ON AN EXTENSION OF THE GRAND FURUTA INEQUALITY

MASATOSHI FUJII AND EIZABURO KAMEI

Received February 12, 2002

ABSTRACT. The grand Furuta inequality says that if $A \geq B > 0$, then

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{G}$$

holds for all $p \geq 1$, $r \geq t$, $s \geq 1$ and $t \in [0,1]$. Very recently Uchiyama gave an extension of the grand Furuta inequality as follows: If $A \geq B \geq C > 0$, then

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{U}$$

holds for all $p \geq 1$, $r \geq t$, $s \geq 1$ and $t \in [0,1]$. The purpose of this short note is to propose a simplifed proof of Uchiyama's extension. Moreover we pose a variant of the grand Furuta inequality motivated by Uchiyama's idea.

1. INTRODUCTION

As a simultaneous extension of the Ando-Hiai inequality [1] and the Furuta inequality [5], Furuta [7] established the grand Furuta inequality, simply GFI, cf. [3]. See also [4], [8], [15] and [17]. For convenience, we denote by $A > 0$ if A is a positive invertible operator on a Hilbert space.

The grand Furuta inequality. *If $A \geq B > 0$, then for each $t \in [0,1]$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{G}$$

holds for all $s \geq 1$, $p \geq 1$ and $r \geq t$.

Very recently, Uchiyama [16] gave an extension of Theorem G which is a version of 3-variables:

Theorem U. *If $A \geq B \geq C > 0$, then for each $t \in [0,1]$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{U}$$

holds for all $s \geq 1$, $p \geq 1$ and $r \geq t$.

It is obtained as an application of the monotonicity of some operator fuctions related to the Furuta inequality. Afterwards, Furuta pointed out that Theorem U easily follows from Theorem G itself by making full use of his original technique, so-called Furuta lemma;

$$(YXY^*)^\alpha = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\alpha-1}X^{\frac{1}{2}}Y^*$$

for $\alpha \in \mathbb{R}$, $X > 0$ and invertible Y. It is expressed as a one-page proof in [9].

On the other hand, we attempted a mean theoretic approach to GFI, in which we proposed the following operator inequality as a key inequality in GFI, [3, Theorem 2]. Recall the notation:

$$A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^sA^{\frac{1}{2}}$$

2000 *Mathematics Subject Classification.* 47A30 and 47A63.

Key words and phrases. Furuta inequality and grand Furuta inequality.

and particularly $\sharp_s = \natural_s$ for $s \in [0,1]$, the s-geometric mean in the sense of the Kubo-Ando theory.

Theorem A. *If $A \geq B > 0$, then*

$$(A) \qquad (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leq B \leq A$$

for $p \geq 1,\ s \geq 1,\ r \geq 0$ and $0 \leq t \leq 1$.

In this note, we want to pay our attention to the roll of Theorem A in Theorem U. Moreover we pose a variant of GFI motivated by Uchiyama's idea.

2. A simple proof of GFI

To make a parallelism between Theorem G and Theorem U clear, we recall a proof of GFI by using Theorem A. For this, we have to cite the Furuta inequality [5] and see [2], [6], [11] and [14]: If $A \geq B \geq 0$, then for each $r \geq 0$

$$(F) \qquad A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

holds for all $p \geq 1,\ r \geq 0$.

For convenience, we cite the original form of the Furuta inequality:

Furuta inequality: *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for p and q such that $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p+r.$$

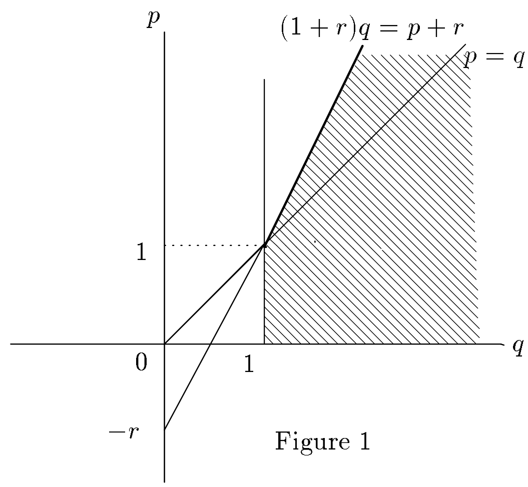

Figure 1

Now we review a proof of GFI:

Proof of GFI. Since $A \geq B > 0$, $p \geq 1,\ s \geq 1,\ r \geq 0$ and $0 \leq t \leq 1$, it follows from Theorem A that

$$D = (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leq B \leq A.$$

So we apply $A \geq D \geq 0$ to the Furuta inequality (F) in the case where $r_1 = r - t$ and $p_1 = (p-t)s + t$: Namely we have

$$(1) \qquad A^{1+r_1} \geq (A^{\frac{r_1}{2}} D^{p_1} A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}},$$

which is just desired inequality (G).

3. A simplified proof of Theorem U

Along with the argument in the preceding section, we enjoy a simplifed proof of Theorem U. An important point in the proof is the (operator) monotonicity of the α-geometric mean for $0 \le \alpha \le 1$.

Proof of Theorem U. Since $B \ge C > 0$, $p \ge 1$, $s \ge 1$, $r \ge 0$ and $0 \le t \le 1$, it follows from Theorem A that

$$(B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}} \le C \le B,$$

so that

$$1 \sharp_{\frac{1}{(p-t)s+t}} (B^t \natural_s C^p) \le B.$$

Moreover we have $B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \le 1$ since $A^t \ge B^t$. Hence it follows that

$$B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{(p-t)s+t}} (B^t \natural_s C^p) \le 1 \sharp_{\frac{1}{(p-t)s+t}} (B^t \natural_s C^p) \le B,$$

so that

$$A^{-t} \sharp_{\frac{1}{(p-t)s+t}} (B^{\frac{-t}{2}} C^p B^{\frac{-t}{2}})^s \le B^{1-t} \le A^{1-t}.$$

In other words, we have

$$D = [A^{\frac{t}{2}} (B^{\frac{-t}{2}} C^p B^{\frac{-t}{2}})^s A^{\frac{t}{2}}]^{\frac{1}{(p-t)s+t}} \le A.$$

Finally we apply $A \ge D \ge 0$ to the Furuta inequality (F) in the case where $r_1 = r - t$ and $p_1 = (p-t)s + t$: Namely we have

$$A^{1+r_1} \ge (A^{\frac{r_1}{2}} D^{p_1} A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}, \tag{2}$$

which is just desired inequality (U).

Remark 1. *Comparing with two proofs, we recognize that two inequalities (G) and (U) have the same structure. As a matter of fact, inequalities (1) and (2) are just the same, in which two D's are different a bit, though.*

4. A variant of GFI

First of all, we remark that (G) in GFI is rephrased as follow: If $A \ge B > 0$, then for each $t \in [0,1]$

$$A^{-r+t} \sharp_{\frac{1+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \le A \tag{G}$$

holds for all $s \ge 1$, $p \ge 1$ and $r \ge t$.

Motivated by Theorem U, we pose a variant of GFI under a weaker condition than that of Theorem U. For this, we use the following inequality shown in [12] and [13]. Recall that $A \gg B$ means the chaotic order, i.e., $\log A \ge \log B$ for $A,\ B > 0$.

Theorem B. *If $A \gg B$ for $A,\ B > 0$, then*

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \le B$$

holds for all $p \ge 1$ and $t \ge 0$.

Theorem 2. *If $A,\ B,\ C > 0$ satisfy $A \gg B$ and $B \ge C$, then for each $t \in [0,1]$*

$$A^{-r+t} \sharp_{\frac{1+r-t}{(p-t)s+r}} (B^t \natural_s C^p) \le (B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}} \le C \le B$$

holds for all $p \ge 1$, $s \ge 1$ and $r \ge t$.

Proof. By Theorem A, we have

$$B_1 = (B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}} \le C \le B.$$

Since $B_1 \ll B \ll A$, we apply Theorem B to the case $p_1 = (p-t)s+t$, $r_1 = r-t$ and $A \gg B_1$. Namely we have

$$A^{-r+t} \sharp_{\frac{1+(r-t)}{(p-t)s+r}} B_1^{p_1} \le B_1,$$

$$A^{-r+t} \sharp_{\frac{1+(r-t)}{(p-t)s+r}} (B^t \natural_s C^p) \le (B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}}.$$

Combining with (3), we have the desired inequality.

Acknowledgement. The authors would like to express their thanks to Professor Furuta for giving an opportunity to read [9] and [10]. The latter is devoted to further discussion on GFI from Uchiyama's view.

References

[1] T.Ando and F.Hiai, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., 197/198(1994), 113-131.
[2] M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory, 23(1990), 67-72.
[3] M.Fujii and E.Kamei, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124**(1996), 2751-2756.
[4] M.Fujii, A.Matsumoto and R.Nakamoto, *A short proof of the best possibility for the grand Furuta inequality*, J.Inequal. Appl.,**4**(1999),339–344.
[5] T.Furuta, $A \ge B \ge 0$ *assures* $(B^rA^pB^r)^{1/q} \ge B^{(p+2r)/q}$ *for* $r \ge 0$, $p \ge 0$, $q \ge 1$ *with* $(1+2r)q \ge p+2r$, Proc. Amer. Math. Soc., **101**(1987), 85–88.
[6] T.Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65**(1989), 126.
[7] T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Alg. and Its Appl., **219**(1995),139–155.
[8] T.Furuta, *Simplified proof of an order preserving operator inequality*, Proc. Japan Acad., **74**, Ser. A(1998), 114.
[9] T.Furuta, *A proof of an order preserving inequality*, Preprint.
[10] T.Furuta, *Order preserving inequalities and related operator functions*, Preprint.
[11] E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., **33**(1988), 883-886.
[12] E.Kamei, *Chaotic order and Furuta inequality*, Sci. Math. Japon, **53**(2001), 289-293.
[13] E.Kamei and M.Nakamura, *Remark on chaotic Furuta inequality*, Sci. Math. Japon., **53**(2001), 535-539.
[14] K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124**(1996), 141–146.
[15] K.Tanahashi, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc., **128**(2000), 511–519.
[16] M.Uchiyama, *Criteria for operator means*, J. Math. Soc. Japan, to appear.
[17] T.Yamazaki, *Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality*, Math. Inequal. Appl., **2**(1999), 473–477.

Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.
E-mail address: `mfujii@@cc.osaka-kyoiku.ac.jp`

Maebashi Institute of Technology, Kamisadori, Maebashi, Gunma 371-0816, Japan
E-mail address: `kamei@@maebashi-it.ac.jp`