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TREE TRANSFORMATIONS DEFINED BY GENERALIZED HYPERSUBSTITUTIONS

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Abstract. Sequences of tree transformations offer a convenient method to describe various manipulations that are commonly performed by compilers and language-based editors. If the considered tree transformations are described by certain mappings defined on the set of all terms, then sequences of tree transformations can be described by products of such mappings. In this paper we use extensions of generalized hypersubstitutions to define tree transformations. This allows us to describe algebraic properties of sets of tree transformations by algebraic properties of the set of all generalized hypersubstitutions.

1 Generalized Hypersubstitutions. By $\{f_i \mid i \in I\}$ we denote an indexed set of operation symbols of type τ where f_i is n_i-ary and $n_i \geq 1$. Let $W_\tau(X)$ be the set of all terms built up by elements of the alphabet $X = \{x_1, \ldots, x_n, \ldots,\}$ and by operation symbols from $\{f_i \mid i \in I\}$. Then an arbitrary mapping

$$\sigma : \{f_i \mid i \in i\} \to W_\tau(X)$$

is called a *generalized hypersubstitution of type* τ (see [Lee-D;00], [Den-L;00]). In [Den-L;00] the following concept of superposition

$$S^m : W_\tau(X)^{m+1} \to W_\tau(X), m \in \mathbf{N}, n \geq 1$$

of terms was defined for all $t_1, \ldots, t_m \in W_\tau(X)$:

(i) If $t = x_i, 1 \leq i \leq m$, then $S^m(x_i, t_1, \ldots, t_m) := t_i$.

(ii) If $t = x_i,\ \ i > m$, then $S^m(x_i, t_1, \ldots, t_m) := x_i$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$ for an $n_i - ary$ operation symbol f_i, then $S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

This kind of superposition means simply to substitute for each occurrence of the variable $x_i, 1 \leq i \leq m$, in the term t the corresponding term $t_i, 1 \leq i \leq m$. If a variable $x_j, j > m$, occurs in t, then nothing has to be substituted.

The extension

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$$

of a generalized hypersubstitution σ is defined inductively by the following steps:

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(i) $\hat{\sigma}[x_i] := x_i \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

This extension is uniquely determined and allows us to define a multiplication, denoted by $\circ_G$, on the set $Hyp_G(\tau)$ of all generalized hypersubstitutions of type τ by

$$\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2. \text{ (Here } \circ \text{ is the usual composition of functions.)}$$

Together with the identity hypersubstitution σ_{id} which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$, one obtains a monoid $(Hyp_G(\tau); \circ_G, \sigma_{id})$. If $Hyp(\tau)$ denotes the set of all arity-preserving hypersubstitutions of type τ then $Hyp(\tau) \subset Hyp_G(\tau)$ forms a submonoid, since the identity hypersubstitution preserves the arity and since the arity is preserved under the product $\circ_G$.

We mention that hypersubstitutions and generalized hypersubstitutions can be used to define the concepts of a *hyperidentity* and of a *strong hyperidentity* in a variety V of algebras of type τ.

Let V be a variety of type τ, let $\mathcal{F}_\tau(X) := (W_\tau(X); (\overline{f_i})_{i \in I})$ be the absolutely free algebra of type τ, and let $\mathcal{F}_V(X) := \mathcal{F}_\tau(X)/IdV$ be the algebra which is relatively free with respect to the variety V, where IdV denotes the set of all identities satisfied in the variety V. Sets Σ of equations of type τ for which there exist varieties V with $\Sigma = IdV$ are called *equational theories*. By $Alg(\tau)$ we denote the class of all algebras of type τ.

Definition 1.1 *Let σ be a generalized hypersubstitution of type τ and let V be a variety of type τ. Then*

$$T_\sigma^V := \{(t, t') \mid t, t' \in W_\tau(X) \text{ and } \hat{\sigma}[t] \approx t' \in IdV\}$$

is called a V-tree transformation defined by the generalized hypersubstitution σ.

Because of $\hat{\sigma}[x] = x$ for every variable $x \in X$ and for every tree transformation we have $\Delta_X \subseteq T_\sigma^V$ (where Δ_X is the diagonal on X). This definition can be generalized to those tree transformations which map terms of a given type to terms of another type. The most important tree transformations are those which are induced by tree transducers ([Gec-S; 84])and we may ask for the class of all tree transducers inducing tree transformations of the form T_σ^V.

2 V-proper and V-inner generalized hypersubstitutions. Our next aim is to consider tree transformations by using of generalized hypersubstitutions with respect to varieties V of algebras of type τ with operation symbols $(f_i)_{i \in I}$.

Definition 2.1 *Let V be a variety of type τ. A generalized hypersubstitution σ of type τ is called a V-proper generalized hypersubstitution if for every identity $s \approx t$ of V the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in V. Let $P_G(V)$ be the set of all V-proper generalized hypersubstitutions. (For the definition of proper hypersubstitutions see [Pło; 94]).)*

Definition 2.2 *Let V be a variety of type τ and assume that $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then*

$$\sigma_1 \sim_V \sigma_2 :\Leftrightarrow \sigma_1(f_i) \approx \sigma_2(f_i) \in Id\ V \text{ for all } i \in I.$$

Clearly, this relation is an equivalence relation on $Hyp_G(\tau)$, but in general it is not a congruence relation. In ([Den-L;01]) the following properties of this relation were proved:

Proposition 2.3 *([Den-L;01]) Let V be a variety of type τ and let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then the following hold:*

(i) *For all* $\sigma_1, \sigma_2 \in Hyp_G(\tau)$ *we have* $\sigma_1 \sim_V \sigma_2 \Leftrightarrow \hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ *for all* $t \in W_\tau(X)$.

(ii) *If* $\sigma_1 \sim_V \sigma_2$ *then* $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV \Leftrightarrow \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.

(iii) $P_G(V)$ *forms a submonoid of* $Hyp_G(\tau)$ *and the set* $P_G(V)$ *is a union of equivalence classes with respect to* $\sim_V$.

(iv) *If* $\sigma_1 \sim_V \sigma_2$ *then* $ker_V\sigma_1 = ker_V\sigma_2$ *(where* $ker_V\sigma := \{(s,t) \mid s,t \in W_\tau(X)$ *and* $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV\}$ *is called the semantical kernel of the generalized hypersubstitution* σ*).* ■

Definition 2.4 *Let* V *be a variety of type* τ *and assume that* $\sigma \in Hyp_G(\tau)$. *A generalized hypersubstitution* σ *of type* τ *is called a* V*-inner generalized hypersubstitution of type* τ *if* $\sigma \sim_V \sigma_{id}$. *Let* $P_G^0(V)$ *be the set of all* V*-inner generalized hypersubstitutions of type* τ.

Then we have:

Proposition 2.5 *Let* V *be a variety of type* τ. *Then*

(i) *If* σ *is a* V*-inner generalized hypersubstitution and if* t *is a term of type* τ *then* $\hat{\sigma}[t] \approx t \in IdV$.

(ii) $P_G^0(V)$ *forms a submonoid of* $P_G(V)$.

Proof. (i) We proceed by induction on the complexity of the term t. If $t = x \in X$, then $\hat{\sigma}[x] = x$ and thus $\hat{\sigma}[x] \approx x \in IdV$.
If $t = f_i(t_1, \ldots, t_{n_i})$ is a composed term and if $\hat{\sigma}[t_j] \approx t_j \in IdV, j = 1, \ldots, n_i$, then $\hat{\sigma}[t] = S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]) \approx S^{n_i}(f_i(x_1, \ldots, x_{n_i}), t_1, \ldots, t_{n_i}) \in IdV$. Here we used the equations $[\hat{\sigma}[t_j]]_{IdV} = [t_j]_{IdV}, [\sigma(f_i)]_{IdV} = [f_i(x_1, \cdots, x_{n_i})]_{IdV}$ and the definition of an operation $\tilde{S}^{n_i}$ corresponding to S^{n_i} on the quotient set $\mathcal{F}_\tau(X)/IdV$ which gives $[S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1]$
$, \ldots, \hat{\sigma}[t_{n_i}])]_{IdV} = [S^{n_i}(f_i(x_1, \ldots, x_{n_i}), t_1, \ldots, t_{n_i})]_{IdV}$. Then we have $\hat{\sigma}[t] \approx t \in IdV$.
(ii) Assume that $\sigma_1, \sigma_2 \in P_G^0(V)$. Then we get $(\sigma_1 \circ_G \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)] \approx \sigma_2(f_i) \approx f_i(x_1, \ldots, x_{n_i}) \in IdV$ for every $i \in I$ and this means $\sigma_1 \circ_G \sigma_2 \sim_V \sigma_{id}$, so $\sigma_1 \circ_G \sigma_2 \in P_G^0(V)$. ■

¿From the definition of T_σ^V and Definition 2.2 the following proposition can easily be derived:

Proposition 2.6 *Let* $\sigma_1, \sigma_2 \in Hyp_G(\tau)$ *and let* V *be a variety of type* τ. *Then* $\sigma_1 \sim_V \sigma_2$ *iff* $T_{\sigma_1}^V = T_{\sigma_2}^V$. ■

If $\sigma_1 \in P_G(V)$ and $\sigma_2 \in Hyp_G(\tau)$, then from $\hat{\sigma}_2[t] \approx t'' \in IdV$ and $\hat{\sigma}_1[t''] \approx t' \in IdV$ we obtain $\hat{\sigma}_1[\hat{\sigma}_2[t]] \approx \hat{\sigma}_1[t''] \approx t' \in IdV$ and therefore $(\sigma_1 \circ_G \sigma_2)\,\hat{}\,[t] \approx t' \in IdV$ and thus $(t,t') \in T_{\sigma_1 \circ_G \sigma_2}^V$. This shows $T_{\sigma_1}^V \circ T_{\sigma_2}^V \subseteq T_{\sigma_1 \circ_G \sigma_2}^V$. It is easy to see that the converse inclusion is also satisfied and therefore we have

$$T_{\sigma_1}^V \circ T_{\sigma_2}^V = T_{\sigma_1 \circ_G \sigma_2}^V \text{ if } \sigma_1 \in P_G(V).$$

We consider the set $\mathcal{T}_{P_G(V)} := \{T_\sigma^V \mid \sigma \in P_G(V)\}$ together with the relational product as binary operation and with $T_{\sigma_{id}}^V := \{(t,t') \mid t \approx t' \in IdV\} = IdV$ as identity element. Then we get

Proposition 2.7 *The monoid $(\mathcal{T}_{P_G(V)};\circ,T^V_{\sigma_{id}})$ is a homomorphic image of $(P_G(V);\circ_G,\sigma_{id})$. A homomorphism is given by $\varphi(\sigma)=T^V_\sigma$ for every $\sigma\in P_G(V)$ and the kernel of this homomorphism is $\sim_V$. Therefore $(\mathcal{T}_{P_G(V)};\circ,T^V_{\sigma_{id}})$ is isomorphic to the quotient monoid $P_G(V)/\sim_V$. (Here $\sim_V$ denotes the restriction of the relation defined in 2.2 to the set $P_G(V)$.)* ∎

This proposition shows also that $\sim_V$ is a congruence relation on the monoid $P_G(V)$. If every generalized hypersubstitution from $Hyp_G(\tau)$ is V-proper, i.e. if $P_G(V)=Hyp(\tau)$, then V is called strongly solid. In this case for every identity $s\approx t$ in V and for every generalized hypersubstitution $\sigma\in Hyp_G(\tau)$, the equation $\hat\sigma[s]\approx\hat\sigma[t]$ is also an identity in V. The set of all strongly solid varieties of type τ forms a complete sublattice of the lattice of all varieties of type τ. Clearly, if V is strongly solid, then $P_G(V)=Hyp_G(\tau)$ and then $(\mathcal{T}_{Hyp_G(\tau)};\circ,T^V_{\sigma_{id}})$ is isomorphic to the monoid $Hyp_G(\tau)/\sim_V$.

3 Properties of Tree Transformations defined by Generalized Hypersubstitutions. Let σ be a V-proper generalized hypersubstitution of type τ and let T^V_σ be the tree transformation defined by σ. In this section we want to find out in which way properties of tree-transformations and properties of hypersubstitutions are related to each other.

Theorem 3.1 *Let V be a variety of type τ and let $\sigma\in P_G(V)$ be a V-proper generalized hypersubstitution of type τ. Then*

(i) T^V_σ is reflexive iff $\sigma\sim_V\sigma_{id}$,

(ii) T^V_σ is symmetric iff $\sigma\circ_G\sigma\sim_V\sigma_{id}$,

(iii) T^V_σ is transitive iff $\sigma\circ_G\sigma\sim_V\sigma$.

Proof. (i) If T^V_σ is reflexive, then we get $\hat\sigma[t]\approx t\in IdV$ for all $t\in W_\tau(X)$. This is valid also for $t=f_i(x_1,\dots,x_{n_i}), i\in I$ and then $\hat\sigma[f_i(x_1,\dots,x_{n_i})]\approx f_i(x_1,\dots,x_{n_i})\in IdV$, i.e. $\hat\sigma[f_i(x_1,\dots,x_{n_i})]\approx\hat\sigma_{id}[f_i(x_1,\dots,x_{n_i})]$ and $\sigma\sim_V\sigma_{id}$. If conversely, $\sigma\sim_V\sigma_{id}$, then we have $\hat\sigma[t]\approx t\in IdV$ for all $t\in W_\tau(X)$, but this means $(t,t)\in T^V_\sigma$ and then T^V_σ is reflexive.
(ii) Since T^V_σ is symmetric, we have $T^V_\sigma=(T^V_\sigma)^{-1}$ and this means $(t,t')\in T^V_\sigma$ iff $(t,t')\in(T^V_\sigma)^{-1}$. Therefore, $\hat\sigma[t]\approx t'\in IdV$ iff $\hat\sigma[t']\approx t\in IdV$. Since σ is V-proper, we obtain $\hat\sigma[\hat\sigma[t]]\approx\hat\sigma[t']\approx t\in IdV$ and this means $\sigma\circ_G\sigma\sim_V\sigma_{id}$.
If, conversely $\sigma\circ_G\sigma\sim_V\sigma_{id}$ and $(t,t')\in T^V_\sigma$, then $\hat\sigma[t]\approx t'\in IdV$ and since σ is V-proper, we have $\hat\sigma[\hat\sigma[t]]\approx\hat\sigma[t']\in IdV$. Now we use $\sigma\circ_G\sigma\sim_V\sigma_{id}$ and obtain $t\approx\hat\sigma[t']\in IdV$ and this means $(t',t)\in T^V_\sigma$ or $(t,t')\in(T^V_\sigma)^{-1}$. This shows $T^V_\sigma\subseteq(T^V_\sigma)^{-1}$ and the opposite inclusion can be shown similarly.
(iii) If T^V_σ is transitive, then $T^V_\sigma\circ T^V_\sigma\subseteq T^V_\sigma$ and therefore $T^V_\sigma\circ T^V_\sigma=T^V_{\sigma\circ_G\sigma}\subseteq T^V_\sigma$. This means, if $(t,t')\in T^V_{\sigma\circ_G\sigma}$, i.e. $t'\approx(\hat\sigma\circ\sigma)\hat{}[t]\in IdV$ then $(t,t')\in T^V_\sigma$, i.e. $t'\approx\hat\sigma[t]\in IdV$. But then $(\hat\sigma\circ\sigma)\hat{}[t]\approx\hat\sigma[t]\in IdV$ for all $t\in W_\tau(X)$ and therefore $\sigma\circ_G\sigma\sim_V\sigma$.
If conversely $\sigma\circ_G\sigma\sim_V\sigma$ and $(t,t')\in T^V_\sigma\circ T^V_\sigma$, then $(\sigma\circ_G\sigma)\,\hat{}[t]\approx t'\in IdV$ and $(\sigma\circ_G\sigma)\,\hat{}[t]\approx\hat\sigma[t]\in IdV$. This means, $\hat\sigma[t]\approx t'\in IdV$ and so $(t,t')\in T^V_\sigma$. Therefore, $T^V_\sigma\circ T^V_\sigma\subseteq T^V_\sigma$. Thus T^V_σ is transitive. ∎

Assume that $(t,t')\in T^V_\sigma\circ(T^V_\sigma)^{-1}$. Then there is a term $t''\in W_\tau(X)$ such that $(t,t'')\in(T^V_\sigma)^{-1}$ and $(t'',t')\in T^V_\sigma$ and this means, there is a term $t''\in W_\tau(X)$ such that $\hat\sigma[t'']\approx t\in IdV$ and $\hat\sigma[t'']\approx t'\in IdV$ and thus $t\approx t'\in IdV$. This shows that $T^V_\sigma\circ(T^V_\sigma)^{-1}\subseteq IdV$.
Assume now that T^V_σ is surjective. We want to show the equality $T^V_\sigma\circ(T^V_\sigma)^{-1}=IdV$,

i.e. we have only to show the inclusion $IdV \subseteq T_\sigma^V \circ (T_\sigma^V)^{-1}$. Assume that $t \approx t' \in IdV$. Since T_σ^V is surjective, to t' there is a term t'' such that $(t'', t') \in T_\sigma^V$, i.e. such that $\hat{\sigma}[t''] \approx t' \in IdV$. Then we have also $\hat{\sigma}[t''] \approx t \in IdV$ and $(t'', t) \in T_\sigma^V$, i.e. $(t, t'') \in (T_\sigma^V)^{-1}$. Altogether, this gives $(t, t') \in T_\sigma^V \circ (T_\sigma^V)^{-1}$ and thus $IdV \subseteq T_\sigma^V \circ (T_\sigma^V)^{-1}$.
This shows that for surjective tree transformations we have $T_\sigma^V \circ (T_\sigma^V)^{-1} = IdV$.
To show the converse, we assume now that $T_\sigma^V \circ (T_\sigma^V)^{-1} = IdV$. Let $t \in W_\tau(X)$ be an arbitrary term of $W_\tau(X)$. We have to show that there is a term $t' \in W_\tau(X)$ with $\hat{\sigma}[t'] \approx t \in IdV$. ¿From $t \approx t \in IdV = T_\sigma^V \circ (T_\sigma^V)^{-1}$ we obtain the existence of a term $t' \in W_\tau(X)$ such that $(t', t) \in T_\sigma^V$, and this shows surjectivity.
If $t = t'$ then $\hat{\sigma}[t] = \hat{\sigma}[t']$ and thus with $t'' := \hat{\sigma}[t]$ we have $\hat{\sigma}[t] \approx t'' \in IdV$ and $\hat{\sigma}[t'] \approx t'' \in IdV$, $\hat{\sigma}[t'] \approx t'' \in IdV$. Thus $(t, t'') \in T_\sigma^V$ and $(t', t'') \in T_\sigma^V$ and therefore $(t, t') \in (T_\sigma^V)^{-1} \circ T_\sigma^V$, so $\triangle_{W_\tau(X)} \subseteq (T_\sigma^V)^{-1} \circ T_\sigma^V$. Assume that T_σ^V is injective. Then from $(t, t'') \in T_\sigma^V$ and $(t', t'') \in T_\sigma^V$ there follows $t = t'$, i.e. from $(t, t') \in (T_\sigma^V)^{-1} \circ T_\sigma^V$ it follows $(t, t') \in \triangle_{W_\tau(X)}$ and thus $(T_\sigma^V)^{-1} \circ T_\sigma^V \subseteq \triangle_{W_\tau(X)}$. So, we see that in the case that T_σ^V is injective, we have $(T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W_\tau(X)}$ and that this equation forces T_σ^V to be injective. Altogether we have

Proposition 3.2 *Let V be a variety of type τ and let $\sigma \in Hyp_G(\tau)$ be a generalized hypersubstitution. Then*

(i) T_σ^V is surjective iff $T_\sigma^V \circ (T_\sigma^V)^{-1} = IdV$

(ii) T_σ^V is injective iff $(T_\sigma^V)^{-1} \circ T_\sigma^V = \triangle_{W\tau(X)}$. ■

Clearly, for the kernel $ker_V\sigma$ we have $ker_V\sigma = (T_\sigma^V)^{-1} \circ T_\sigma^V$. In the case that T_σ^V is injective, i.e. if $ker_V\sigma = \triangle_{W_\tau(X)}$, it is an equational theory since $\triangle_{W_\tau(X)}$ is the set of all equations which are satisfied in every algebra of type τ.

An answer when $ker_V\sigma$ is an equational theory can be found in [Den-L;01].

Theorem 3.3 *([Den-L;01]) Let σ be a generalized hypersubstitution of type τ. Then the semantical kernel $ker_V\sigma$ of σ with respect to a nontrivial variety V is an equational theory iff σ maps no operation symbol f_i (which is n_i-ary) to a variable different from $x_1, \cdots, x_{n_i}$.* ■

4 Tree Transformations with respect to Strongly Solid Varieties of Semigroups.

We mentioned already that an identity $s \approx t$ in a variety V is called a strong hyperidentity if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in V for every generalized hypersubstitution σ of type τ and that a variety is called strongly solid if every identity in V is a strong hyperidentity in V. By Definition 2.1 for a strongly solid variety we have $P_G(V) = Hyp_G(\tau)$ and in this case instead of the monoid $\mathcal{T}_{P_G(V)}$ we can consider $\mathcal{T}_{Hyp_G(\tau)}$. By Proposition 2.7 this monoid is isomorphic to the quotient monoid $Hyp_G(\tau)/\sim_V$. In [Den-L; 00] all strongly solid varieties of semigroups were determined.

Theorem 4.1 *([Den-L;00]) The varieties*
$Rec := Mod(\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3\})$
and
$V_{big} := Mod(\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1x_2^2 \approx x_1^2x_2 \approx x_1x_2, x_1x_2x_3x_4 \approx x_1x_3x_2x_4\})$
are the only nontrivial strongly solid varieties of semigroups. ■

Our aim is to apply the results of chapter 3 to these varieties. We start with the variety Rec. It is easy to see that
$Hyp_G(2)/\sim_{Rec}= \{[\sigma_{x_1}]_{\sim_{Rec}}, [\sigma_{x_2}]_{\sim_{Rec}}, [\sigma_{x_1x_2}]_{\sim_{Rec}}, [\sigma_{x_2x_1}]_{\sim_{Rec}}, [\sigma_{x_1^2}]_{\sim_{Rec}}, [\sigma_{x_2^2}]_{\sim_{Rec}}\} \cup \{[\sigma_{x_i}]_{\sim_{Rec}} \mid i > 2\} \cup \{[\sigma_{x_ix_j}]_{\sim_{Rec}} \mid i \text{ or } j \text{ is greater than } 2\}$. Here σ_t for a term $t \in W_{(2)}(X)$ denotes the hypersubstitution which maps the binary operation symbol to the term t. Instead of the congruence classes we may consider only its representatives. The multiplication $\circ_G$ is described by the following table (here the classes are denoted by their representatives).

$\circ_G$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1x_2}$	$\sigma_{x_2x_1}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	σ_{x_i}, $(i > 2)$	$\sigma_{x_ix_j}$, $(i \text{ or } j > 2)$
σ_{x_1}	σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_i}, $(i > 2)$	σ_{x_i}
σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_2}	σ_{x_1}	σ_{x_1}	σ_{x_2}	σ_{x_i}, $(i > 2)$	σ_{x_j}, $(j > 2)$
$\sigma_{x_1x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1x_2}$	$\sigma_{x_2x_1}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	σ_{x_i}, $(i > 2)$	$\sigma_{x_ix_j}$, $(i \text{ or } j > 2)$
$\sigma_{x_2x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_2x_1}$	$\sigma_{x_1x_2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	σ_{x_i}, $(i > 2)$	$\sigma_{x_jx_i}$, $(i \text{ or } j > 2)$
$\sigma_{x_1^2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	σ_{x_i}, $(i > 2)$	$\sigma_{x_i^2}$
$\sigma_{x_2^2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	σ_{x_i}, $(i > 2)$	$\sigma_{x_j^2}$, $(j > 2)$
σ_{x_i}, $(i > 2)$	σ_{x_1}	σ_{x_2}	σ_{x_i}	σ_{x_i}	σ_{x_i}	σ_{x_i},	σ_{x_i}, $(i > 2)$	σ_{x_i}
$\sigma_{x_ix_j}$, $(i \text{ or } j > 2)$	σ_{x_1}	σ_{x_2}	$\sigma_{x_ix_j}$				σ_{x_i}, $(i > 2)$	

For the products which do not occur in the table we have

$$\sigma_{x_ix_j} \circ_G \sigma_{x_2x_1} = \begin{cases} \sigma_{x_ix_2} & if \quad i > 2, j = 1 \\ \sigma_{x_ix_1} & if \quad i > 2, j = 2 \\ \sigma_{x_2x_j} & if \quad i = 1, j > 2 \\ \sigma_{x_1x_j} & if \quad i = 2, j > 2 \\ \sigma_{x_ix_j} & if \quad i, j > 2, \end{cases}$$

$$\sigma_{x_ix_j} \circ_G \sigma_{x_1^2} = \begin{cases} \sigma_{x_ix_1} & if \quad i > 2, j = 1 \\ \sigma_{x_ix_1} & if \quad i > 2, j = 2 \\ \sigma_{x_1x_j} & if \quad i = 1, j > 2 \\ \sigma_{x_1x_j} & if \quad i = 2, j > 2 \\ \sigma_{x_ix_j} & if \quad i, j > 2, \end{cases}$$

$$\sigma_{x_ix_j} \circ_G \sigma_{x_2^2} = \begin{cases} \sigma_{x_ix_2} & if \quad i > 2, j = 1 \\ \sigma_{x_ix_2} & if \quad i > 2, j = 2 \\ \sigma_{x_2x_j} & if \quad i = 1, j > 2 \\ \sigma_{x_2x_j} & if \quad i = 2, j > 2 \\ \sigma_{x_ix_j} & if \quad i, j > 2, \end{cases}$$

$$\sigma_{x_i x_j} \circ_G \sigma_{x_i x_j} = \begin{cases} \sigma_{x_i^2} & if \quad i > 2, j = 1 \\ \sigma_{x_i x_j} & if \quad i > 2, j = 2 \\ \sigma_{x_i x_j} & if \quad i = 1, j > 2 \\ \sigma_{x_j^2} & if \quad i = 2, j > 2 \\ \sigma_{x_i x_j} & if \quad i, j > 2. \end{cases}$$

Proposition 2.7 and the fact that the variety Rec is strongly solid imply that the monoid $(\mathcal{T}_{P_G(Rec)}; \circ, T_{id})$ is isomorphic to $(Hyp(2)/\sim_{Rec}; \circ_G, \sigma_{id})$. Our table shows that the set of all idempotent elements is given by

$Idem = \{\sigma_{x_i} \mid x_i \in X\} \cup \{\sigma_{x_i^2} \mid x_i \in X\} \cup \{\sigma_{x_i x_2} \mid i > 2, x_i \in X\} \cup \{\sigma_{x_1 x_j} \mid j > 2, x_j \in X\} \cup \{\sigma_{x_i x_j} \mid i, j > 2, i \neq j, x_i, x_j \in X\} \cup \{\sigma_{x_1 x_2}\}$.

Further we see that there are no left-zero elements. Every generalized hypersubstitution of the form σ_{x_i} (for an arbitrary variable $x_i \in X$) is a right-zero element. Now we want to describe the tree transformations corresponding to these hypersubstitutions. By $leftmost(t)$ and by $rightmost(t)$ we denote the first and the last variable, respectively, of the term t. $T^{Rec}_{\sigma_{x_1}} = \{(t, x_k) \mid t \in W_{(2)}(X) \text{ and } \text{leftmost}(t) = x_k\}$,

$T^{Rec}_{\sigma_{x_2}} = \{(t, x_k) \mid t \in W_{(2)}(X) \text{ and } \text{rightmost}(t) = x_k\}$,

$T^{Rec}_{\sigma_{x_1 x_2}} = \{(t, t) \mid t \in W_{(2)}(X)\} = \triangle_{W_{(2)}(X)}$,

$T^{Rec}_{\sigma_{x_2 x_1}} = \{(t, t^d) \mid t \in W_{(2)}(X) \text{ where } t^d \text{ is the term dual to } t\}$,

(The dual term t^d is defined inductively by $x^d := x$ for variables and $f(t_1, t_2)^d := f(t_2^d, t_1^d)$.)

$T^{Rec}_{\sigma_{x_1^2}} = \{(t, x_k^2) \mid t \in W_{(2)}(X) \text{ and } \text{leftmost}(t) = x_k \text{ if } t \notin X\} \cup X^2$,

$T_{\sigma_{x_2^2}} = \{(t, x_k^2) \mid t \in W_{(2)}(X) \text{ and } \text{rightmost}(t) = x_k \text{ if } t \notin X\} \cup X^2$,

$T^{Rec}_{\sigma_{x_i}} = \{(t, x_i) \mid t \in W_{(2)}(X) \text{ and if } t \notin X\} \cup X^2$, if $i > 2$.

For $i = 1$ and $j > 2$ we have

$T^{Rec}_{\sigma_{x_1 x_j}} = \{(t, x_k x_j) \mid t \in W_{(2)}(X) \text{ and } \text{leftmost}(t) = x_k \text{ if } t \notin X\} \cup X^2$,

for $i = 2$ and $j > 2$ we have

$T^{Rec}_{\sigma_{x_2 x_j}} = \{(t, x_k x_j) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and } \text{rightmost}(t) = x_k\} \cup X^2$,

for $i > 2$ and $j = 1$ we have

$T^{Rec}_{\sigma_{x_i x_1}} = \{(t, x_i x_k) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and } \text{leftmost}(t) = x_k\} \cup X^2$,

for $i > 2$ and $j = 2$ we have

$T^{Rec}_{\sigma_{x_i x_2}} = \{(t, x_i x_k) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and } \text{rightmost}(t) = x_k\} \cup X^2$,

and for $i, j > 2$

$T^{Rec}_{\sigma_{x_i x_j}} = \{(t, x_i x_j) \mid t \in W_{(2)}(X) \text{ and } t \notin X\} \cup X^2$.

As a second example now we consider the variety V_{big}. Using the identities of V_{big} we determine all elements of $Hyp(2)/\sim_{V_{big}}$. Instead of the congruence classes we will give a system of representatives and obtain

$Hyp(2)/\sim_{V_{big}} = \{\sigma_{x_1}, \sigma_{x_2}\} \cup \{\sigma_{x_1^2}, \sigma_{x_2^2}, \sigma_{x_1 x_2}, \sigma_{x_2 x_1}, \sigma^2_{x_j} (j > 2)\} \cup \{\sigma_{x_1 x_2 x_1}, \sigma_{x_2 x_1 x_2}\}$
$\cup \{\sigma_{x_1 x_{i_1} \cdots x_{i_k}} \mid i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_2 x_{i_1} \cdots x_{i_k}} \mid i_1, \ldots, i_k > 2, k \geq 1\}$
$\cup \{\sigma_{x_1 x_2 x_{i_1} \cdots x_{i_k}} \mid i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_2 x_1 x_{i_1} \cdots x_{i_k}} \mid i_1, \ldots, i_k > 2, k \geq 1\}$
$\cup \{\sigma_{x_{i_1} \cdots x_{i_k} x_1} \mid i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_{i_1} \cdots x_{i_k} x_2} \mid i_1, \ldots, i_k > 2, k \geq 1\}$
$\cup \{\sigma_{x_{i_1} \cdots x_{i_k} x_1 x_2} \mid i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_{i_1} \cdots x_{i_k} x_2 x_1} \mid i_1, \ldots, i_k > 2, k \geq 1\}$
$\cup \{\sigma_{x_1 x_{i_1} \cdots x_{i_k} x_1} \mid i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_2 x_{i_1} \cdots x_{i_k} x_2} \mid i_1, \ldots, i_k > 2, k \geq 1\}$
$\cup \{\sigma_{x_1 x_{i_1} \cdots x_{i_k} x_2} \mid i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_2 x_{i_1} \cdots x_{i_k} x_1} \mid i_1, \ldots, i_k > 2, k \geq 1\}$
$\cup \{\sigma_{x_1 x_2 x_{i_1} \cdots x_{i_k} x_1} \mid i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_2 x_1 x_{i_1} \cdots x_{i_k} x_2} \mid i_1, \ldots, i_k > 2, k \geq 1\}$
$\cup \{\sigma_{x_i x_1 x_2 x_{i_1} \cdots x_{i_k}} \mid i, i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_i x_1 x_{i_1} \cdots x_{i_k}} \mid i, i_1, \ldots, i_k > 2, k \geq 1\}$

$\cup\, \{\sigma_{x_i x_2 x_{i_1} \cdots x_{i_k}} \mid i, i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_{i_1} \cdots x_{i_k}} \mid i_1, \ldots, i_k > 2, k \geq 1\}.$

We set $m := x_{i_1} \cdots x_{i_k}$ and let $j \in \{i, i_1, \ldots, i_k\}$. The multiplication in the monoid $Hyp(2)/\sim_{V_{big}}$ is given by the following tables.

$\circ_G$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1 x_2}$	$\sigma_{x_2 x_1}$	$\sigma_{x_j^2}$	$\sigma_{x_1 x_2 x_1}$
σ_{x_1}	σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_j}	σ_{x_1}
σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_2}	σ_{x_1}	σ_{x_j}	σ_{x_1}
$\sigma_{x_1^2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_j^2}$	$\sigma_{x_1^2}$
$\sigma_{x_2^2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_j^2}$	$\sigma_{x_1^2}$
$\sigma_{x_1 x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1 x_2}$	$\sigma_{x_2 x_1}$	$\sigma_{x_j^2}$	$\sigma_{x_1 x_2 x_1}$
$\sigma_{x_2 x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_2 x_1}$	$\sigma_{x_1 x_2}$	$\sigma_{x_j^2}$	$\sigma_{x_1 x_2 x_1}$
$\sigma_{x_j^2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$
$\sigma_{x_1 x_2 x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1 x_2 x_1}$	$\sigma_{x_2 x_1 x_2}$	$\sigma_{x_j^2}$	$\sigma_{x_1 x_2 x_1}$
$\sigma_{x_2 x_1 x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_2 x_1 x_2}$	$\sigma_{x_1 x_2 x_1}$	$\sigma_{x_j^2}$	$\sigma_{x_1 x_2 x_1}$
$\sigma_{x_1 m}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_j m}$	$\sigma_{x_1 m}$
$\sigma_{x_2 m}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_2 m}$	$\sigma_{x_1 m}$	$\sigma_{x_j m}$	$\sigma_{x_1 m}$
$\sigma_{x_1 x_2 m}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_1 x_2 m}$	$\sigma_{x_2 x_1 m}$	$\sigma_{x_j m}$	$\sigma_{x_1 x_2 m}$
$\sigma_{x_2 x_1 m}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_2 x_1 m}$	$\sigma_{x_1 x_2 m}$	$\sigma_{x_j m}$	$\sigma_{x_1 x_2 m}$
$\sigma_{m x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{m x_1}$	$\sigma_{m x_2}$	$\sigma_{m x_1}$	$\sigma_{m x_2}$	$\sigma_{m x_j}$	$\sigma_{m x_1}$
$\sigma_{m x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{m x_1}$	$\sigma_{m x_2}$	$\sigma_{m x_2}$	$\sigma_{m x_1}$	$\sigma_{m x_j}$	$\sigma_{m x_1}$
$\sigma_{m x_1 x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{m x_1}$	$\sigma_{m x_2}$	$\sigma_{m x_1 x_2}$	$\sigma_{m x_2 x_1}$	$\sigma_{m x_j}$	$\sigma_{m x_2 x_1}$
$\sigma_{m x_2 x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{m x_1}$	$\sigma_{m x_2}$	$\sigma_{m x_2 x_1}$	$\sigma_{m x_1 x_2}$	$\sigma_{m x_j}$	$\sigma_{m x_2 x_1}$
$\sigma_{x_1 m x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_j m x_j}$	$\sigma_{x_1 m x_1}$
$\sigma_{x_2 m x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_1 m x_1}$	$\sigma_{x_j m x_j}$	$\sigma_{x_1 m x_1}$
$\sigma_{x_1 m x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_1 m x_2}$	$\sigma_{x_2 m x_1}$	$\sigma_{x_j m x_j}$	$\sigma_{x_1 m x_2 x_1}$
$\sigma_{x_2 m x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_2 m x_1}$	$\sigma_{x_1 m x_2}$	$\sigma_{x_j m x_j}$	$\sigma_{x_1 m x_2 x_1}$
$\sigma_{x_1 x_2 m x_1}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_1 x_2 m x_1}$	$\sigma_{x_2 x_1 m x_2}$	$\sigma_{x_j m x_j}$	$\sigma_{x_1 x_2 m x_1}$
$\sigma_{x_2 x_1 m x_2}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_2 x_1 m x_2}$	$\sigma_{x_1 x_2 m x_1}$	$\sigma_{x_j m x_j}$	$\sigma_{x_1 x_2 m x_1}$
$\sigma_{x_i x_1 x_2 m}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_i x_1 m}$	$\sigma_{x_i x_2 m}$	$\sigma_{x_i x_1 x_2 m}$	$\sigma_{x_i x_2 x_1 m}$	$\sigma_{x_i x_j m}$	$\sigma_{x_i x_1 x_2 m}$
$\sigma_{x_i x_1 m}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_i x_1 m}$	$\sigma_{x_i x_2 m}$	$\sigma_{x_i x_1 m}$	$\sigma_{x_i x_2 m}$	$\sigma_{x_i x_j m}$	$\sigma_{x_i x_1 m}$
$\sigma_{x_i x_2 m}$	σ_{x_1}	σ_{x_2}	$\sigma_{x_i x_1 m}$	$\sigma_{x_i x_2 m}$	$\sigma_{x_i x_2 m}$	$\sigma_{x_i x_1 m}$	$\sigma_{x_i x_j m}$	$\sigma_{x_i x_1 m}$
σ_{m}	σ_{x_1}	σ_{x_2}	σ_{m}	σ_{m}	σ_{m}	σ_{m}	σ_{m}	σ_{m}

$\circ_G$	$\sigma_{x_2 x_1 x_2}$	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_1 x_2 m}$	$\sigma_{x_2 x_1 m}$	$\sigma_{m x_1}$
σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	$\sigma_{x_{i_1}}$
σ_{x_2}	σ_{x_2}	$\sigma_{x_{i_k}}$	$\sigma_{x_{i_k}}$	$\sigma_{x_{i_k}}$	$\sigma_{x_{i_k}}$	σ_{x_1}
$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_{i_1}^2}$
$\sigma_{x_2^2}$	$\sigma_{x_2^2}$	$\sigma_{x_{i_k}^2}$	$\sigma_{x_{i_k}^2}$	$\sigma_{x_{i_k}^2}$	$\sigma_{x_{i_k}^2}$	$\sigma_{x_1^2}$
$\sigma_{x_1 x_2}$	$\sigma_{x_2 x_1 x_2}$	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_1 x_2 m}$	$\sigma_{x_2 x_1 m}$	$\sigma_{m x_1}$
$\sigma_{x_2 x_1}$	$\sigma_{x_2 x_1 x_2}$	$\sigma_{m^d x_1}$	$\sigma_{m^d x_2}$	$\sigma_{m^d x_2 x_1}$	$\sigma_{m^d x_1 x_2}$	$\sigma_{x_1 m^d}$
$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$
$\sigma_{x_1 x_2 x_1}$	$\sigma_{x_2 x_1 x_2}$	$\sigma_{x_1 m x_1}$	$\sigma_{x_2 m x_2}$	$\sigma_{x_1 x_2 m}$	$\sigma_{x_2 m x_1 x_2}$	$\sigma_{m x_1 x_{i_1}}$
$\sigma_{x_2 x_1 x_2}$	$\sigma_{x_2 x_1 x_2}$	$\sigma_{m^d x_1 x_{i_k}}$	$\sigma_{m^d x_2 x_{i_k}}$	$\sigma_{m^d x_1 x_2 x_{i_k}}$	$\sigma_{m^d x_2 x_1 x_{i_k}}$	$\sigma_{x_1 m^d x_1}$
$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	$\sigma_{x_1 m}$	$\sigma_{x_2 m}$	σ_{m}
$\sigma_{x_2 m}$	$\sigma_{x_2 m}$	$\sigma_{x_{i_k} m}$	$\sigma_{x_{i_k} m}$	$\sigma_{x_{i_k} m}$	$\sigma_{x_{i_k} m}$	$\sigma_{x_1 m}$

$\circ_G$	$\sigma_{x_2x_1x_2}$	σ_{x_1m}	σ_{x_2m}	$\sigma_{x_1x_2m}$	$\sigma_{x_2x_1m}$	σ_{mx_1}
$\sigma_{x_1x_2m}$	$\sigma_{x_2x_1m}$	σ_{x_1m}	σ_{x_2m}	$\sigma_{x_1x_2m}$	$\sigma_{x_2x_1m}$	$\sigma_{mx_1x_{i_k}}$
$\sigma_{x_2x_1m}$	$\sigma_{x_2x_1m}$	$\sigma_{m^dx_1x_{i_k}}$	$\sigma_{m^dx_2x_{i_k}}$	$\sigma_{m^dx_2x_1x_{i_k}}$	$\sigma_{m^dx_1x_2x_{i_k}}$	σ_{x_1m}
σ_{mx_1}	σ_{mx_2}	σ_{mx_1}	σ_{mx_2}	σ_{mx_1}	σ_{mx_2}	$\sigma_{mx_{i_1}}$
σ_{mx_2}	σ_{mx_2}	σ_m	σ_m	σ_m	σ_m	σ_{mx_1}
$\sigma_{mx_1x_2}$	$\sigma_{mx_1x_2}$	$\sigma_{mx_1x_{i_k}}$	$\sigma_{mx_2x_{i_k}}$	$\sigma_{mx_1x_2x_{i_k}}$	$\sigma_{mx_2x_1x_{i_k}}$	σ_{mx_1}
$\sigma_{mx_2x_1}$	$\sigma_{mx_1x_2}$	σ_{mx_1}	σ_{mx_2}	$\sigma_{mx_2x_1}$	$\sigma_{mx_1x_2}$	$\sigma_{mx_1x_{i_1}}$
$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_1}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{mx_{i_1}}$
$\sigma_{x_2mx_2}$	$\sigma_{x_2mx_2}$	$\sigma_{x_{i_k}m}$	$\sigma_{x_{i_k}m}$	$\sigma_{x_{i_k}m}$	$\sigma_{x_{i_k}m}$	$\sigma_{x_1mx_1}$
$\sigma_{x_1mx_2}$	$\sigma_{x_2mx_1x_2}$	σ_{x_1m}	σ_{x_2m}	$\sigma_{x_1x_2m}$	$\sigma_{x_2x_1m}$	σ_{mx_1}
$\sigma_{x_2mx_1}$	$\sigma_{x_2mx_1x_2}$	$\sigma_{m^dx_1}$	$\sigma_{m^dx_2}$	$\sigma_{m^dx_2x_1}$	$\sigma_{m^dx_1x_2}$	$\sigma_{x_1m^d}$
$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{mx_1x_{i_1}}$
$\sigma_{x_2x_1mx_2}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{m^dx_1x_{i_k}}$	$\sigma_{m^dx_2x_{i_k}}$	$\sigma_{m^dx_2x_1x_{i_k}}$	$\sigma_{m^dx_1x_2x_{i_k}}$	$\sigma_{x_1mx_1}$
$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_2x_1m}$	$\sigma_{x_ix_1m}$
$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	σ_{x_im}
$\sigma_{x_ix_2m}$	$\sigma_{x_ix_2m}$	σ_{x_im}	σ_{x_im}	σ_{x_im}	σ_{x_im}	$\sigma_{x_ix_1m}$
σ_m	σ_m	σ_m	σ_m	σ_m	σ_m	σ_m

$\circ_G$	σ_{mx_2}	$\sigma_{mx_1x_2}$	$\sigma_{mx_2x_1}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_2}$
σ_{x_1}	$\sigma_{x_{i_1}}$	$\sigma_{x_{i_1}}$	$\sigma_{x_{i_1}}$	σ_{x_1}	σ_{x_2}	σ_{x_1}
σ_{x_2}	σ_{x_2}	σ_{x_2}	σ_{x_1}	σ_{x_1}	σ_{x_2}	σ_{x_2}
$\sigma_{x_1^2}$	$\sigma_{x_{i_1^2}}$	$\sigma_{x_{i_1}^2}$	$\sigma_{x_{i_1}^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$
$\sigma_{x_2^2}$	$\sigma_{x_2^2}$	$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_2^2}$
$\sigma_{x_1x_2}$	$\sigma_{x_1mx_2}$	$\sigma_{mx_1x_2}$	$\sigma_{mx_2x_1}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_2}$
$\sigma_{x_2x_1}$	$\sigma_{x_2m^d}$	$\sigma_{x_2x_1m^d}$	$\sigma_{x_1x_2m^d}$	$\sigma_{x_1m^dx_1}$	$\sigma_{x_2m^dx_2}$	$\sigma_{x_2m^dx_1}$
$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$
$\sigma_{x_1x_2x_1}$	$\sigma_{mx_2x_{i_1}}$	$\sigma_{mx_1x_2x_{i_1}}$	$\sigma_{mx_2x_1x_{i_1}}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_2x_1}$
$\sigma_{x_2x_1x_2}$	$\sigma_{x_2m^dx_2}$	$\sigma_{x_2m^dx_1x_2}$	$\sigma_{x_1m^dx_2x_1}$	$\sigma_{x_1m^dx_1}$	$\sigma_{x_2m^dx_2}$	$\sigma_{x_2m^dx_1x_2}$
σ_{x_1m}	σ_m	σ_m	σ_m	σ_{x_1m}	σ_{x_2m}	σ_{x_1m}
σ_{x_2m}	σ_{x_2m}	σ_{x_2m}	σ_{x_1m}	σ_{x_1m}	σ_{x_2m}	σ_{x_2m}
$\sigma_{x_1x_2m}$	$\sigma_{mx_2x_{i_k}}$	$\sigma_{mx_1x_2x_{i_k}}$	$\sigma_{mx_2x_1x_{i_k}}$	σ_{x_1m}	σ_{x_2m}	$\sigma_{x_1x_2}$
$\sigma_{x_2x_1m}$	σ_{x_2m}	$\sigma_{x_2x_1m}$	$\sigma_{x_1x_2m}$	σ_{x_1m}	σ_{x_2m}	$\sigma_{x_2x_1m}$
σ_{mx_1}	$\sigma_{mx_{i_1}}$	$\sigma_{mx_{i_1}}$	$\sigma_{mx_{i_1}}$	σ_{mx_1}	σ_{mx_2}	σ_{mx_1}
σ_{mx_2}	σ_{mx_2}	σ_{mx_2}	σ_{mx_1}	σ_{mx_1}	σ_{mx_2}	σ_{mx_2}
$\sigma_{mx_1x_2}$	σ_{mx_12}	$\sigma_{mx_1x_2}$	$\sigma_{mx_2x_1}$	σ_{mx_1}	σ_{mx_2}	$\sigma_{mx_1x_2}$
$\sigma_{mx_2x_1}$	$\sigma_{mx_2x_{i_1}}$	$\sigma_{mx_2x_1x_{i_1}}$	$\sigma_{mx_1x_2x_{i_1}}$	σ_{mx_1}	σ_{mx_2}	$\sigma_{mx_2x_1}$
$\sigma_{x_1mx_1}$	$\sigma_{mx_{i_1}}$	$\sigma_{mx_{i_1}}$	$\sigma_{mx_{i_1}}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_1}$
$\sigma_{x_2mx_2}$	$\sigma_{x_2mx_2}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_1}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_2mx_2}$
$\sigma_{x_1mx_2}$	σ_{mx_2}	$\sigma_{mx_1x_2}$	$\sigma_{x_1mx_1}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_2mx_2}$
$\sigma_{x_2mx_1}$	$\sigma_{x_2m^d}$	$\sigma_{x_2x_1m^d}$	$\sigma_{x_1x_2m^d}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_2mx_1}$
$\sigma_{x_1x_2mx_1}$	$\sigma_{mx_2x_{i_1}}$	$\sigma_{mx_1x_2x_{i_1}}$	$\sigma_{mx_2x_1x_{i_1}}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1x_2mx_1}$
$\sigma_{x_2x_1mx_2}$	$\sigma_{x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_2x_1mx_2}$
$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_2x_1m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1x_2m}$
$\sigma_{x_ix_1m}$	σ_{x_im}	σ_{x_im}	σ_{x_im}	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1m}$
$\sigma_{x_ix_2m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_2m}$
σ_m	σ_m	σ_m	σ_m	σ_m	σ_m	σ_m

$\circ_G$	$\sigma_{x_2mx_1}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$
σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}	σ_{x_i}	σ_{x_i}	σ_{x_i}
σ_{x_2}	σ_{x_1}	σ_{x_1}	σ_{x_2}	$\sigma_{x_{i_k}}$	$\sigma_{x_{i_k}}$	$\sigma_{x_{i_k}}$
$\sigma_{x_1^2}$	$\sigma_{x_1^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_i^2}$	$\sigma_{x_i^2}$	$\sigma_{x_i^2}$
$\sigma_{x_2^2}$	$\sigma_{x_1^2}$	$\sigma_{x_1^2}$	$\sigma_{x_2^2}$	$\sigma_{x_{i_k}^2}$	$\sigma_{x_{i_k}^2}$	$\sigma_{x_{i_k}^2}$
$\sigma_{x_1x_2}$	$\sigma_{x_2mx_1}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$
$\sigma_{x_2x_1}$	$\sigma_{x_1m^dx_2}$	$\sigma_{x_1m^dx_2x_1}$	$\sigma_{x_2m^dx_1x_2}$	$\sigma_{m^dx_2x_1x_i}$	$\sigma_{m^dx_1x_i}$	$\sigma_{m^dx_2x_i}$
$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_j^2}$
$\sigma_{x_1x_2x_1}$	$\sigma_{x_2mx_1x_2}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_ix_1x_2mx_i}$	$\sigma_{x_ix_1mx_i}$	$\sigma_{x_ix_2mx_i}$
$\sigma_{x_2x_1x_2}$	$\sigma_{x_1m^dx_2x_1}$	$\sigma_{x_1m^dx_2x_1}$	$\sigma_{x_1m^dx_1x_2}$	$\sigma_{m^dx_ix_1x_2x_{i_k}}$	$\sigma_{m^dx_ix_1x_{i_k}}$	$\sigma_{m^dx_ix_2x_{i_k}}$
σ_{x_1m}	σ_{x_2m}	σ_{x_1m}	σ_{x_2m}	σ_{x_im}	σ_{x_im}	σ_{x_im}
σ_{x_2m}	σ_{x_1m}	σ_{x_1m}	σ_{x_2m}	$\sigma_{x_{i_k}m}$	$\sigma_{x_{i_k}m}$	$\sigma_{x_{i_k}m}$
$\sigma_{x_1x_2m}$	$\sigma_{x_2x_1m}$	$\sigma_{x_1x_2m}$	$\sigma_{x_2x_1m}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$
$\sigma_{x_2x_1m}$	$\sigma_{x_1x_2m}$	$\sigma_{x_1x_2m}$	$\sigma_{x_2x_1m}$	$\sigma_{m^dx_2x_1x_ix_{i_k}}$	$\sigma_{m^dx_1x_ix_{i_k}}$	$\sigma_{m^dx_2x_ix_{i_k}}$
σ_{mx_1}	σ_{mx_1}	σ_{mx_1}	σ_{mx_2}	σ_{mx_i}	σ_{mx_i}	σ_{mx_i}
σ_{mx_2}	σ_{mx_1}	σ_{mx_1}	σ_{mx_2}	σ_m	σ_m	σ_m
$\sigma_{mx_1x_2}$	$\sigma_{mx_2x_1}$	$\sigma_{mx_2x_1}$	$\sigma_{mx_1x_2}$	$\sigma_{mx_ix_1x_2x_{i_k}}$	$\sigma_{mx_ix_1x_{i_k}}$	$\sigma_{mx_ix_2x_{i_k}}$
$\sigma_{mx_2x_1}$	$\sigma_{mx_1x_2}$	$\sigma_{mx_2x_1}$	$\sigma_{mx_1x_2}$	$\sigma_{mx_2x_1x_i}$	$\sigma_{mx_1x_i}$	$\sigma_{mx_2x_i}$
$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_imx_i}$	$\sigma_{x_imx_i}$	$\sigma_{x_imx_i}$
$\sigma_{x_2mx_2}$	$\sigma_{x_1mx_1}$	$\sigma_{x_1mx_1}$	$\sigma_{x_2mx_2}$	$\sigma_{x_{i_k}m}$	$\sigma_{x_{i_k}m}$	$\sigma_{x_{i_k}m}$
$\sigma_{x_1mx_2}$	$\sigma_{x_2mx_1}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$
$\sigma_{x_2mx_1}$	$\sigma_{x_1mx_2}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_ix_1x_2mx_i}$	$\sigma_{x_ix_1mx_i}$	$\sigma_{x_ix_2mx_i}$
$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_{i_k}x_2x_1x_im}$	$\sigma_{x_{i_k}x_1x_im}$	$\sigma_{x_{i_k}x_2x_im}$
$\sigma_{x_2x_1mx_2}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_2x_1mx_2}$	$\sigma_{x_ix_1x_2mx_i}$	$\sigma_{x_ix_1mx_i}$	$\sigma_{x_ix_2mx_i}$
$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_2x_1m}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_2x_1m}$	$\sigma_{x_ix_1x_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$
$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	σ_{x_im}	σ_{x_im}	σ_{x_im}
$\sigma_{x_ix_2m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_1m}$	$\sigma_{x_ix_2m}$	σ_{x_im}	σ_{x_im}	σ_{x_im}
σ_m	σ_m	σ_m	σ_m	σ_m	σ_m	σ_m

$\circ_G$	σ_m	$\circ_G$	σ_m
σ_{x_1}	$\sigma_{x_{i_1}}$	σ_{mx_1}	$\sigma_{mx_{i_1}}$
σ_{x_2}	$\sigma_{x_{i_k}}$	σ_{mx_2}	σ_m
$\sigma_{x_1^2}$	$\sigma_{x_{i_1}}$	$\sigma_{mx_1x_2}$	σ_m
$\sigma_{x_2^2}$	$\sigma_{x_{i_k}^2}$	$\sigma_{mx_2x_1}$	$\sigma_{mx_{i_1}}$
$\sigma_{x_1x_2}$	σ_m	$\sigma_{x_1mx_1}$	$\sigma_{mx_{i_1}}$
$\sigma_{x_2x_1}$	σ_{m^d}	$\sigma_{x_2mx_2}$	$\sigma_{x_{i_k}m}$
$\sigma_{x_j^2}$	$\sigma_{x_j^2}$	$\sigma_{x_1mx_2}$	σ_m
$\sigma_{x_1x_2x_1}$	$\sigma_{mx_{i_1}}$	$\sigma_{x_2mx_1}$	$\sigma_{mx_{i_1}}$
$\sigma_{x_2x_1x_2}$	$\sigma_{m^dx_{i_k}}$	$\sigma_{x_1x_2mx_1}$	$\sigma_{x_{i_k}m}$
σ_{x_1m}	σ_m	$\sigma_{x_2x_1mx_2}$	$\sigma_{mx_{i_1}}$
σ_{x_2m}	$\sigma_{x_{i_k}m}$	$\sigma_{x_ix_1x_2m}$	σ_{x_im}
$\sigma_{x_1x_2m}$	σ_m	$\sigma_{x_ix_1m}$	σ_{x_im}
$\sigma_{x_2x_1m}$	$\sigma_{x_{i_km}}$	$\sigma_{x_ix_2m}$	σ_{x_im}
σ_m	σ_{x_m}		

This calculation gives us the set of all idempotent elements of $Hyp(2)/\sim_{V_{big}}$.
$Idem = \{\sigma_{x_l} \mid x_l \in X\} \cup \{\sigma_{x_l^2} \mid x_l \in X\} \cup \{\sigma_{x_1x_2}\} \cup \{\sigma_{x_j^2} \mid j \in \{i, i_1, \cdots, i_k\}, i, i_1, \cdots, i_k$ $> 2, k \geq 1\} \cup \{\sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2}\} \cup \{\sigma_{x_1m}, \sigma_{x_1x_2m}, \sigma_{mx_2}, \sigma_{mx_1x_2}, \sigma_{x_1mx_1}, \sigma_{x_2mx_2}, \sigma_{x_1mx_2},$

$\sigma_{x_1x_2mx_1}, \sigma_{x_2x_1mx_2}, \sigma_m \mid m = x_{i_1} \cdots x_{i_k}; i_1, \ldots, i_k > 2, k \geq 1\} \cup \{\sigma_{x_ix_1x_2m} \mid m = x_{i_1} \cdots x_{i_k}; i, i_1, \ldots, i_k > 2, k \geq 1\}$.

Now we want to describe the tree transformations corresponding to these hypersubstitutions.

$T^{V_{big}}_{\sigma_{x_1}} = \{(t, x_k) \mid t \in W_{(2)}(X) \text{ and leftmost}(t) = x_k\}$,
$T^{V_{big}}_{\sigma_{x_2}} = \{(t, x_k) \mid t \in W_{(2)}(X) \text{ and rightmost}(t) = x_k\}$,
$T^{V_{big}}_{\sigma_{x_1x_2}} = \{(t, t) \mid t \in W_{(2)}(X)\} = \triangle_{W_{(2)}(X)}$,
$T^{V_{big}}_{\sigma_{x_2x_1}} = \{(t, t^d) \mid t \in W_{(2)}(X) \text{ where } t^d \text{ is the term dual to } t\}$,
$T^{V_{big}}_{\sigma_{x_1^2}} = \{(t, x_k^2) \mid t \in W_{(2)}(X) \text{ and leftmost}(t) = x_k \text{ if } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_2^2}} = \{(t, x_k^2) \mid t \in W_{(2)}(X) \text{ and rightmost}(t) = x_k \text{ if } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_j^2}}(j > 2) = \{(t, x_j^2) \mid t \in W_{(2)}(X) \text{ and if } t \notin X\} \cup X^2$
$T^{V_{big}}_{\sigma_{x_1x_2x_1}} = \{(t, tx_p) \mid t \in W_{(2)}(X) \text{ and leftmost}(t) = x_p \text{ if } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_2x_1x_2}} = \{(t, x_pt) \mid t \in W_{(2)}(X) \text{ and rightmost}(t) = x_p \text{ if } t \notin X\} \cup X^2$.
For $m = x_{i_1} \cdots x_{i_k}; i_1, \cdots, i_k > 2, k \geq 1$ we have
$T^{V_{big}}_{\sigma_{x_1m}} = \{(t, x_pm) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and leftmost}(t) = x_k\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_2m}} = \{(t, x_pm) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and rightmost}(t) = x_k\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_1x_2m}} = \{(t, tm) \mid t \in W_{(2)}(X) \text{ and } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_2x_1m}} = \{(t, t^dm) \mid t \in W_{(2)}(X) \text{ and } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{mx_1}} = \{(t, mx_p) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and leftmost}(t) = x_k\} \cup X^2$,
$T^{V_{big}}_{\sigma_{mx_2}} = \{(t, mx_p) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and rightmost}(t) = x_k\} \cup X^2$,
$T^{V_{big}}_{\sigma_{mx_1x_2}} = \{(t, mt) \mid t \in W_{(2)}(X) \text{ and } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{mx_2x_1}} = \{(t, mt^d) \mid t \in W_{(2)}(X) \text{ and } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_1mx_1}} = \{(t, x_pmx_p) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and leftmost}(t) = x_k\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_2mx_2}} = \{(t, x_pmx_p) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and rightmost}(t) = x_k\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_1mx_2}} = \{(t, qmx_{j_r}) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and } t \text{ is of the form } qx_{j_r} \text{ where } q = x_{j_1}x_{j_2} \cdots x_{j_{r-1}}\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_2mx_1}} = \{(t, x_{j_r}mq^d) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and } t \text{ is of the form } qx_{j_r} \text{ where } q = x_{j_1}x_{j_2} \cdots x_{j_{r-1}}\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_1x_2mx_1}} = \{(t, tmx_p) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and leftmost}(t) = x_p\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_2x_1mx_1}} = \{(t, t^dmx_p) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and rightmost}(t) = x_p\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_ix_1x_2m}} = \{(t, x_itm) \mid t \in W_{(2)}(X) \text{ and } t \notin X\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_ix_1m}} = \{(t, x_ix_pm) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and leftmost}(t) = x_p, i > 2\} \cup X^2$,
$T^{V_{big}}_{\sigma_{x_ix_2m}} = \{(t, x_ix_pm) \mid t \in W_{(2)}(X) \text{ and } t \notin X \text{ and rightmost}(t) = x_p, i > 2\} \cup X^2$,
$T^{V_{big}}_{\sigma_m} = \{(t, m) \mid t \in W_{(2)}(X) \text{ and } t \notin X\} \cup X^2$.

References [Den-L; 00] K. Denecke, S. Leeratanavalee, *Equational theories generated by generalized hypersubstitutions,* preprint 2000, submitted to the Conf. Volume of the Sixth International Conference on Discrete Mathematics and Applications.

[Den-L; 01], K. Denecke, S. Leeratanavalee, *M-solid polynomial varieties of semigroups, East-West Journal of Mathematics,* Vol. 3, No. 1 (2001) pp.37-48.

[Gec-S; 84] F. Gécseg, M. Steinby, *Tree Automata*, Akadémiai Kiadó, Budapest 1984.

[Lee-D; 00] S. Leeratanavalee, K. Denecke, *Generalized hypersubstitutions and strongly solid varieties,* Shaker-Verlag Aachen 2000, pp. 135-146.

[Pło; 94] J. Płonka, *Proper and Inner Hypersubstitutions of Varieties*, Proceedings of the International Conference Summer School on General Algebra and Ordered Sets, Olomouc 1994, 106-116.

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