# UNIFORM SPREAD COMPLETION FOR FRAMES

Hlengani J. Siweya* and D. Baboolal

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Abstract. Following the equivalence of a complete uniform spread (in the new sense) and Hunt's complete uniform spread, we construct a point-free uniform spread completion, a completion we show is also unique up to equivalence.

## 1. Introduction

In [10], Fox introduced spreads (now commonly referred to as *Fox's spreads*) as an encompassing concept for branched and unbranched coverings. One of the important contributions brought about by Fox's spreads is the construction of the *completion of a spread* between locally connected $T_1$-spaces. Motivated by Fox's spreads, Hunt [11] extended the notion of a spread to that of a uniform spread between uniformly locally connected spaces. Fox's canonical completion uses *spread points* (a term suggested by Hunt) whereas Hunt achieved his uniform spread completion via minimal Cauchy filters. To justify the use of minimal Cauchy filters, Hunt established that these filters are equivalent to his spread points. See Hunt [13].

Uniform spreads and spread uniformities were introduced into the frame-theoretic setting by the author in [19]. In this article, we introduce complete uniform spreads into this setting, then show that our notion of a complete uniform spread coincides with that of Hunt. Following the equivalence of a complete uniform spread (in our sense) and Hunt's complete uniform spread, we construct a point-free uniform spread completion, a completion we show is also unique up to equivalence.

The difference between our approach and Hunt's in the construction is that, unlike Hunt, ours is dependent on the *Banaschewski-Pultr uniform frame completion* (a term already suggested in Siweya [18]). Our approach has room for the application of certain notions introduced in and results established in [17]. We shall, even though briefly, have occasion to mention some of the results here-under.

The basic concepts are dealt with in Section 2. Included are properties of *uniform local connectedness with respect to along* (Proposition 2.6), and the equivalence of Hunt's notion of completeness of a uniform spread and that of ours we introduce in this section (see Theorem 2.10). In Proposition 2.8, we look at conditions under which a uniformity and a spread uniformity relate to each other.

In Section 3, the main result is the "pointless" construction of the uniform spread completion whose proof rests both on the Banaschewski-Pultr uniform frame completion (see [6])

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*correspondence author

and the fact that a uniform spread $h : L \longrightarrow M$ lifts to a uniform spread $Ch : CL \longrightarrow CM$ (Proposition 3.1). This then paves the way for the establishment of other Hunt-type results.

## 2. Preliminaries

In an effort to keep this article short, we assume familiarity with the notions of uniform frame, uniform homomorphism, and any other uniform concept as enunciated in [6, 7, 9, 14, 16]. For general knowledge on frames, we refer to [3, 15].

Following Baboolal-Banaschewski [2], given a frame $L$, we say an element $z \in L$ is *connected* if, whenever $z = x \vee y$ with $x \wedge y = 0$, then $x = 0$ or $y = 0$. The frame $L$ is *connected* whenever its top element $e$ is connected; and $L$ is *locally connected* if for each $x \in L$ we have

$$x = \bigvee\{y \in L \mid y \textit{ is connected},\ y \leq x\},$$

An element $y \in L$ is a *component* of $x \in L$ if $y \leq x$ is maximally connected. For a component $y$ of $x$ we write $y \leq_c x$.

**Definition 2.1**

(a) A frame homomorphism $h : L \longrightarrow M$ between locally connected frames is called a *localic spread* if $\bigcup\{S_u \mid u \in L\}$ is a basis for $M$, where for each $u \in L$,

$$S_u = \{x \in M \mid x \leq_c h(u)\}.$$

(b) Given an onto frame homomorphism $h : L \longrightarrow M$, we say that M is *locally connected with respect to L along h* if there is a basis B of L such that $h(b)$ is connected for each $b \in B$.

**Proposition 2.2.** Let $h : L \longrightarrow M$ be a dense homomorphism with $M$ locally connected with respect to $h$. Then $h_* : M \longrightarrow L$ preserves disjoint binary joins. In particular,

$$h_*(\bigvee_{u_i \in M} u_i) \quad = \quad \bigvee_{u_i \in M} h_*(u_i),$$

if the $u_i \in M$ are pairwise disjoint.

**Proof.** Pick $x, y \in M$ with $x \wedge y = 0$, and a basis $B$ of $L$ such that

$$h_*(x \vee y) = \bigvee_{s \in B} s \qquad (h(s) \textit{ is connected}).$$

We have $h(s) \leq x \vee y$. Now since $h(s)$ is connected with $x \wedge y = 0$, it follows that $h(s) \leq x$ or $h(s) \leq y$ so that $s \leq h_*(x)$ or $s \leq h_*(y)$. Thus

$$s \leq h_*(x) \vee h_*(y)$$

Now, taking joins over all such $s$, we have

$$\begin{aligned} h_*(x \vee y) \quad &= \quad \bigvee_{s \in B} s \qquad (h(s) \textit{ is connected}) \\ &\leq \quad h_*(x) \vee h_*(y). \end{aligned}$$

so that $h_*(x \vee y) = h_*(x) \vee h_*(y)$. $\square$

According to Baboolal [1], the *fine uniformity* on a uniform frame $L$ is the uniformity $\mathfrak{U}_F L$ generated by all normal covers, i.e. those covers $A$ of $L$ such that $A = A_1$ in some sequence $(A_n)_n$ of covers of $L$ such that $A_{n+1} \leq^* A_n$ for all $n = 1, 2, 3, \ldots$. Also, a uniform frame $(L, \mathfrak{U}L)$ is *uniformly locally connected* if each cover $U \in \mathfrak{U}L$ is refined by a uniform cover $V \in \mathfrak{U}L$ each of whose elements is connected.

Now consider an onto uniform homomorphism $h : L \longrightarrow M$ which is a spread, where $L$ carries the fine uniformity $\mathfrak{U}_F L$.

**Lemma 2.3** (See also [17]). The collection

$$\mathfrak{B} = \{T_U \mid U \in \mathfrak{U}_F L\}$$

is a basis for a uniformity on $M$, where for each $U \in \mathfrak{U}_F L$,

$$T_U = \{z \in M \mid z \leq_c h(u),\ \textit{for some } u \in U\}.$$

The uniformity $\mathfrak{U}^h M$ so generated is called the *spread uniformity on $M$* and $h$ is called a *uniform spread*. For a uniform spread $h : (L, \mathfrak{U}L) \longrightarrow (M, \mathfrak{U}^h M)$ we write $(L, h, M)$.

**Definition 2.4.** Let $h : (L, \mathfrak{U}L) \longrightarrow (M, \mathfrak{U}M)$ be a surjective homomorphism. We say that $M$ is *uniformly locally connected with respect to $L$ along $h$* if there is a basis $\mathcal{B}$ for the uniformity $\mathfrak{U}L$ such that each cover $h[B]$ is connected for $B \in \mathcal{B}$, where

$$h[B] = \{h(b) \mid b \in \mathcal{B}\}.$$

**Proposition 2.5.** Let $h : L \longrightarrow M$ be a surjective homomorphism. If $M$ is uniformly locally connected with respect to $L$ along $h$, then $M$ is uniformly locally connected. So, in particular, if $M$ is uniformly locally connected with respect to $L$ along $h$ then $L$ is uniformly locally connected.

**Proposition 2.6.** Under the hypothesis of the previous proposition, if $h : L \longrightarrow M$ is a dense surjection, then $M$ is uniformly locally connected if and only if, $M$ is uniformly locally connected with respect to $L$ along $h$.

**Proof.** Sufficiency follows from the previous proposition.

For the converse, suppose that $M$ is uniformly locally connected. Then $\mathfrak{U}M$ is generated by all $C \in \mathfrak{U}M$ such that $C$ is a connected cover. We claim that the collection

$$\{h_*[C] \mid C \in \mathfrak{U}M,\ C \textit{ connected}\}$$

is a basis for $\mathfrak{U}L$, and $h \circ h_*[C] = C$.

We first show that $h_*[C] \in \mathfrak{U}L$ : For, let $C \in \mathfrak{U}M$. Then there exists a uniform cover $A \in \mathfrak{U}L$ such that $h[A] \leq C$, so that $A \leq h_*[C]$ which proves that $h_*[C] \in \mathfrak{U}L$.

Take $A \in \mathfrak{U}L$. Find $B \in \mathfrak{U}L$ such that $B \leq^* A$. Then $h[B] \in \mathfrak{U}M$ . We claim that $h_* \circ h[B] \leq A$ : Pick any $b \in B$. Then $Bb \leq a$, for some $a \in A$ which implies that $b \triangleleft_{\mathfrak{U}L} a$; thus $b \prec a$ and so $b^* \vee a = e$. Now

$$h(h_* \circ h(b) \wedge b^*) = h(b) \wedge h(b^*) = 0.$$

Since $h$ is dense, this implies that $(h_* \circ h(b)) \wedge b^* = 0$. Now since $b^* \vee a = e$ it follows that $h_* \circ h(b) \leq a$, which gives $h_* \circ h[B] \leq A$. Therefore, since $M$ is uniformly locally connected there is a connected cover $C \in \mathfrak{U}M$ such that $C \leq h[B]$ so that $h_*[C] \leq h_* \circ h[B]$. Hence $h_*[C] \leq A$. □

**Definition 2.7.** An *extension* of a uniform spread $(L, h, M)$ is a uniform spread $(L, g, N)$ together with a dense surjection $f : N \longrightarrow M$, where $M$ is uniformly locally connected with respect to $N$ along $f$ such that $f \circ g = h$. A uniform spread $(L, h, M)$ is said to be *complete* if, whenever $(L, g, N)$ is a uniform spread and $f : N \longrightarrow M$ is a dense surjection with $M$ uniformly locally connected with respect to $N$ along $f$ such that the following triangle

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[dr, "g"'] & M \\ & N \arrow[u, "f"'] \end{tikzcd}$$

commutes, then $f$ is an isomorphism. Then a *completion* of a uniform spread is a complete extension of the uniform spread.

**Proposition 2.8.** Let $h : (L, \mathfrak{U}L) \longrightarrow (M, \mathfrak{U}^h M)$ be a uniform spread. If $g : (L, \mathfrak{U}L) \longrightarrow (N, \mathfrak{U}^g N)$ is a uniform spread and $f : (N, \mathfrak{U}N) \longrightarrow (M, \mathfrak{U}^h M)$ is a dense surjection such that $h = f \circ g$, then $\mathfrak{U}N \subseteq \mathfrak{U}^g N$ (i.e. $f : (N, \mathfrak{U}^g N) \longrightarrow (M, \mathfrak{U}^h M)$ is also a dense surjection).

**Proof.** To show that $\mathfrak{U}N \subseteq \mathfrak{U}^g N$, we need only show that for each $A \in \mathfrak{U}N$ then $A \in \mathfrak{U}^g N$, i.e. there exists $T_W \in \mathfrak{U}^g N$ satisfying $T_W \leq A$.

Given $A \in \mathfrak{U}N$, there exists $T_V \in \mathfrak{U}^h M$ such that $f_*[T_V] \leq A$ with $V \in \mathfrak{U}_F L$. It suffices to find $T_W \in \mathfrak{U}^g N$ such that $T_W \leq f_*[T_V]$. Now since $g$ is a uniform spread there is $T_W \in \mathfrak{U}^g N$ with $T_W \leq g[V]$ for $W \in \mathfrak{U}_F L$.

We claim that $T_W \leq f_*[T_V]$: Take $y \in T_W$. Then $y \leq_c g(w)$ for some $w \in W$. But then $y \leq g(v)$ for some $v \in V$, so $f(y) \leq f \circ g(v) = h(v)$. Since $M$ is locally connected [2], we may assume that

$$h(v) = \bigvee a \quad (a \leq_c h(v)).$$

Since $f$ is dense onto and $M$ is locally connected with respect to $L$ along $f$ [17], $f_*$ preserves pairwise disjoint joins, so

$$f_* \circ h(v) = \bigvee f_*(a) \quad (a \leq_c h(v)),$$

whence

$$y \leq \bigvee f_*(a) \quad (a \leq_c h(v)).$$

Since the $f_*(a)$ are pairwise disjoint, the connectedness of $y$ implies that $y \leq f_*(a)$, for some $a \leq_c h(v)$. Thus $T_W \leq f_*[T_V]$ as desired. □

Hunt defines a uniform spread $(L, h, M)$ to be *complete* whenever $M$ is a complete uniform frame with respect to the induced spread uniformity $\mathfrak{U}^h M$. In the following result we show that Hunt's definition is equivalent to ours.

**Theorem 2.9.** Let $(L, \mathfrak{U}L)$ be a complete uniform frame, let $h : (L, \mathfrak{U}L) \longrightarrow (M, \mathfrak{U}M)$ be a uniform homomorphism and let $\rho : (N, \mathfrak{U}N) \longrightarrow (M, \mathfrak{U}M)$ be a dense surjection. Then there is a unique uniform homomorphism $g : L \longrightarrow N$ such that $\rho \circ g = h$:

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[dr, dashed, "g"] & M \\ & N \arrow[u, "\rho"'] \end{tikzcd}$$

In particular, if $L$ carries the fine uniformity (so that it is also complete with respect to) $\mathfrak{U}_F L$, the homomorphism $g : L \longrightarrow N$ is a uniform spread.

**Proof.** See [19, Proposition 2.1]. □

**Theorem 2.10.** Given a uniform spread $h : (L, \mathfrak{U}L) \longrightarrow (M, \mathfrak{U}^h M)$, then $h$ is a complete uniform spread if and only if $(M, \mathfrak{U}^h M)$ is a complete uniform frame, i.e. $h$ is complete in the sense of Definition 2.7 if and only if it is complete in the sense of Hunt.

**Proof.**
$(\Longrightarrow)$ : Suppose that $h : (L,\, \mathfrak{U}L) \longrightarrow (M,\, \mathfrak{U}^h M)$ is complete and that $p : (N,\, \mathfrak{U}N) \longrightarrow (M,\, \mathfrak{U}^h M)$ is a dense surjection. Since $L$ is complete it follows that $h$ extends to a uniform spread, say $g : L \longrightarrow (N,\, \mathfrak{U}^g N)$ by the construction in Theorem 2.9 - such an extension is precisely the composition

$$g = \gamma_N \circ (Cp)^{-1} \circ Ch \circ \gamma_L^{-1},$$

and (trivially) $p \circ g = h$.

Since $p$ is a dense surjection, it follows (Proposition 2.4) that $p : (N,\, \mathfrak{U}^g N) \longrightarrow (M,\, \mathfrak{U}^h M)$ is also a dense surjection. Now because $M$ is uniformly locally connected, it follows that $M$ is uniformly locally connected with respect to $N$ along $p$ (Proposition 2.6). Therefore, since $h$ is complete and since $p \circ g = h$ with $g$ being a uniform spread, it is implied that $p$ is an isomorphism, thus $M$ is a complete uniform frame. □

## 3. Uniform spread completion

The main result of this section is the existence theorem for Hunt's uniform spread completion. Recall (Hunt [13]) that the construction of a uniform spread completion in the case of uniform spaces was arrived at through the use of minimal Cauchy filters. Crucial to Hunt's construction, was a result to the effect that in $\mathbb{U}$**nif**, there is a bijective correspondence between minimal Cauchy filters and the set of *spread points*. In this connection, it must be remembered that Fox [10] used *spread points* (a term suggested by Hunt) to obtain a canonical completion of a spread between locally connected $T_1$-spaces.

In our construction, we appeal to the Banaschewski-Pultr uniform frame completion [6]. We do not know whether the uniform frame completions of Isbell [14] and Kříž [16] would lead to the result arrived at here. Indeed, the Banachewski-Pultr uniform frame completion can also be used for other generalizations. See e.g. Siweya [18].

For completeness, given a uniform frame $L$ we denote its Banaschewski-Pultr uniform completion by $CL$, the dense surjection $CL \longrightarrow L$ by $\gamma_L$, the quotient (frame) homomorphism $\mathfrak{R}L \longrightarrow CL$ by $\nu_L$ and by $k_L : L \longrightarrow \mathfrak{R}L$ the map defined by

$$k_L(a) = \{x \in L \mid x \lhd a\},$$

where $\mathfrak{R}L$ is the *Samuel compactification of* $L$. See [6]. A uniform spread $h : L \longrightarrow M$ lifts to a uniform spread $Ch : CL \longrightarrow CM$, in the sense of the following

**Proposition 3.1.** Given a uniform spread $h : L \longrightarrow M$, the induced uniform homomorphism $Ch : CL \longrightarrow CM$ making the rectangle

$$\begin{tikzcd} CL \arrow[r, "Ch"] \arrow[d, "\gamma_L"'] & CM \arrow[d, "\gamma_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

commutative is a uniform spread whose generators are

$$T^L[U] = \{\nu_M \circ k_M(x) \in CM \mid \nu_M \circ k_M(x) \leq_c Ch\left(\nu_L \circ k_L(u)\right),\ for\ some\ u \in U\},$$

*for* $U \in \mathfrak{U}_F L$, *where* $x \leq_c h(u)$.

**Proof.**
$(i)$ Given $\nu_L \circ k_L[A] \in \mathfrak{U}_{CL}$, pick $\nu_L \circ k_L[B] \in \mathfrak{U}_{CL}$ such that

$$\nu_L \circ k_L[B] \leq^* \nu_L \circ k_L[A].$$

It is to be shown that $T^L[B] \leq^* T^L[A]$. Taking $\nu_M \circ k_M(x) \in T^L[B]$, we have

$$\nu_M \circ k_M(x) \leq_c Ch\left(\nu_L \circ k_L(u_x)\right),$$

for some $u_x \in B$, with $x \leq_c h(u_x)$. Furthermore,

$$\nu_L \circ k_L[B] \leq^* \nu_L \circ k_L[A]$$

implies that

$$\left(\nu_L \circ k_L[B]\right) \nu_L \circ k_L(u_x) \leq \nu_L \circ k_L(y),$$

for some $y \in A$. We also have

$$\begin{aligned} \nu_M \circ k_M(x) &\leq Ch\left(\nu_L \circ k_L(u_x)\right) \\ &\leq Ch\left(\nu_L \circ k_L[B]\right) Ch\left(\nu_L \circ k_L(u_x)\right) \\ &\leq Ch\left(\nu_L \circ k_L(y)\right). \end{aligned}$$

Now pick a component

$$\nu_M \circ k_M(r) \leq_c Ch\left(\nu_L \circ k_L(y)\right)$$

which satisfies

$$\nu_M \circ k_M(x) \leq \nu_M \circ k_M(r).$$

It remains to show that

$$T^L[B]\left(\nu_M \circ k_M(x)\right) \leq \nu_M \circ k_M(r).$$

Take $\nu_M \circ k_M(w) \in T^L[B]$ such that $\nu_M \circ k_M(w) \wedge \nu_M \circ k_M(x) \neq 0$. Then

$$\nu_M \circ k_M(w) \leq_c Ch\left(\nu_L \circ k_L(u_w)\right),$$

for some $u_w \in B$ with $w \leq_c h(u_w)$. From

$$\nu_M \circ k_M(w) \wedge \nu_M \circ k_M(x) \neq 0 \textit{ and } \nu_M \circ k_M(x) \leq Ch\left(\nu_L \circ k_L(u_x)\right)$$

we find that

$$Ch\left(\nu_L \circ k_L(u_x)\right) \wedge Ch\left(\nu_L \circ k_L(u_w)\right) \neq 0$$

and then

$$\nu_L \circ k_L(u_x) \wedge \nu_L \circ k_L(u_w) \neq 0.$$

But we also have

$$\nu_L \circ k_L(u_w) \leq \left(\nu_L \circ k_L[B]\right) \nu_L \circ k_L(u_x),$$

so that

$$Ch\left(\nu_L \circ k_L(u_w)\right) \leq Ch\left(\nu_L \circ k_L[B]\right) Ch\left(\nu_L \circ k_L(u_x)\right).$$

Since

$$Ch\left(\nu_L \circ k_L[B]\right) Ch\left(\nu_L \circ k_L(u_x)\right) \leq Ch\left(\nu_L \circ k_L(y)\right),$$

it follows from the above relation that

$$Ch\left(\nu_L \circ k_L(u_w)\right) \leq Ch\left(\nu_L \circ k_L(y)\right).$$

Now the relation

$$\nu_M \circ k_M(w) \leq_c Ch\left(\nu_L \circ k_L(u_w)\right)$$

gives rise to

$$\nu_M \circ k_M(w) \leq Ch\left(\nu_L \circ k_L(y)\right);$$

and then

$$\nu_M \circ k_M(w) \wedge \nu_M \circ k_M(x) \neq 0$$

ensures that

$$\nu_M \circ k_M(w) \wedge \nu_M \circ k_M(r) \neq 0.$$

So, since $\nu_M \circ k_M(w)$ is connected (and!) below $Ch\left(\nu_L \circ k_L(y)\right)$ and

$$\nu_M \circ k_M(r) \leq_c Ch\left(\nu_L \circ k_L(y)\right),$$

we must have

$$\nu_M \circ k_M(w) \leq \nu_M \circ k_M(r).$$

Hence it has been shown that

$$T^L[B]\nu_M \circ k_M(x) \leq \nu_M \circ k_M(r),$$

that is, $T^L[B] \leq^* T^L[A]$ as was to be shown.
$(ii)$ Take $\nu_M \circ k_M(x) \in CM$. Since $CM$ is uniform,

$$\begin{aligned} & \nu_M \circ k_M(x) \\ = & \bigvee\{\nu_M \circ k_M(y) \in CM \mid \left(\nu_M \circ k_M[A]\right) \nu_M \circ k_M(y) \\ \leq & \nu_M \circ k_M(x)\}, \end{aligned}$$

for some $A \in \mathfrak{U}M$. Find a cover $\nu_L \circ k_L[U]$ in $\mathfrak{U}_{CL}$ such that

$$Ch\left(\nu_L \circ k_L[U]\right) \leq \nu_M \circ k_M[A].$$

It will be shown that

$$T^L[U]\nu_M \circ k_M(y) \leq \nu_M \circ k_M[A].$$

Pick $\nu_M \circ k_M(z) \in T^L[U]$ such that

$$\nu_M \circ k_M(z) \wedge \nu_M \circ k_M(y) \neq 0.$$

Then

$$\nu_M \circ k_M(z) \leq_c Ch\left(\nu_L \circ k_L(u_z)\right),$$

for some $u_z \in U$ with $U \in \mathfrak{U}_F L$ where $z \leq_c h(u_z)$. We have

$$Ch\left(\nu_L \circ k_L[U]\right) \leq \nu_M \circ k_M[A]$$

so that

$$Ch\left(\nu_L \circ k_L(u_z)\right) \leq \nu_M \circ k_M(a_u),$$

for some $a_u \in A$. Then

$$\nu_M \circ k_M(z) \leq \nu_M \circ k_M(a_u),$$

so that

$$T^L[U]\nu_M \circ k_M(y) \leq \nu_M \circ k_M(a_u),$$

for some $a_u \in A$. Hence, for each $\nu_M \circ k_M(x) \in CM$, we have

$$\nu_M \circ k_M(x) = \bigvee\{\nu_M \circ k_M(y) \in CM \mid T^L[U]\nu_M \circ k_M(y) \leq \nu_M \circ k_M(x)\},$$

for $U \in \mathfrak{U}L$, i.e. some $\nu_L \circ k_L[U] \in \mathfrak{U}_{CL}$. $\square$

**Theorem 3.2 (Existence Theorem).** Every uniform spread $(L, h, M)$ has a uniform spread completion, unique up to equivalence.

**Proof.** Given a uniform spread $(L, h, M)$, consider the Banaschweski-Pultr uniform frame completion $(CM, \gamma_M)$ of $M$. Find a unique uniform spread $(L, g, CM)$ such that $\gamma_M \circ g = h$ (see Theorem 2.9).

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[dr, "g"'] & M \\ & CM \arrow[u, "\gamma_M"'] \end{tikzcd}$$

Now $M$ is uniformly locally connected with respect to $CM$ along $\gamma_M$. By completeness of $L$, the morphism $\gamma_L : CL \longrightarrow L$ is an isomorphism, and so $g = Ch \circ \gamma_L^{-1}$, with $Ch$ being the induced uniform spread $CL \longrightarrow CM$. Clearly, $Ch \circ \gamma_L^{-1}$ is a uniform spread. Since $CM$ is a complete uniform frame, $(Ch \circ \gamma_L^{-1}, CM, \gamma_M)$ is a uniform spread completion.

For the uniqueness of the uniform spread completion $(Ch \circ \gamma_L^{-1}, CM, \gamma_M)$ of $(L, h, M)$, consider another uniform spread completion $(k, N, f)$ with $g = Ch \circ \gamma_L^{-1}$:

$$\begin{tikzcd} & CM \arrow[dr, "\gamma_M"] & \\ L \arrow[ur, "g"] \arrow[dr, "k"'] & & M \\ & N \arrow[ur, "f"'] & \end{tikzcd}$$

The required isomorphism $CM \longrightarrow N$ is $\psi = \gamma_N \circ (Cf)^{-1}$ whose uniqueness follows from the fact that $f$ is monic (see [3]). $\square$

**Remarks 3.3.** If $(g, (N, \mathfrak{U}^g N), f)$ is a uniform spread completion of a uniform spread $(L, h, M)$,

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[dr, "g"'] & M \\ & N \arrow[u, "f"'] \end{tikzcd}$$

then

(i) $(N, \mathfrak{U}^g N)$ is a complete uniform frame;

(ii) $((N, \mathfrak{U}^g N), f)$ is a uniform completion of $M$.

Moreover, if $(L, h, M)$ is a uniform spread and $f : N \longrightarrow M$ is a dense surjection with $N$ complete, it follows from Proposition 2.10 and Theorem 2.8 that the unique uniform spread extension $! : L \longrightarrow (N, \mathfrak{U}^! N)$ is a uniform spread completion of $h$.

For the rest of the paper, we need the following concept

**Definition 3.4.** A pair $(f, k)$ of frame homomorphisms is called a *spread homomorphism* of a spread $(L, h, M)$ into another spread $(H, t, K)$ if the following rectangle commutes:

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "f"'] & M \arrow[d, "k"] \\ H \arrow[r, "t"'] & K \end{tikzcd}$$

And whenever both $f$ and $g$ are frame isomorphisms,the pair $(f, g)$ is called a *spread isomorphism.*

**Proposition 3.5 (Extension Theorem).** Let $(k, l) : (L_1, h_1, M_1) \longrightarrow (L_2, h_2, K_2)$ be a uniform spread homomorphism, let $(g_i, N_i, f_i)$ be a uniform spread completion of $(L_i, h_i, M_i)$, for $i = 1, 2$. Then there exists a unique uniform homomorphism $m : N_1 \longrightarrow N_2$ such that
(a) $f_2 \circ m = l \circ f_1$, and
(b) $(k, m) : (L_1, g_1, N_1) \longrightarrow (L_2, g_2, N_2)$ is a uniform spread homomorphism:

$$\begin{tikzcd} N_1 \arrow[rrr, dashed, "m"] \arrow[dr, bend right, "f_1"'] & & & N_2 \arrow[dl, bend left, "f_2"] \\ & L_1 \arrow[ul, "g_1"'] \arrow[r, "k"] \arrow[d, "h_1"] & L_2 \arrow[ur, "g_2"] \arrow[d, "h_2"] & \\ & M_1 \arrow[r, "l"'] & M_2 & \end{tikzcd}$$

**Proof.** Both the uniform homomorphism $Cf_2$ and the dense surjection $\gamma_{N_1}$ are isomorphisms. Since $N_1$ is complete, a unique uniform spread $m : N_1 \longrightarrow N_2$ exists, where

$$m = \gamma_{N_2} \circ (Cf_2)^{-1} \circ Cl \circ Cf_1 \circ \gamma_{N_1}^{-1}$$

such that $f_2 \circ m = q$, where $q = l \circ \gamma_{M_1} \circ Cf_1 \circ \gamma_{N_1}^{-1}$:

$$\begin{tikzcd} N_1 \arrow[rrrr, dashed, "m"] \arrow[ddr, bend right, "f_1"'] & & & & N_2 \arrow[ddl, bend left, "f_2"] \\ & L_1 \arrow[ul, "g_1"'] \arrow[rr, "k"] \arrow[d, "h_1"] & & L_2 \arrow[ur, "g_2"] \arrow[d, "h_2"] & \\ CN_1 \arrow[uu, "\gamma_{N_1}"] \arrow[dr, "Cf_1"'] & M_1 \arrow[rr, "l"'] & & M_2 & CN_2 \arrow[uu, "\gamma_{N_2}"'] \arrow[dl, "Cf_2"] \\ & CM_1 \arrow[u, "\gamma_{M_1}"] \arrow[rr, "Cl"'] & & CM_2 \arrow[u, "\gamma_{M_2}"'] & \end{tikzcd}$$

Now commutativity of the left rectangle in the figure above yields

$$f_2 \circ m = l \circ (\gamma_{M_1} \circ Cf_1) \circ \gamma_{N_1}^{-1} = l \circ (f_1 \circ \gamma_{N_1}) \circ \gamma_{N_1}^{-1} = l \circ f_1$$

which takes care of (a). We easily check that

$$f_2 \circ (g_2 \circ k) = f_2 \circ (m \circ g_1)$$

from which a familiar result ensures that $g_2 \circ k = m \circ g_1$. Hence $(m, k)$ is a uniform spread homomorphism. That $m$ is unique with these properties follows from the fact that $f_2$ is a monomorphism. □

**Corollary 3.6.** If $(k, l)$ is a uniform spread isomorphism, so is $(m, k)$. □

**Theorem 3.7 (Uniqueness Theorem).** If $(g_i, N_i, f_i)$ is uniform spread completion of a uniform spread $(L, h, M)$ for $i = 1, 2$ then $(L, g_1, N_1) \cong (L, g_2, N_2)$.

**Proof.** In the diagram below, both triangles and the lower rectangle commute:

$$\begin{tikzcd}
N_1 \arrow[rrr, dashed, "m"] \arrow[drr, bend right, "f_1"'] & & & N_2 \arrow[dl, bend left, "f_2"] \\
& L \arrow[ul, "g_1"'] \arrow[r, "id_L"] \arrow[d, "h"] & L \arrow[ur, "g_2"] \arrow[d, "h"] & \\
& M \arrow[r, "id_M"'] & M &
\end{tikzcd}$$

From Proposition 3.6 a unique uniform spread $(N_1, m, N_2)$ such that $f_2 \circ m = id_M \circ f_1$ exists. Now $f_2 \circ (m \circ g_1) = f_2 \circ (g_2 \circ id_L)$ so that $m \circ g_1 = g_2 \circ id_L$. Thus the upper rectangle commutes. By the corollary, $m$ is an isomorphism, so $(L, g_1, N_1) \cong (L, g_2, N_2)$. □

Concerning the unique *fill-in* at the top of the following commutative diagram,

$$\begin{tikzcd}
CM \arrow[r, "Ch"] \arrow[d, "\gamma_M"'] & CL \arrow[d, "\gamma_L"] \\
M \arrow[r, "h"'] & L
\end{tikzcd}$$

assume that $(M, h, L)$ is a complete uniform spread, that $(CM, g, N)$ is a uniform spread and $f : N \longrightarrow CL$ is a dense surjection with $CL$ uniformly locally connected with respect to $N$ along $f$ such that $Ch = f \circ g$. Then the dense surjection $\gamma_M$ is an isomorphism, and $l = g \circ \gamma_M^{-1} : M \longrightarrow N$ is a uniform spread. Since $k = \gamma_L \circ f$ is a dense surjection with $k \circ l = h$, and $L$ is uniformly locally connected with respect to $N$ along $\gamma_L \circ f$, it follows by the completeness of $h$ that $\gamma_L \circ f$ is an isomorphism. This implies that $f$ is an isomorphism.

We then have

**Proposition 3.8.** If $(M, h, L)$ is a complete spread, then so is the induced uniform spread $(CM, Ch, CL)$; in particular, the dense surjection $\gamma_L$ is an isomorphism.

**Proof.** The second part follows from $h = \gamma_L \circ (Ch \circ \gamma_M^{-1})$. □

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Department of Mathematics, University of the North, Private Bag X1106, Sovenga, 0727, South Africa (e-mail: siweyah@unin.unorth.ac.za)

Department of Mathematics, University of Durban-Westville, Private Bag X54001, Durban, 4000, South Africa (e-mail: dbab@pixie.udw.ac.za)