2(1-\beta)^2\left\{(3-\beta)\alpha-(1+\beta)\right\} \\ &\textit{and}\quad \frac{ABC}{D}=\frac{\alpha^2(\alpha-\beta)^2}{(1+\alpha)(1+\beta)\left\{(3-\beta)\alpha-(1+\beta)\right\}}. \end{aligned} \tag{22}$$

Further assume that

$$\beta \le \alpha,$$

and write

$$\underline{\alpha}=\frac{-1+\sqrt{2-\beta^2}}{1-\beta} \quad \textit{and} \quad \overline{\alpha}=\frac{1+\beta^2}{1+2\beta-\beta^2}.$$

Then

$$\beta \le \underline{\alpha} \le \overline{\alpha} < 1 \tag{23}$$

and the following results hold. ($\lambda,\ \mu$ and ν are defined similarly as (21) in Therem 4.2).

(i) *If ($\beta \le$) $\alpha \le \underline{\alpha}$, then*

$$T(a,b;p) \le M_1^2 M_2^2 C_1\left(\frac{1}{4}-\nu^2\right).$$

(ii) *If $\underline{\alpha} < \alpha < \overline{\alpha}$, then $D>0$ and*

$$T(a,b;p) \le M_1^2 M_2^2 \max\left\{\frac{ABC}{D}-C\mu^2-\frac{D}{4C}\lambda^2,\ C_1\left(\frac{1}{4}-\nu^2\right)\right\}.$$

(iii) *If $\overline{\alpha} \le \alpha \le 1$, then*

$$T(a,b;p) \le M_1^2 M_2^2 C_1\left(\frac{1}{4}-\nu^2\right).$$

Proof. By Lemma 2.3, we have to compute the maximum or an upper bound of $T_p(a,b)=T(a,b;p)$ at points (a,b) such that

$$a=(\overbrace{m_1,\dots,m_1}^{s},\overbrace{M_1,\dots,M_1}^{n-s}), \quad \text{and} \quad b=(\overbrace{m_2,\dots,m_2}^{t},\overbrace{M_2,\dots,M_2}^{n-t}), \tag{24}$$

where s and t are integers in I_n^*.

We may again assume that $M_1 = M_2 = 1$, so that $m_1 = \alpha$ and $m_2 = \beta$. It is essential to consider the problem when $\beta < \alpha$. Now the first case is

Case I: $0 \le t \le s \le n$. Let

$$a^{(s)} = (\overbrace{\alpha, \dots, \alpha}^{t}, \overbrace{\alpha, \dots, \alpha}^{s-t}, \overbrace{1, \dots, 1}^{n-s}) \quad \text{and} \quad b^{(t)} = (\overbrace{\beta, \dots, \beta}^{t}, \overbrace{1, \dots, 1}^{s-t}, \overbrace{1, \dots, 1}^{n-s}).$$

Then by the similar argument as in Lemma 4.1 (cf. (17)), we have

$$\begin{aligned} T(a^{(s)}, b^{(t)}; p) &= \alpha^2(1-\beta)^2 P_t(P_s - P_t) + (\alpha-\beta)^2 P_t(1-P_s) \\ &\quad + (1-\alpha)^2(P_s - P_t)(1-P_s) \\ &= AP_t(P_s - P_t) + BP_t(1-P_s) + C(P_s - P_t)(1-P_s). \end{aligned}$$

First note that $A, B, C > 0$ (cf. $\beta < \alpha$) and by definition

$$\begin{aligned} \tilde{A} = B + C - A &= (\alpha-\beta)^2 + (1-\alpha)^2 - \alpha^2(1-\beta)^2 \\ &= (1-\alpha)\left\{1 + \beta^2 - (1 + 2\beta - \beta^2)\alpha\right\}, \end{aligned}$$

so that $\tilde{A} > 0$ if (and only if) $1 + \beta^2 - (1 + 2\beta - \beta^2)\alpha > 0$, or equivalently

$$\alpha < \overline{\alpha} = \frac{1+\beta^2}{1 + 2\beta - \beta^2}.$$

Here, it is not difficult to see $\beta < \overline{\alpha} < 1$. Next we have

$$\tilde{B} = C + A - B = (1-\alpha)(1-\beta)\left\{(1+\alpha)\beta + 1 - \alpha\right\} > 0$$

and

$$\tilde{C} = A + B - C = (1-\beta)\left\{(1-\beta)\alpha^2 + 2\alpha - (1+\beta)\right\},$$

so that $\tilde{C} > 0$ if (and only if) $(1-\beta)\alpha^2 + 2\alpha - (1+\beta) > 0$, or equivalently

$$(1 >)\ \alpha > \underline{\alpha} = \frac{-1 + \sqrt{2 - \beta^2}}{1 - \beta}.$$

Here, by an elementary computation we can see $\underline{\alpha} < \overline{\alpha} < 1$, so that we have (23). Now from Lemma 2.4 we have the following three subcases.

I-(1): If $(\beta <)\ \alpha \le \underline{\alpha}$, then $\tilde{A}, \tilde{B} > 0$, $\tilde{C} \le 0$, so that

$$T(a, b; p) \le C\left(\frac{1}{4} - \nu^2\right) \le C_1\left(\frac{1}{4} - \nu^2\right).$$

I-(2): If $\underline{\alpha} < \alpha < \overline{\alpha}$, then $\tilde{A}, \tilde{B}, \tilde{C} > 0$, so that

$$T(a, b; p) \le \frac{ABC}{D} - C\mu^2 - \frac{D}{4C}\lambda^2.$$

Here, by an elementary computation we can see that

$$D = (1+\alpha)(1+\beta)(1-\alpha)^2(1-\beta)^2\left\{(3-\beta)\alpha - (1+\beta)\right\}$$

and

$$\frac{ABC}{D} = \frac{\alpha^2(\alpha-\beta)^2}{(1+\alpha)(1+\beta)\{(3-\beta)\alpha-(1+\beta)\}}.$$

I-(3): If $\overline{\alpha} \le \alpha < 1$, then $\tilde{A} \le 0$, $\tilde{B} > 0$ and $\tilde{C} > 0$, so that

$$T(a,b;p) \le A\left(\frac{1}{4}-\nu^2\right) \le C_1\left(\frac{1}{4}-\nu^2\right).$$

Case II: $0 \le s \le t \le n$. Let

$$a^{(s)} = (\overbrace{\alpha,\dots,\alpha}^{s},\overbrace{1,\dots,1}^{t-s},\overbrace{1,\dots,1}^{n-t}) \quad \text{and} \quad b^{(t)} = (\overbrace{\beta,\dots,\beta}^{s},\overbrace{\beta,\dots,\beta}^{t-s},\overbrace{1,\dots,1}^{n-t}).$$

Then similarly as in Case I, we have

$$\begin{aligned} T(a^{(s)},b^{(t)};p) &= \beta^2(1-\alpha)^2P_s(P_t-P_s)+(\alpha-\beta)^2P_s(1-P_t) \\ &\quad +(1-\beta)^2(P_t-P_s)(1-P_t) \\ &= A_1P_s(P_t-P_s)+B_1P_s(1-P_t)+C_1(P_t-P_s)(1-P_t). \end{aligned}$$

For the signs of the constants $\tilde{A}_1$, $\tilde{B}_1$ and $\tilde{C}_1$, we have

$$\begin{aligned} \tilde{A}_1 = B_1+C_1-A_1 &= (1-\beta)\left\{1+\alpha^2-\beta(1+2\alpha-\alpha^2)\right\} \\ &\ge (1-\beta)\left\{1+\alpha^2-\alpha(1+2\alpha-\alpha^2)\right\} \\ &= (1-\beta)(1+\alpha)(1-\alpha)^2 > 0, \end{aligned}$$

$$\tilde{B}_1 = C_1+A_1-B_1 = (1-\alpha)(1-\beta)^2 > 0$$

and

$$\begin{aligned} \tilde{C}_1 = A_1+B_1-C_1 &= (1-\alpha)\left\{-1+2\beta+\beta^2-\alpha(1+\beta^2)\right\} \\ &\le (1-\alpha)\left\{-1+2\beta+\beta^2-\beta(1+\beta^2)\right\} \\ &= -(1-\alpha)(1-\beta)(1-\beta^2) \le 0. \end{aligned}$$

Hence by Lemma 2.4 we have

$$T(a,b;p) \le C_1\left(\frac{1}{4}-\nu^2\right).$$

Summarizing Cases I and II, we obtain the desired facts in the theorem. □

Theorem 5.2 *With the same notations and the same assumptions as in Theorem* 5.1*,*

$$T(a,b;p) \le \frac{M_1^2M_2^2C_1}{4} = \frac{M_1^2M_2^2(1-\beta)^2}{4}.$$

If there is an integer $t=t_0$ such that $P_{t_0}=1/2$, then

$$T_{max}(= \textit{the maximum of } T(a,b;p)) = \frac{M_1^2M_2^2C_1}{4},$$

which is attained at (a',b') such that

$$a' = (\overbrace{M_1,\dots,M_1}^{n}) \quad \textit{and} \quad b' = (\overbrace{m_2,\dots,m_2}^{t_0},\overbrace{M_2,\dots,M_2}^{n-t_0}).$$

Proof. By Theorem 5.1, we have only to show that if $\underline{\alpha} < \alpha < \overline{\alpha}$, (or if $\tilde{A}$, $\tilde{B}$ and $\tilde{C} > 0$) then

$$\frac{ABC}{D} < \frac{C_1}{4}, \tag{25}$$

or $\frac{ABC}{D} < \frac{B+C}{4}$ because

$$B + C = (\alpha - \beta)^2 + (1 - \alpha)^2 < (1 - \beta)^2 = C_1.$$

Since

$$\frac{B+C}{4} - \frac{ABC}{D} = \frac{(B+C)D - 4ABC}{4D},$$

we have to show $(B+C)D - 4ABC > 0$. Note that $D = 4BC - \tilde{A}^2$ and $A = B + C - \tilde{A}$, so that we have

$$\begin{aligned}(B+C)D - 4ABC &= (B+C)(4BC - \tilde{A}^2) - 4(B + C - \tilde{A})BC \\ &= \tilde{A}\left\{A(B+C) - (B-C)^2\right\} \\ &\geq \tilde{A}\left\{A^2 - (B-C)^2\right\} \quad (\text{cf. } B + C > A) \\ &= \tilde{A}\tilde{B}\tilde{C} > 0.\end{aligned}$$

□

Remark 5.3 *Related to Theorem* 5.2 (*and also Theorem* 4.3), *we ask if the value* $T_p(a", b") = T(a", b"; p) = \frac{M_1^2 M_2^2 ABC}{D}$ *at the point* $(a", b")$ *with*

$$a" = (\overbrace{m_1, \ldots, m_1}^{s_0}, \overbrace{M_1, \ldots, M_1}^{n-s_0}) \quad \textit{and} \quad b" = (\overbrace{m_2, \ldots, m_2}^{t_0}, \overbrace{M_2, \ldots, M_2}^{n-t_0})$$

is the maximum of $T_p(a, b)$, *whenever* $(\tilde{A}, \tilde{B}, \tilde{C} > 0$ *and*) *there are integers* $s = s_0,\ t = t_0$ *satisfying*

$$P_{t_0} = \frac{C\tilde{C}}{D} \quad \textit{and} \quad P_{s_0} - P_{t_0} = \frac{B\tilde{B}}{D}.$$

Unfortunately, this is not true. In fact, if $P_{t_0} = \frac{C\tilde{C}}{D}$ *is 'sufficiently near' to* $1/2$, *then for the point* (a', b') *with*

$$a' = (\overbrace{M_1, \ldots, M_1}^{n}) \quad \textit{and} \quad b' = (\overbrace{m_2, \ldots, m_2}^{t_0}, \overbrace{M_2, \ldots, M_2}^{n-t_0})),$$

we have

$$\begin{aligned}T_p(a', b') &= M_1^2 M_2^2 T(a^{(n)}, b^{(t_0)}; p) = C_1 P_{t_0}(1 - P_{t_0}) \\ &= C_1\left\{\frac{1}{4} - \left(\frac{1}{2} - P_{t_0}\right)^2\right\} = \frac{C_1}{4} - C_1\epsilon^2 > \frac{ABC}{D} \quad \left(\epsilon = \left|\frac{1}{2} - P_{t_0}\right|\right)\end{aligned}$$

by the inequality (25).

Concernig the preceding remark, as a numerical example, let $M_1 = M_2 = 1,\ m_1 = \alpha = \frac{7}{10}$ and $m_2 = \beta = \frac{1}{2}$, then $A = \frac{49}{400},\ B = \frac{1}{25},\ C = \frac{9}{100},\ C_1 = \frac{1}{4},\ D = \frac{2295}{400^2}, \ldots$ If we put

$n = 3$ and $p = (p_1, p_2, p_3) = \left(\frac{C\tilde{C}}{D}, \frac{B\tilde{B}}{D}, \frac{A\tilde{A}}{D}\right) = \left(\frac{1044}{2295}, \frac{1104}{2295}, \frac{147}{2295}\right)$, then for $s_0 = 2,\ t_0 = 1$, that is, for $a" = (\frac{7}{10}, \frac{7}{10}, 1),\ b" = (\frac{1}{2}, 1, 1)$, we have

$$T(a", b"; p) = \frac{ABC}{D} = \frac{196}{6375} = 0.0307...$$

On the other hand, for $s_0 = 0,\ t_0 = 1$, that is, for $a' = (1, 1, 1),\ b' = (\frac{1}{2}, 1, 1)$, we have

$$T(a', b'; p) = C_1 P_1 (1 - P_1) = \frac{4031}{65025} = 0.0619... > \frac{ABC}{D}.$$

Corollary 5.4 *With the same notations and the same assumptions as in Theorem 5.1, in particular, if the weight $p = (p_1, \ldots, p_n)$ is uniform, that is, $p_1 = \cdots = p_n = 1/n$, and if n is even, then*

$$T_{max} = \frac{M_1^2 M_2^2 (1-\beta)^2}{4}.$$

6 A concluding remark We can show corresponding continuous or measurable versions of all results in this paper. For example, corresponding to Theorem 3.2, we obtain the following:

Theorem 6.1 *Let f and g be positive measurable functions on a finite measure space (Ω, μ) with $\mu(\Omega) = 1$. Assume that $m_1 \le f \le M_1,\ m_2 \le g \le M_2,\ 0 < m_1 < M_1$ and $0 < m_2 < M_2$. Further assume that $\alpha = m_1/M_1 \ge m_2/M_2 = \beta$. Then*

$$\begin{aligned} &\int_\Omega f^2 d\mu \int_\Omega g^2 d\mu - \left(\int_\Omega fg d\mu\right)^2 \\ &\le M_1^2 M_2^2 \sup_{X \subset \Omega} \left\{ \frac{(1-\alpha\beta)^2}{4} (1-\mu(X))^2 + (1-\beta)^2 \mu(X)(1-\mu(X)) \right\} \\ &\quad \left(\le \frac{(M_1 M_2 - m_1 m_2)^2}{3} \right). \end{aligned}$$

To sketch the proof, let $\{X_1, ..., X_n\}$ be a decomposition of measurable sets in Ω and let $x_k \in X_k$ $(k = 1,\ \ldots,\ n)$. Then from Theorem 3.2 we have

$$\begin{aligned} &\sum_{k=1}^n f(x_k)^2 \mu(X_k) \sum_{k=1}^n g(x_k)^2 \mu(X_k) - \left(\sum_{k=1}^n f(x_k) g(x_k) \mu(X_k)\right)^2 \\ &\le M_1^2 M_2^2 \sup_{X \subset \Omega} \left\{ \frac{(1-\alpha\beta)^2}{4} (1-\mu(X))^2 + (1-\beta)^2 \mu(X)(1-\mu(X)) \right\}. \end{aligned}$$

Taking the limit of the decomposition we obtain the desired inequality.

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* FACULTY OF EDUCATION, TOYAMA UNIVERSITY, GOFUKU, TOYAMA 930-8555, JAPAN

E-mail address: s-izumino@@h5.dion.ne.jp

** Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, CROATIA

E-mail address: pecaric@@element.hr