ON THREE ζ-TYPES OF MAXIMAL S-SUBSETS OF AN S-SET.

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Abstract. Let $\Gamma(M)$ be the set of S-subsets of a centered S-set M with a zero, where S is a semigroup. In general, minimal ζ-subsets of an S-set M fall into three types, where ζ is a conjugate map on $\Gamma(M)$. Now, for the ζ-core K_ζ of a maximal S-subset K of an S-set M, the $\overline{\zeta}$-socle of M/K_ζ consists of the only minimal $\overline{\zeta}$-subset of M/K_ζ , where $\overline{\zeta}$ is a conjugate map on $\Gamma(M/K_\zeta)$ naturally induced by ζ. Here we use this fact to introduce the three ζ-types of maximal S-subsets of M and we give a characterization of a maximal S-subset of M of ζ-type $i(i = 1, 2, 3)$. Now, it is known that a finite group G is solvable if and only if every maximal subgroup of G is c-normal in G. On the other hand, a concept of a c_ζ-subset of an S-set is analogous to that of a c-normal subgroup of a group and here we show that for any maximal S-subset K of an S-set M, K is a c_ζ-subset of M if and only if K is either of ζ-type 1 or of ζ-type 2. Continuously, we give some properties about an S-set whose maximal S-subset is always a c_ζ-subset.

1 Introduction. Throughout this paper, let M be a centered (right) S-set, where S is a semigroup with a zero. We denote by $\Gamma(M)$ and $\Gamma_{\max}(M)$ the set of S-subsets of M and the set of maximal S-subsets of M, respectively. Let ζ be a conjugate map on $\Gamma(M)$. If $K \in \Gamma_{\max}(M)$, then M/K_ζ is a $\overline{\zeta}$-primtive S-set and so $\mathrm{Soc}_{\overline{\zeta}}(M/K_\zeta)$ is a minimal $\overline{\zeta}$-subset of M/K_ζ, where K_ζ is the ζ-core of K and $\overline{\zeta}$ is a conjugate map on $\Gamma(M/K_\zeta)$ naturally induced by ζ. In general, minimal ζ-subsets of an S-set M fall into three different types (cf. [2, Lemma 4.1]). Thereby, we say that a maximal S-subset K of M is of ζ-type $i(i = 1, 2, 3)$ if $\mathrm{Soc}_{\overline{\zeta}}(M/K_\zeta)$ is of type $i(i = 1, 2, 3)$ as a minimal $\overline{\zeta}$-subset of M/K_ζ. In Section 2, we give a chracterization to the three ζ-types of maximal S-subsets of an S-set.

In [3], we introduced a concept of a c_ζ-subset of an S-set, which is analogous to that of a c-normal subgroup of a group. In Section 3, we show that for any maximal S-subset K of M, K is a c_ζ-subset of M if and only if K is either of ζ-type 1 or of ζ-type 2. Furthermore, we define an S-set M to be ζ-monolithic if each maximal S-subset of M is a c_ζ-subset of M. On the other hand, it is well known that a finite group G is solvable if and only if every maximal subgroup of G is c-normal in G([4, Theorem 3]). This fact motivates us to take an interest in a ζ-monolithic S-set and we give some properties with respect to a ζ-monolithic S-set. One is relevant to a heredity on the ζ-monolithics of an S-set and the other is relevant to the nilpotency of ζ.

2 Three ζ-types of maximal S-subsets. In this paper, S will denote a semigroup with a zero 0. Each (right) S-set M is assumed to be centered, that is, M contains an element $\theta = \theta s = m0$ for all $m \in M$ and $s \in S$. This element θ will be called the *zero* of M. Unless otherwise noted terminology and notations will be as found in [3] and [5]. Hence $\Gamma(M)$

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always denotes the set of S-subsets of M. Furthermore, $\Gamma_{\max}(M)$ also denotes the set of maximal S-subsets of M. Now, we list some definitions with respect to a conjugate map on $\Gamma(M)$.

Definition 2.1. A map $\zeta : \Gamma(M) \to \Gamma(M)$ is said to be a *conjugate map* on $\Gamma(M)$ if for any $L \in \Gamma(M)$, $\zeta(L) = \cup \left\{\zeta(uS^1) \mid u \in L\right\}$ and $\zeta^2(L) \subseteq L$, that is, $\zeta(\zeta(L)) \subseteq L$.

In the rest of this paper, ζ denotes always a conjugate map on $\Gamma(M)$.

Definition 2.2. An S-subset L of M is said to be a ζ-*subset* of M if $\zeta(L) \subseteq L$. We denote by $\Gamma_\zeta(M)$ the set of ζ-subsets of M.

Definition 2.3. For any $L \in \Gamma(M)$, the ζ-*core* of L in M is defined to be
$L_\zeta = \cup \left\{aS^1 \mid a \in L \text{ with } \zeta(aS^1) \subseteq L\right\}$.

We note that L_ζ is the greatest ζ-subset of M contained in L (cf. [3, Lemma3.1]).

Definition 2.4. For the Rees factor S-set M/L with $L \in \Gamma_\zeta(M)$ and for conjugate map ζ on $\Gamma(M)$, the map $\zeta_L : \Gamma(M/L) \to \Gamma(M/L)$ is defined by $\zeta_L(K') = \iota(\zeta(\iota^{-1}(K'))$ for all $K' \in \Gamma(M/L)$, where ι is the natural map from M to M/L.

We note that ζ_L is a conjugate map on $\Gamma(M/L)$ (cf. [2, Proposition 3.2]).

Definition 2.5. An S-set M is said to be ζ-*primitive* if there is a $K \in \Gamma_{\max}(M)$ such that $K_\zeta = \{\theta\}$.

Definition 2.6. The ζ-*socle* $\mathrm{Soc}_\zeta(M)$ of M is defined to be the union of minimal ζ-subsets of M, with the stipulation that $\mathrm{Soc}_\zeta(M) = \{\theta\}$ if there are no minimal ζ-subsets of M.

Here, we recall that, for a $K \in \Gamma_{\max}(M)$, M/K_ζ is a ζ_{K_ζ}-primitive S-set and $\mathrm{Soc}_{\zeta_{K_\zeta}}(M/K_\zeta)$ consists of the only minimal ζ_{K_ζ}-subset of M/K_ζ(cf. [3, Remark A and Lemma 3.3]).

In general, a minimal ζ-subset N of an S-set M is of one of the folowing types (cf. [2, Lemma 4.1]):

(1) N is a simple S-subset of M and $\zeta(N) = \{\theta\}$;

(2) N is a simple S-subset of M and $\zeta(N) = N$;

(3) there is a simple S-subset L of M such that $N = L \cup \zeta(L)$, $L \cap \zeta(L) = \{\theta\}$, $L = \zeta^2(L)$ and $\zeta(L)$ is also a simple S-subset of M.

Definition 2.7. Let $K \in \Gamma_{\max}(M)$. If $\mathrm{Soc}_{\zeta_{K_\zeta}}(M/K_\zeta)$ is a minimal ζ_{K_ζ}-subset of M/K_ζ of type $i, i \in \{1, 2, 3\}$, then K is said to be of ζ-*type* i.

In the rest of this section, we give a characterization to maximal S-subsets of an S-set in connection with the ζ-types. Let $K \in \Gamma_{\max}(M)$. Set $s(K) = aS^1$ for an $a \in M$ with $a \notin K$. Since $aS^1 \cup K = M$, $s(K)$ is independent of the choice of such a. Set $m_\zeta(K) = s(K) \cup \zeta(s(K))$ and $d_\zeta(K) = m_\zeta(K) \cap K_\zeta$. Then $m_\zeta(K)$, $d_\zeta(K) \in \Gamma_\zeta(M)$. Furthermore, we abbreviate $m_\zeta(K)$ and $d_\zeta(K)$ to $m(K)$ and $d(K)$ respectively, if there is no danger of confusion.

Proposition 2.8. *Let* $K \in \Gamma_{\max}(M)$. *Then the following conditions are equivalent*:

(1) *K is of ζ-type* 1;

(2) $\zeta^2(M) \subseteq K$;

(3) $\zeta(m_\zeta(K)) \subseteq d_\zeta(K)$.

Proof. Set $\overline{\zeta} = \zeta_{K_\zeta}$ and denote by $\overline{\theta}$ the zero of M/K_ζ.

(1) $\leftrightarrow$ (2) (i) Suppose that M/K_ζ is a simple S-set. Then $K = K_\zeta$. Furthermore, $\overline{\zeta}(M/K_\zeta) = \overline{\zeta}^2(M/K_\zeta)$. Thus K is of ζ-type 1 if and only if $\overline{\zeta}^2(M/K_\zeta) = \{\overline{\theta}\}$, that is, $\zeta^2(M) \subseteq K$. Hence our assertion holds.

(ii) Suppose that M/K_ζ is a nonsimple S-set. Then $\mathrm{Soc}_{\overline{\zeta}}(M/K_\zeta) = \overline{\zeta}(K/K_\zeta) \vee \overline{\zeta}^2(K/K_\zeta)$ by [3. Lemma 3.3]. Hence K is of ζ-type 1 if and only if $\overline{\zeta}^2(K/K_\zeta) = \{\overline{\theta}\}$. On the other hand, $M/K_\zeta = K/K_\zeta \vee \overline{\zeta}(K/K_\zeta)$ by [3, Lemma 3.2]. Hence $\overline{\zeta}^2(K/K_\zeta) = \{\overline{\theta}\}$ if and only if $\overline{\zeta}^2(M/K_\zeta) = \{\overline{\theta}\}$, that is, $\zeta^2(M) \subseteq K$ because $\zeta^2(M) \in \Gamma_\zeta(M)$. Thus our assertion holds.

(2) $\rightarrow$ (3) Let $\zeta(m(K)) \nsubseteq K$. Then $s(K) \subseteq \zeta(m(K))$ and so $\zeta(s(K)) \subseteq \zeta^2(m(K)) \subseteq K$. Thus $\zeta(m(K)) = \zeta(s(K)) \cup \zeta^2(s(K)) \subseteq K$, a contradiction. Hence $\zeta(m(K)) \subseteq K$. Since $\zeta(m(K)) \in \Gamma_\zeta(M)$, $\zeta(m(K)) \subseteq m(K) \cap K_\zeta = d(K)$

(3) $\rightarrow$ (2) Since $m(K) \cup K = M, \zeta^2(M) = \zeta^2(m(K)) \cup \zeta^2(K)$. On the other hand, $\zeta^2(m(K)) \subseteq \zeta(d(K)) \subseteq d(K) \subseteq K$ and $\zeta^2(K) \subseteq K$. Thus $\zeta^2(M) \subseteq K$.

Proposition 2.9. Let $K \in \Gamma_{\max}(M)$. *Then the following conditions are equivalent*:

(1) *K is of ζ-type* 2;

(2) $\zeta(K) \subseteq K$ *and* $\zeta(M) \nsubseteq K$;

(3) $\zeta(K) \subseteq K$ *and* $\zeta^2(M) \nsubseteq K$;

(4) $\zeta(s(K)) = s(K)$.

Proof. Set $\overline{\zeta} = \zeta_{K_\zeta}$.

(1) $\leftrightarrow$ (2) $\leftrightarrow$ (3) Suppose that $\mathrm{Soc}_{\overline{\zeta}}(M/K_\zeta)$ is a simple S-subset of M/K_ζ and M/K_ζ is a nonsimple S-set. Then, by [3, Lemma 3.3], $\mathrm{Soc}_{\overline{\zeta}}(M/K_\zeta) = \overline{\zeta}(K/K_\zeta)$ and $\overline{\zeta}^2(K/K_\zeta) = \{\overline{\theta}\}$. In this case, K is of ζ-type 1. On the other hand, if K is of ζ-type 2, then $\mathrm{Soc}_{\overline{\zeta}}(M/K_\zeta)$ is a simple S-subset of M/K_ζ. Hence, if K is of ζ-type 2, then M/K_ζ is a simple S-set. Thus K is of ζ-type 2 if and only if M/K_ζ is a simple S-set and $\overline{\zeta}(M/K_\zeta) = M/K_\zeta$, that is, $K = K_\zeta$ and $\zeta(M) \nsubseteq K$. In this case, we can substitute $\overline{\zeta}^2(M/K_\zeta) = M/K_\zeta$ for $\overline{\zeta}(M/K_\zeta) = M/K_\zeta$, that is, $\zeta^2(M) \nsubseteq K$. Hence our assertion holds.

(2) $\rightarrow$ (4) Since $M = s(K) \cup K, \zeta(M) = \zeta(s(K)) \cup \zeta(K) \nsubseteq K$, and $\zeta(K) \subseteq K$. Hence $\zeta(s(K)) \nsubseteq K$. Therefore $s(K) \subseteq \zeta(s(K)) \subseteq \zeta^2(s(K)) \subseteq s(K)$, that is, $s(K) = \zeta(s(K))$.

(4) $\rightarrow$ (2) Assume that $\zeta(K) \nsubseteq K$. Then $s(K) \subseteq \zeta(K)$ and so $s(K) = \zeta(s(K)) \subseteq \zeta^2(K) \subseteq K$, a contradiction. Hence $\zeta(K) \subseteq K$. Furthermore, since $\zeta(s(K)) = s(K) \nsubseteq K$, $\zeta(M) \nsubseteq K$.

Proposition 2.10. *Let $K \in \Gamma_{\max}(M)$. Then the following conditions are equivalent*:

(1) *K is of ζ-type* 3;

(2) $\zeta(K) \nsubseteq K$ *and* $\zeta^2(M) \nsubseteq K$;

(3) $\zeta(m_\zeta(K)) \nsubseteq d_\zeta(K)$ *and* $\zeta(s(K)) \neq s(K)$.

Proof. Since K is of ζ-type 3 if and only if K is neither of ζ-type 1 nor of ζ-type 2, our assertion follows at once from Proposition 2.8 and 2.9.

Throrem 2.11. *Let $K \in \Gamma_{\max}(M)$. Then $m_\zeta(K)/d_\zeta(K)$ is a minimal $\zeta_{d_\zeta(K)}$-subset of $M/d_\zeta(K)$. Furthermore, $m_\zeta(K)/d_\zeta(K)$ is of type i as a minimal $\zeta_{d_\zeta(K)}$-subset of $M/d_\zeta(K)$ if and only if K is of ζ-type $i (i = 1, 2, 3)$.*

Proof. Let $x \in m(K)$ with $x \notin d(K)$. Then $x \notin K_\zeta$. Hence, if $x \in K$, then $\zeta(xS^1) \nsubseteq K$ and so $s(K) \subseteq \zeta(xS^1)$. In this case, $m(K) = s(K) \cup \zeta(s(K)) \subseteq \zeta(xS^1) \cup \zeta^2(xS^1) \subseteq \zeta(xS^1) \cup xS^1 \subseteq \zeta(m(K)) \cup m(K) = m(K)$. Hence $xS^1 \cup \zeta(xS^1) = m(K)$. On the other hand, if $x \notin K$ then $xS^1 \cup \zeta(xS^1) = m(K)$ is clear. This shows that $m(K)/d(K)$ is a minimal $\overline{\zeta}$-subset of $M/d(K)$, where $\overline{\zeta} = \zeta_{d(K)}$.

Now, a minimal $\overline{\zeta}$-subset $m(K)/d(K)$ of $M/d(K)$ is of type 1 if and only if $\overline{\zeta}(m(K)/d(K)) = \{\overline{\theta}\}$, that is, K is of ζ-type 1 by Proposition 2.8, where $\overline{\theta}$ is the zero of $M/d(K)$. Next, a minimal $\overline{\zeta}$-subset $m(K)/d(K)$ of $M/d(K)$ is of type 2 if and only if $m(K)/d(K)$ is a simple S-subset of $M/d(K)$ and $\overline{\zeta}(m(K)/d(K)) = m(K)/d(K)$. In this case, $s(K) \cup d(K) = m(K)$ and $\zeta(s(K)) \cup d(K) = s(K) \cup d(K)$. Hence $s(K) \subseteq \zeta(s(K)) \subseteq \zeta^2(s(K)) \subseteq s(K)$, that is, $s(K) = \zeta(s(K))$. Thus K is of ζ-type 2 by Proposition 2.9. Conversely, assume that K is of ζ-type 2. By Proposition 2.9, $s(K) = \zeta(s(K)) = m(K)$. Let $x \in m(K)$ and $x \notin d(K)$. Assume that $x \in K$. Since $x \notin K_\zeta, \zeta(xS^1) \nsubseteq K$ and so $s(K) \subseteq \zeta(xS^1)$. Hence $s(K) = \zeta(s(K)) \subseteq \zeta^2(xS^1) \subseteq xS^1 \subseteq K$, a contradiction. Thus $x \notin K$ and so $xS^1 = s(K)$. Hence $m(K)/d(K)$ is a simple S-subset of $M/d(K)$ and $\bar{\zeta}(m(K)/d(K)) = m(K)/d(K)$. Hence a minimal $\bar{\zeta}$-subset $m(K)/d(K)$ of $M/d(K)$ is of type 2. At the same time, we know that a minimal $\bar{\zeta}$-subset $m(K)/d(K)$ of $M/d(K)$ is of type 3 if and only if K is of ζ-type 3.

Corollary 2.12. *Let $K \in \Gamma_{\max}(M)$. Then $d_\zeta(K) = \{\theta\}$ if and only if $m_\zeta(K)$ is a minimal ζ-subset of M.*

Proof. 'Only if' part. This follows from Theorem 2.11.
'If' part. Let $d(K) \neq \{\theta\}$. Since $d(K) \subseteq m(K)$ and $m(K)$ is a minimal ζ-subset of $M, d(K) = m(K)$. Hence $m(K) \subseteq K_\zeta$, a contradiction. Hence $d(K) = \{\theta\}$.

3. ζ-monolithic S-sets. It is well known that a finite group G is solvable if and only if every maximal subgroup of G is c-normal in G(cf. [4, Theorem 3.1]). On the other hand, we introduced a concept of a c_ζ-subset of an S-set which is analogous to that of a c-normal subgroup of a group (cf. [3]). From this point of view we take an interest in an S-set M such that each maximal S-subset of M is a c_ζ-subset of M.

Definition 3.1. (cf. [1, Definition 3.1]) An S-set M is said to be ζ-*monolithic* if $\Gamma_{\max}(M) \neq \emptyset$ and each maximal S-subset of M is a c_ζ-subset of M.

Theorem 3.2. *Let $K \in \Gamma_{\max}(M)$. Then K is a c_ζ-subset of M if and only if K is either of ζ-type 1 or of ζ-type 2.*

Proof. 'Only if' part. Sincce K is a c_ζ- subset of M, we have $m(K) \cap K \subseteq K_\zeta$. Assume that K is of ζ-type 3. Then $\zeta(s(K)) \neq s(K)$ by Proposition 2.10. If $s(K) \subset \zeta(s(K))$, then $\zeta(s(K)) \subseteq \zeta^2(s(K)) \subseteq s(K)$, a contradiction. Hence $s(K) \nsubseteq \zeta(s(K))$. Thus $\zeta(s(K)) \subseteq K$.

Since $m(K)\cap K \subseteq K_\zeta, \zeta(s(K)) \subseteq K_\zeta$. Thus $\zeta^2(M) = \zeta^2(s(K)\cup K) = \zeta^2(s(K))\cup\zeta^2(K) \subseteq K$, a contradiction to Proposition 2.10. Hence K is either of ζ-type 1 or of ζ-type 2.
'If' part. Assume that K is of ζ-type 1. By Proposition 2.8, $\zeta(m(K)) \subseteq K$. Thus $\zeta(m(K)\cup K) \subseteq K$ and so $m(K)\cap K \subseteq K_\zeta$, that is, K is a c_ζ subset of M. Next, assume that K is of ζ-type 2. Then $K \in \Gamma_\zeta(M)$ by Proposition 2.9. Hence K is a c_ζ-subset of M.

Without reference to Theorem 3.2, we shall hereinafter use this result. First, we investigate a heredity on the ζ monolithics of an S-set.

Definition 3.3. For any $L \in \Gamma(M)$ and any conjugate map ζ on $\Gamma(M)$, the map $\zeta|_L : \Gamma(L) \to \Gamma(L)$ is defined by $\zeta|_L(H) = L \cap \zeta(H)$ for all $H \in \Gamma(L)$.

If $L \in \Gamma_\zeta(M)$, then $\zeta|_L(H) = \zeta(H)$ for all $H \in \Gamma(L)$ and so, in this case, we use the notation ζ for $\zeta|_L$.

Proposition 3.4. *Let ζ be a conjugate map on $\Gamma(M)$ and let $L \in \Gamma(M)$. Then $\zeta|_L$ is a conjugate map on $\Gamma(L)$.*

Proof. Let $H \in \Gamma(L)$. Then $\zeta|_L{}^2(H) = \zeta|_L(L\cap\zeta(H)) = L\cap\zeta(L\cap\zeta(H)) \subseteq \zeta^2(H) \subseteq H$ and $\zeta|_L(H) = L\cap\zeta(H) = L\cap\left\{\cup\{\zeta(aS^1)|a\in H\}\right\} = \cup\{L\cap\zeta(aS^1)|a\in H\} = \cup\{\zeta|_L(aS^1)|a\in H\}$. Hence $\zeta|_L$ is a conjugate map on $\Gamma(M)$.

An S-subset L of M is said to be *maximal sensitive* in M if $\Gamma_{\max}(L) \neq \emptyset$ and, for any $H \in \Gamma_{\max}(L)$, there is a $K \in \Gamma_{\max}(M)$ such that $H = L\cap K$.

Theorem 3.5. *Let an S-set M be ζ-monolithic.*

(1) *For any $L \in \Gamma(M)$, if L is maximal sensitive in M, then L is also $\zeta|_L$-monolithic.*

(2) *For any $L \in \Gamma_\zeta(M)$, if $\Gamma_{\max}(M/L) \neq \emptyset$, then M/L is also ζ_L-monolithic.*

Proof. (1) Let $H \in \Gamma_{\max}(L)$. Then there is a $K \in \Gamma_{\max}(M)$ such that $H = L\cap K$. Since M is ζ-monolithic, there is an $N \in \Gamma_\zeta(M)$ such that $K\cup N = M$ and $K\cap N \subseteq K_\zeta$. Then $L = L\cap(K\cup N) = (L\cap K)\cup(L\cap N) = H\cup(L\cap N)$. Furthermore, it is clear that $L\cap N \in \Gamma_{\zeta|_L}(L)$. Since $H\cap(L\cap N) \subseteq K\cap N = K_\zeta\cap N \in \Gamma_\zeta(M), H\cap(L\cap N) \subseteq H_{\zeta|_L}$. Therefore H is a $c_{\zeta|_L}$-subset of L and so L is $\zeta|_L$-monolithic.
(2) This follows at once from Proposition 2.8 and 2.9.

Theorem 3.6. *For any $L \in \Gamma_\zeta(M)$, if L is ζ-monolithic and M/L is ζ_L-monolithic, then M is ζ-monolithic.*

Proof. Let $K \in \Gamma_{\max}(M)$. If $L \subseteq K$, then K/L is a c_{ζ_L}-subset of M/L and so K is a c_ζ-subset of M by Proposition 2.8 and 2.9. Let $L \nsubseteq K$. Then $L\cap K \in \Gamma_{\max}(L)$. Hence there is an $N \in \Gamma_\zeta(L)$ such that $N\cup(L\cap K) = L$ and $N\cap(L\cap K) \subseteq (L\cap K)_\zeta$. Then $N \subseteq K$ implies $L \subseteq K$, a contradition. Hence $N \nsubseteq K$ and so $N\cup K = M$. Furthermore, $N\cap K = (N\cap L)\cap K = N\cap(L\cap K) \subseteq (L\cap K)_\zeta \subseteq K_\zeta$. Thus K is a c_ζ-subset of M. Hence M is ζ-monolithic.

Corollary 3.7. *Let $K \in \Gamma_{\max}(M)\cap\Gamma_\zeta(M)$ which is maximal sensitive in M. Then M is ζ-monolithic if and only if K is ζ-monolithic.*

Proof. 'Only if' part. This follows from Theorem 3.5.

'If' part. In this case, $K = K_\zeta$. Hence M/K_ζ is a simple S-set and so M/K_ζ is ζ_{K_ζ}-monolithic.Moreover, K_ζ is ζ-monolithic by the assumption. Hence M is ζ-monolithic by Theorem 3.6.

Next, we investigate a connection between the nilpotency of a conjugate map ζ on $\Gamma(M)$ and the ζ-monolithics on M. We recall that for any $L \in \Gamma(M)$, a cojugate map ζ on $\Gamma(M)$ is said to be *nilpotent* on L if $\zeta^n(L) = \{\theta\}$ for some positive integer n.

Now, for any $\triangle \subseteq \Gamma_{\max}(M)$, we define $\Phi(\triangle) = \cap\{K|K \in \triangle\}$ if $\triangle \neq \emptyset$: otherwise, we let $\Phi(\triangle) = M$. Let $\Gamma_{1,\zeta}(M)$ be the set of maximal S-subsets of M of ζ-type 1 and set $\Phi_{1,\zeta}(M) = \Phi(\Gamma_{1,\zeta}(M))$. Furthermore, let $\Gamma_{2,3,\zeta}(M)$ be the set of maximal S-subsets of M, which are either of ζ-type 2 or of ζ-type 3 and set $\Phi_{2,3,\zeta}(M) = \Phi(\Gamma_{2,3,\zeta}(M))$. Finally, set $\Phi_{\max}(M) = \Phi(\Gamma_{\max}(M))$. The subscript ζ in those notations is deleted if there is no danger of confusion.

Theorem 3.8. *Let ζ be a cojugate map on $\Gamma(M)$. Then the following properties hold*:

(1) *ζ is nilpotent on M if and only if ζ is nilpotent on $\Phi_{\max}(M)$ and all maximal S-subsets of M are of ζ-type* 1.

(2) *ζ is nilpotent on $\Phi_{\max}(M)$ if and only if ζ is nilpotent on $\Phi_{2,3,\zeta}(M)$. In this case, $\Phi_{2,3,\zeta}(M)$ is the greatest S-subset of M in the set of S-subsets of M on which ζ is nilpotent. Furthermore, $\Phi_{2,3,\zeta}(M) \in \Gamma_\zeta(M)$.*

Proof. (1) If ζ is nilpotent on M, then for each $K \in \Gamma_{\max}(M), \zeta_{K_\zeta}$ is nilpotent on $\mathrm{Soc}_{\zeta_{K_\zeta}}(M/K_\zeta)$. Hence K is of ζ-type 1. It is clear that ζ is nilpotent on $\Phi_{\max}(M)$. The converse is a direct consequence of Proposition 2.8.

(2) 'Only if' part. If $\Gamma_{2,3}(M) = \emptyset$, then ζ is nilpotent on M by (1). Let $\Gamma_{2,3}(M) \neq \emptyset$. By Proposition 2.8, $\zeta^2(\Phi_{2,3}(M)) \subseteq \Phi_1(M)$. Hence $\zeta^2(\Phi_{2,3}(M)) \subseteq \Phi_1(M) \cap \Phi_{2,3}(M) = \Phi_{\max}(M)$. Thus ζ is nilpotent on $\Phi_{2,3}(M)$.

'If' part. This is clear.

Proceeding to the last assertion of our theorem, let $H \in \Gamma(M)$ on which ζ is nilpotent. We will show that $H \subseteq \Phi_{2,3}(M)$. Suppose that $H \nsubseteq \Phi_{2,3}(M)$. Then there is a $K \in \Gamma_{2,3}(M)$ with $H \nsubseteq K$. Now $H \cup K = M$ and $m(K) \subseteq H \cup \zeta(H)$. Since ζ is nilpotent on $H \cup \zeta(H), \zeta$ is nilpotent on $m(K)$. Thus $\zeta(m(K)) \subseteq d(K)$ because $m(K)/d(K)$ is a minimal $\zeta_{d(K)}$-subset of $M/d(K)$ by Theorem 2.11. Hence K is of ζ-type 1 by Proposition 2.8, a contradiction. Thus $H \subseteq \Phi_{2,3}(M)$. Therefore, $\Phi_{2,3}(M)$ is the greatest S-subset of M in the set of S-subset of M on which ζ is nilpotent. Furthermore, ζ is nilpotent on $\Phi_{2,3}(M) \cup \zeta(\Phi_{2,3}(M))$ and so it equals $\Phi_{2,3}(M)$. Hence $\Phi_{2,3}(M) \in \Gamma_\zeta(M)$.

Theorem 3.9. *Let M be a ζ-primitive S-set. Then M is ζ-monolithic if and only if there is a maximal S-subset K of M such that $K_\zeta = \{\theta\}$ and ζ is nilpotent on K.*

In this case, if M is a nonsimple S-set, then $\zeta^2(M) = \{\theta\}$.

Proof. If M is a simple S-set, then $\{\theta\}$ is the only maximal S-subset of M and so our assertion holds clearly. Suppose that M is a nonsimple S-set.
'Only if' part. Assume that M is ζ-monolithic. Now, there is a $K \in \Gamma_{\max}(M)$ with $K \neq \{\theta\}$ and $K_\zeta = \{\theta\}$. Hence $\zeta(K) \nsubseteq K$. On the other hand, K is either of ζ-type 1 or of ζ-type 2. Hence K is of ζ-type 1 by Proposition 2.9. Thus $\zeta^2(M) \subseteq K$ by Proposition 2.8. Since $\zeta^2(M) \in \Gamma_\zeta(M), \zeta^2(M) = \{\theta\}$ folows from $K_\zeta = \{\theta\}$.

'If' part. Let $K \in \Gamma_{\max}(M)$ such that $K_\zeta = \{\theta\}$ and ζ is nilpotent on K. Since $M = \zeta(K) \vee K$ by [3, Lemma 3.2], ζ is nilpotent on M. Hence M is ζ-monolithic by Theorem 3.8.

Cprpllary 3.10. *Let an S-set M be ζ-monolithic and let $L, K \in \Gamma_{\max}(M)$ with $L_\zeta = K_\zeta$. Then $L = K$.*

Proof. Set $H = L_\zeta = K_\zeta$. If $H = L$, then $L = K_\zeta \subseteq K$, that is, $L = K$. Without loss of generality, we assume that $H \neq L$ and $H \neq K$. Then M/H is ζ_H-primitive and ζ_H-monolithic by Theorem 3.5. Hence ζ_H is nilpotent on M/H by Theorem 3.9. Moreover, $(L/H)_{\zeta_H} = (K/H)_{\zeta_H} = \{\bar{\theta}\}$, where $\bar{\theta}$ is the zero of M/H. Hence we have $L/H = K/H$ by [3, Corollary 3.5]. Thus $L = K$.

If ζ is nilpotent on M, then M is ζ-monolithic by Theorem 3.8. Hence Corollary 3.10 is an extension of [3, Corollary 3.5].

Now, set $\Gamma_{d_\zeta}(M) = \{K | K \in \Gamma_{\max}(M) \text{ with } d_\zeta(K) = \{\theta\}\}$ and $\Phi_{d_\zeta}(M) = \Phi(\Gamma_{d_\zeta}(M))$. Furthermore, we denote by $\Phi_\zeta(M)$ the ζ-core of $\Phi_{\max}(M)$.

Theorem 3.11. (1) *For any S-set $M, \Phi_{d_\zeta}(M)$ is the smallest S-subset of M in the set of S-subsets H of M such that $Soc_\zeta(M) \cup H = M$.*

(2) *For any S-set M such that $\Phi_\zeta(M) = \{\theta\}$, if M is ζ-monolithic, then $M = \mathrm{Soc}_\zeta(M) \vee \Phi_{d_\zeta}(M)$.*

Proof. (1) If $\mathrm{Soc}_\zeta(M) = \{\theta\}$, then $\Gamma_{d_\zeta}(M) = \emptyset$ by Corollary 2.12, that is , $\Phi_{d_\zeta}(M) = M$. Thus our assertion holds. Suppose that $\mathrm{Soc}_\zeta(M) \neq \{\theta\}$. Let $H \in \Gamma(M)$ such that $\mathrm{Soc}_\zeta(M) \cup H = M$. If $\mathrm{Soc}_\zeta(M) \subseteq H$, then $H = M$ and so $\Phi_{d_\zeta}(M) \subseteq H$. Assume that $\mathrm{Soc}_\zeta(M) \nsubseteq H$. Let Ω be the set of simple S-subsets L of M satisfying the following conditions:

(i) L is a simple S-subset of M such that $L \nsubseteq H$;

(ii) $L \cup \zeta(L)$ is a minimal ζ-subset of M.

Then $\Omega \neq \emptyset$. Let $L \in \Omega$ and set $L^\wedge = H \cup \{\cup\{A | A \in \Omega \text{ with } A \neq L\}\}$. Then $L^\wedge \cup L = H \cup \mathrm{Soc}_\zeta(M) = M$ and $L^\wedge \cap L = \{\theta\}$. Hence $L^\wedge \in \Gamma_{d_\zeta}(M)$ and so $\Phi_{d_\zeta}(M) \subseteq L^\wedge$. On the other hand, $H = \cap\{L^\wedge | L \in \Omega\}$. Hence $\Phi_{d_\zeta}(M) \subseteq H$. Next, we shall show that $\mathrm{Soc}_\zeta(M) \cup \Phi_{d_\zeta}(M) = M$. If $\Phi_{d_\zeta}(M) = M$, then it is clear. Assume that $\Phi_{d_\zeta}(M) \neq M$, that is, $\Gamma_{d_\zeta}(M) \neq \emptyset$. Let $K \in \Gamma_{d_\zeta}(M)$. Then $m(K)$ is a minimal ζ-subset of M by Corollary 2.12 and $m(K) \cup K = M$. Hence $\mathrm{Soc}_\zeta(M) \cup K = M$. Therefore $\mathrm{Soc}_\zeta(M) \cup \{\cap\{K | K \in \Gamma_{d_\zeta}(M)\}\} = M$, that is, $\mathrm{Soc}_\zeta(M) \cup \Phi_{d_\zeta}(M) = M$.

(2) If $\mathrm{Soc}_\zeta(M) = \{\theta\}$, then $\Gamma_{d_\zeta}(M) = \emptyset$, that is $\Phi_{d_\zeta}(M) = M$. Thus our assertion holds. Let $\mathrm{Soc}_\zeta(M) \neq \{\theta\}$ and let L be a minimal ζ-subset of M. Since $\Phi_\zeta(M) = \{\theta\}, L \nsubseteq \Phi_\zeta(M)$ and so there is a $K \in \Gamma_{\max}(M)$ with $L \nsubseteq K$. Then $L = m(K)$ and so $K \in \Gamma_{d_\zeta}(M)$ by Corollary 2.12. If K is of ζ-type 1, then $\zeta(m(K)) = \{\theta\}$ by Proposition 2.8 and so $m(K) = s(K)$. If K is of ζ-type 2, then $m(K) = s(K)$ by Proposition 2.9. Therefore $L = s(K)$ is a simple S-subset of M. Thus $L \cap K = \{\theta\}$ and so $L \cap \Phi_{d_\zeta}(M) = \{\theta\}$. Hence $\mathrm{Soc}_\zeta(M) \cap \Phi_{d_\zeta}(M) = \{\theta\}$. Thus $\mathrm{Soc}_\zeta(M) \vee \Phi_{d_\zeta}(M) = M$ by (1).

Here, we handle examples with respect to a decision of ζ-types of maximal S-subsets of an S-set (cf. Proposition 2.8, 2.9 and 2.10), a nilpotency of a conjugate map (cf. Theorem 3.9) and a decomposition of a ζ-monolithic S-set (cf. Theorem 3.11).

Let f be an S-endomorphism of M. The map $\zeta_f : \Gamma(M) \to \Gamma(M)$ is defined by $\zeta_f(L) = \cup\{f(uS^1) \cap f^{-1}(uS^1) \mid u \in L\}$ for all $L \in \Gamma(M)$. Then ζ_f is a conjugate map on $\Gamma(M)$. Here, any semigroup S is considered a (right) S-set by its multiplication. For any $\alpha \in S$, the S-endomorphism $\lambda_\alpha : S \to S$ is defined by $\lambda_\alpha(x) = \alpha x$ for all $x \in S$.

Example 3.12. Let S be a band with a zero such that $\Gamma_{\max}(S) \neq \emptyset$. For an $a \in S$, set $\zeta = \zeta_{\lambda_a}$. Let $u \in S$ and $x \in \lambda_a(uS) \cap \lambda_a^{-1}(uS)$. Then there are $s, t \in S$ such that $x = aus$ and $ax = ut$. In this case, $x = aus = a^2us = ax = ut$. Thus $\zeta(uS) \subseteq uS$. Hence each S-subset of S is always a ζ-subset of S. This shows by Proposition 2.10 that S itself is a ζ-monolithic S-set.

Example 3.13. Let S be a commutative semigroup with a zero such that $\Gamma_{\max}(S) \neq \emptyset$. For an $a \in S$, set $\zeta = \zeta_{\lambda_a}$. By the same way as Example 3.12, we know that S itself is a ζ-monolithic S-set.

Example 3.14. Let $S = \{0, a, b, c\}$ be a semigroup with the multiplication table:

	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	b	c

Then S has only two maximal S-subsets $K_1 = \{0, a, b\}$ and $K_2 = \{0, b, c\}$.

(1) Set $f = \lambda_a$ and $\zeta = \zeta_f$. Then $\zeta(aS) = \{0\}, \zeta(bS) = \{0, a\}$ and $\zeta(cS) = \{0, a\}$.

(i) Since $\zeta^2(S) = \{0\} \subseteq K_1 \cap K_2, K_1$ and K_2 are of ζ-type 1 by Proposition 2.8 and so S is ζ-monolithic.

(ii) Since $(K_2)_\zeta = \{0\}$, S is ζ-primitive and $\zeta^2(S) = \{0\}$ (cf. Theorem 3.9).

(iii) Since $\Phi_\zeta(S) = (K_1 \cap K_2)_\zeta = \{0\}, S = \mathrm{Soc}_\zeta(S) \vee \Phi_{d_\zeta}(S)$ (cf. Theorem 3.11). In fact, $\mathrm{Soc}_\zeta(S) = \{0, a\}$ and $\Phi_{d_\zeta}(S) = K_2$.

(2) Set $f = \lambda_c$ and $\zeta = \zeta_f$. Then $\zeta(aS) = \{0\}, \zeta(bS) = \{0, b\}$ and $\zeta(cS) = \{0, b, c\}$.

(i) Since $\zeta(S) = \zeta^2(S) = \{0, b, c\}$ and $\zeta(K_1) = \{0, b\}, \zeta(S) \nsubseteq K_1$ and $\zeta(K_1) \subseteq K_1$. Thus K_1 is of ζ-type 2 by Proposition 2.9. Moreover, $\zeta^2(S) \subseteq K_2$ and so K_2 is of ζ-type 1 by Proposition 2.8. Hence S is ζ-monolithic.

(ii) Since $K_1, K_2 \in \Gamma_\zeta(S), S$ is not ζ-primitive. Furthermore, $\zeta^2(S) \neq \{0\}$ (cf. Theorem 3.9).

(iii) Since $\mathrm{Soc}_\zeta(S) = aS \cup bS = K_1$ and $\Phi_{d_\zeta}(S) = K_2$, $\mathrm{Soc}_\zeta(S) \cap \Phi_{d_\zeta}(S) = \{0, b\} \neq \{0\}$. In this case, $\Phi_\zeta(S) = \{0, b\} \neq \{0\}$ (cf. Theorem 3.11).

(3) Let $f : S \to S$ be an S-endomorphism defined by $f(0) = 0,\ f(a) = b, f(b) = a$ and $f(c) = a$. Set $\zeta = \zeta_f$. Then $\zeta(aS) = \{0, b\}, \zeta(bS) = \{0, a\}$ and $\zeta(cS) = \{0, a\}$.

(i) Since $\zeta^2(S) = K_1, K_1$ is of ζ-type 1 by Proposition 2.8. Since $\zeta^2(S) \nsubseteq K_2$ and $\zeta(K_2) = \{0, a\} \nsubseteq K_2, K_2$ is of ζ-type 3 by Proposiiton 2.10 and so S is not ζ-monolithic.

(ii) Since $(K_2)_\zeta = \{0\}$, S is ζ-primitive. However, $\zeta^2(S) = \{0, a, b\} \neq \{0\}$ (cf. Theorem 3.9).

(iii) Since $\mathrm{Soc}_\zeta(S) = K_1$ and $\Phi_{d_\zeta}(S) = K_2$, $\mathrm{Soc}_\zeta(S) \cap \Phi_{d_\zeta}(S) = \{0, b\} \neq \{0\}$. However, $\Phi_\zeta(S) = \{0\}$ (cf. Theorem 3.11).

Example 3.15. Let $S = \{0, a, b, c, d\}$ be a simigroup with the multiplication table:

	0	a	b	c	d
0	0	0	0	0	0
a	0	0	a	0	0
b	0	0	b	0	0
c	0	0	0	0	c
d	0	0	0	0	d

Then S has only four maximal S-subsets $K_1 = \{0, a, b, c\}, K_2 = \{0, a, b, d\}, K_3 = \{0, a, c, d\}$, and $K_4 = \{0, b, c, d\}$.

Let $f : S \to S$ be an S-endomorphism defined by $f(0) = 0, f(a) = b, f(b) = a, f(c) = d$ and $f(d) = c$. Set $\zeta = \zeta_f$. Then $\zeta(aS) = \{0, b\}, \zeta(bS) = \{0, a\}, \zeta(cS) = \{0, d\}$ and $\zeta(dS) = \{0, c\}$.

(i) Since $\zeta(K_i) \nsubseteq K_i$ and $\zeta^2(S) = S \nsubseteq K_i, K_i$ is of ζ-type 3 ($i = 1, 2, 3, 4$) and so S is not ζ-monolithic.

(ii) Now, $(K_1)_\zeta = (K_2)_\zeta = \{0, a, b\}$. However, $K_1 \neq K_2$(cf. Corollary 3.10).

(iii) Since $\mathrm{Soc}_\zeta(S) = \{0, a, b\} \cup \{0, c, d\} = S$ and $\Phi_{d_\zeta}(S) = \cap\{K_i \mid i = 1, 2, 3, 4\} = \{0\}, S = \mathrm{Soc}_\zeta(S) \vee \Phi_{d_\zeta}(S)$. Moreover, $\Phi_\zeta(S) = \{0\}$. On the other hand, S is not ζ-monolithic. This shows that the inverse of Theorem 3.11 does not necessary hold.

References

[1] A. Ballester-Bolinches, $\mathfrak{h}$-*normalizers and local definitions of saturated formations of finite groups*, Isr. J. Math. **67**(1989), 312–326.

[2] Z. Goseki, *The ρ-socle of an S-set*, Math. Japonica **51**(2000), 313–319.

[3] Z. Goseki, *ζ-primitive S-sets*, Semigroup Forum **61**(2000), 405–413.

[4] Y. Wang, *C-normality of groups and its properties*, J. Algebra **180**(1996), 954–965.

[5] H. J. Weinert, *S-sets and semigroups of quotients*, Semigroup Forum **19**(1980), 1–78.

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