SEQUENTIAL ESTIMATIONS OF SOME VECTOR IN LINEAR REGRESSION MODEL UNDER A LINEX LOSS

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Abstract. We consider two sequential problems: minimum risk problem and bounded risk problem under linex loss function. We shall show that the least square estimate in linear regression model is improved by another estimator asymptotically.

1. Introduction

In this paper, we consider the problem of estimating sequentially under LINEX loss function, the vector of regression parameters in a linear regression model in which the errors are assumed to be independent and identically distributed as normal with mean 0 and unknown variance. The LINEX loss function was first proposed by Varian (1975) who showed that it is asymptotically equivalent to the squared loss function and thus provides a more general loss function. Also Zellner (1986) has considered the problem under the asymmetric loss function. Recently Takada and Nagao (2001) considered the problem of estimating the mean vector of a multivariate normal distribution under LINEX loss function when the covariance is unknown. In this paper, we obtain the results for regression parameters. It is shown that the least square estimate of the vector of regression parameters under LINEX loss function is not asymptotically admissible by providing an improved estimator. It may be noted that this problem differs from obtaining sequentially fixed radius confidence intervals for the mean vector and vector of regression parameters considered by Srivastava (1967, 1971) and Nagao and Srivastava (2001).

2. Linear regression model

We consider the model $y_i = x_i'\beta + \epsilon_i$, where known x_i and unknown $\beta = (\beta_1, \cdots, \beta_p)'$ are $p \times 1$ vectors and ϵ_i $(i = 1, 2, \cdots)$ are i.i.d. random variables having a normal distribution with mean zero and variance σ^2. We assume that the rank of $X_n = (x_1, \cdots, x_n)'$ is p $(\le n)$. Here we use the following loss function when we have sample size n.

$$L(d^{(n)}, \beta) = \sum_{i=1}^{p} b_i \{\exp(a_i^{(n)}(d_i^{(n)} - \beta_i)) - a_i^{(n)}(d_i^{(n)} - \beta_i) - 1\},$$

where $a_i^{(n)} = a_i/(nw_{ii,n})^{1/2}$ $(i = 1, \cdots, p)$ with $(X_n'X_n)^{-1} = (w_{ij,n})$ and $b_i > 0$ and $a_i \neq 0$ are known values. Also $d^{(n)} = (d_1^{(n)}, \cdots, d_p^{(n)})'$ is estimate of β based on sample size n. When σ^2 is known, we consider the estimate $d^{(n)} = \hat{\beta}_n - \dfrac{\lambda_n}{2n}$, where $\hat{\beta}_n = (X_n'X_n)^{-1}X_n'Y_n$ with $Y_n = (y_1, \cdots, y_n)'$ and $\lambda_n = \sigma^2(a_1(nw_{11,n})^{1/2}, \cdots, a_p(nw_{pp,n})^{1/2})'$. Here we note that $\hat{\beta}_n$ is a least square estimator of β. Then we have $\mathrm{E}L(d^{(n)}, \beta) = \sum_{i=1}^{p} b_i \dfrac{a_i^2\sigma^2}{2n}$. On the other hand, $\mathrm{E}L(\hat{\beta}_n, \beta) = \sum_{i=1}^{p} b_i\{\exp(\dfrac{a_i^2\sigma^2}{2n}) - 1\}$. Since $\mathrm{E}L(d^{(n)}, \beta) < \mathrm{E}L(\hat{\beta}_n, \beta)$, $\hat{\beta}_n$ is

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not admissible. We consider the case that σ^2 is unknown. Let $\tilde{\beta}_n = \hat{\beta}_n - \frac{\hat{\lambda}_n}{2n}$, where $\hat{\lambda}_n = \hat{\sigma}_n^2(a_1(nw_{11,n})^{1/2}, \cdots, a_p(nw_{pp,n})^{1/2})'$ with $\hat{\sigma}_n^2 = \frac{1}{n-p}\sum_{i=1}^n (y_i - x_i'\hat{\beta}_n)^2$.

$$\begin{aligned} \mathrm{E}L(\tilde{\beta}_n, \beta) &= \textstyle\sum_{i=1}^p b_i \mathrm{E}\{\exp(\frac{a_i^2}{2n}(\sigma^2 - \hat{\sigma}_n^2)) + \frac{a_i^2\sigma^2}{2n} - 1\} \\ &= \textstyle\sum_{i=1}^p b_i \{\exp(\frac{a_i^2\sigma^2}{2n})(1 + \frac{a_i^2\sigma^2}{n(n-p)})^{-(n-p)/2} + \frac{a_i^2\sigma^2}{2n} - 1\}. \end{aligned}$$

Since $(1 + \frac{a_i^2\sigma^2}{2n\nu})^{-\nu} \le (1 + \frac{a_i^2\sigma^2}{2n})^{-1}$ for $\nu > 0$, we have

$$\mathrm{E}L(\hat{\beta}_n, \beta) - \mathrm{E}L(\tilde{\beta}_n, \beta) \ge \sum_{i=1}^p b_i \frac{\ell_i}{1+\ell_i}\{\exp(\ell_i) - \ell_i - 1\} \ge 0$$

with $\ell_i = \frac{a_i^2\sigma^2}{2n}$. Thus $\hat{\beta}_n$ is not admissible.

3. Sequential estimators

We shall show that the least square estimate is asymptotically improved by another estimate even if we are in considering the estimate problem of β in sequential situations. At first we consider the problem of finding the sample size such that $R_n = \mathrm{E}L(d^{(n)}, \beta) + cn$ minimizes where positive number c is a cost of one sample. We call this problem a minimum risk problem in this paper. Then we have $R_n = \sum_{i=1}^p b_i \frac{a_i^2\sigma^2}{2n} + cn$. The minimum sample size is $n_c = \left(\frac{\sigma^2}{2c}\sum_{i=1}^p b_i a_i^2\right)^{1/2} = (\frac{A}{2c})^{1/2}\sigma$, where $A = \sum_{i=1}^p b_i a_i^2$. Then we have $R_{n_c} = 2cn_c$. Unfortunately σ^2 is unknown. So we define the stopping time

$$T_c = \inf\{n \ge m \mid n \ge \ell_n(\frac{A}{2c})^{1/2}\hat{\sigma}_n\},$$

where $m > p$ and $\ell_n = 1 + \frac{\ell}{n} + o(n^{-1})$. When $T_c = n$, we estimate β by $\tilde{\beta}_n = \hat{\beta}_n - \frac{\hat{\lambda}_n}{2n}$. Let $R_{T_c} = \mathrm{E}\{L(\tilde{\beta}_{T_c}, \beta) + cT_c\}$. We evaluate the regret $R_{T_c} - R_{n_c}$ in the later. Next we consider another problem. Let $W > 0$ be a known positive number. We want that $\mathrm{E}(d^{(n)}, \beta) \le W$, then $n_W = \frac{A}{2W}\sigma^2$. Also this problem is called a bounded risk problem. Thus we define with $m > p$

$$T_W = \inf\{n \ge m \mid n \ge \ell_n \frac{A}{2W}\hat{\sigma}_n^2\}$$

The following two lemmas were used by Albert(1966) and Srivastava (1967, 1971) for sequentially obtaining fixed radius confidence intervals for the mean vector and the vector of regression parameters. They have extended Chow and Robbins (1965) to the linear regression parameters.

Lemma 3.A. $\hat{\beta}_n$ and $\{\hat{\sigma}_m^2, \cdots, \hat{\sigma}_n^2\}$ are independent for $m > p$, where $\hat{\sigma}_k^2 = \frac{1}{k-p}\sum_{i=1}^{k}(y_i - x_i'\hat{\beta}_k)^2$ with $\hat{\beta}_k = (X_k'X_k)^{-1}X_k'Y_k$.

An outline of proof. Since $(y_1 - x_1'\hat{\beta}_k, \cdots, y_k - x_k'\hat{\beta}_k)' = (I_k - X_k(X_k'X_k)^{-1}X_k')Y_k$, then $\mathrm{Cov}(\hat{\beta}_n, (I_k - X_k(X_k'X_k)^{-1}X_k', 0)Y_n) = \sigma^2(X_n'X_n)^{-1}X_n'\begin{pmatrix} I_k - X_k(X_k'X_k)^{-1}X_k' \\ 0 \end{pmatrix}$. Let $X_n = \begin{pmatrix} X_k \\ \tilde{X}_{n-k} \end{pmatrix}$, then $X_n'\begin{pmatrix} I_k - X_k(X_k'X_k)^{-1}X_k' \\ 0 \end{pmatrix} = (X_k', \tilde{X}_{n-k}')$
$\times\begin{pmatrix} I_k - X_k(X_k'X_k)^{-1}X_k' \\ 0 \end{pmatrix} = X_k' - X_k' = 0$. Thus $\hat{\beta}_n$ and $\hat{\sigma}_k^2$ $(k = m, \cdots, n)$ are independent.

Lemma 3.B. Let $\tilde{\sigma}_n^2 = (n-p)\hat{\sigma}_n^2$, then we have $\tilde{\sigma}_n^2 = U_1 + \cdots + U_{n-p}$, where U_i $(i = 1, \cdots, n-p)$ are independent and identically distributed random variables and each distribution is σ^2 times chi-square distribution with one degree of freedom.

An outline of proof. Let $C_k = (X_k'X_k)^{-1}$, then we have $C_n = C_{n-1} - \frac{1}{1+\Delta_n}C_{n-1}x_n x_n' C_{n-1}$, where $\Delta_n = x_n'C_{n-1}x_n$. Then $\hat{\beta}_n = \hat{\beta}_{n-1} + \frac{1}{1+\Delta_n}C_{n-1}x_n$
$\times(y_n - x_n'\hat{\beta}_{n-1})$. Then we have

$$\begin{aligned} \tilde{\sigma}_n^2 &= (n-p)\hat{\sigma}_n^2 = \tilde{\sigma}_{n-1}^2 + \frac{1}{1+\Delta_n}(y_n - x_n'\hat{\beta}_{n-1})^2 = \frac{1}{1+\Delta_n}(y_n - x_n'\hat{\beta}_{n-1})^2 \\ &+\frac{1}{1+\Delta_{n-1}}(y_{n-1} - x_{n-1}'\hat{\beta}_{n-2})^2 + \cdots + \frac{1}{1+\Delta_{p+1}}(y_{p+1} - x_{p+1}'\hat{\beta}_p)^2. \end{aligned}$$

Since $\frac{1}{\sqrt{(1+\Delta_k)}}(y_k - x_k'\hat{\beta}_{k-1})$ is a normal distribition with mean zero and variance σ^2, we have desired result by Hogg-Craig's theorem (1958) or Craig's (1943) theorem. Also see, e.g. Srivastava and Khatri (1979, p.67).

Thus we define

$$t_a = \inf\{n \geq m - p \mid \sum_{i=1}^{n} U_i < an^{\alpha}L(n)\},$$

where $L(n) = 1 + \frac{L_0}{n} + o(n^{-1})$. Stopping time of this kind has been introduced by Woodroofe (1977).

Let $N_a = t_a + p$. Then $N_a = T_c$ or $N_a = T_W$,
for T_c, $a = \frac{2c}{A}$, $L_0 = 2(p-\ell)$, $n_a = (\frac{A}{2c})^{1/2}\sigma$, $\alpha = 3$,
and for T_W, $a = \frac{2W}{A}$, $L_0 = p - \ell$, $n_a = (\frac{A\sigma^2}{2W})$, $\alpha = 2$.
Let $\gamma = \frac{1}{1-\alpha}$ and $n_a = (\frac{\sigma^2}{a})^{\gamma}$. $F(x) = \Pr(U_1 \leq x) \leq Bx^{1/2}$ for $x \geq 0$. From Woodroofe (1977), we have the following lemmas.

Lemma 3.1. When $a \to 0$, we have $\frac{N_a}{n_a} \to 1$, $\frac{N_a - n_a}{\sqrt{n_a}} \to N(0, 2\gamma^2)$.

Lemma 3.2. For $0 < \epsilon < 1$, we have $\Pr(N_a \le \epsilon n_a) = O(n_a^{-(m-p)/(2\gamma)})$.

Lemma 3.3. If $\frac{1}{2}(m-p) > \gamma$, then we have $(\frac{N_a - n_a}{\sqrt{n_a}})^2$ is uniformly integrable.

Lemma 3.4. If $\frac{1}{2}(m-p) > 2\gamma$, then we have $\lim_{a\to 0} \mathrm{E}\{\frac{(N_a-n_a)^2}{N_a}\} = 2\gamma^2$.

Let $R_a = a t_a^\alpha L(t_a) - \sum_{i=1}^{t_a} U_i$, the R_a converges in law to H and $\nu = \mathrm{E}(H) = \frac{\gamma\sigma^2}{2}[(\alpha-1)^2+2] - \sum_{n=1}^{\infty} n^{-1}\mathrm{E}((S_n - n\alpha\sigma^2)^+)$, where $S_n = \sum_{k=1}^n U_k$.

Lemma 3.5. If $\frac{1}{2}(m-p) > \gamma$, we have

$$\mathrm{E}(t_a) = n_a + \gamma\nu - \gamma L_0 - \alpha\gamma^2 + o(1).$$

From the Lemma 3.A, we have

$$\begin{aligned} \mathrm{E}L(\tilde\beta_N, \beta) &= \textstyle\sum_{i=1}^p b_i \mathrm{E}\{\exp(\frac{a_i^2}{2N}(\sigma^2 - \hat\sigma_N^2)) + \frac{a_i^2\hat\sigma_N^2}{2N} - 1\} \\ &= \textstyle\sum_{i=1}^p b_i \mathrm{E}\{\exp(\frac{a_i^2}{2N}(\sigma^2 - \hat\sigma_N^2)) + \frac{a_i^2}{2N}(\hat\sigma_N^2 - \sigma^2) - 1\} + \mathrm{E}(\frac{A\sigma^2}{2N}). \end{aligned} \tag{3.1}$$

Proposition 3.1. If $\frac{1}{2}(m-p) > 2\gamma$, we have

$$\mathrm{E}L(\tilde\beta_N, \beta) = \mathrm{E}(\frac{A\sigma^2}{2N}) + o(n_a^{-2}).$$

Proof. From (3.1), we shall show that

$$\mathrm{E}\{\exp(\frac{a_i^2}{2N}(\sigma^2 - \hat\sigma_N^2)) + \frac{a_i^2}{2N}(\hat\sigma_N^2 - \sigma^2) - 1\} = o(n_a^{-2}).$$

Let $C = \{N > \epsilon n_a\} \cap \{|\hat\sigma_N^2 - \sigma^2| \le \delta\}$ for $0 < \epsilon < 1, \delta > 0$. $\bar C$ stands for the complement of C. $\Pr(\bar C) \le \Pr(N \le \epsilon n_a) + \Pr(\sup_{n \ge \epsilon n_a} |\hat\sigma_n^2 - \sigma^2| > \delta)$. Since $\{|\hat\sigma_n^2 - \sigma^2|^q\}$ is a reverse submartingale for $q > 1$, then

$$\Pr(\sup_{n\ge\epsilon n_a} |\hat\sigma_n^2 - \sigma^2| > \delta) \le \frac{1}{\delta^{2q}}\mathrm{E}(\hat\sigma_n^2 - \sigma^2)^{2q} = O(n_a^{-q})$$

for any number $q > 1$. By Lemma 3.2, $\Pr(\bar C) = O(n_a^{-(m-p)/(2\gamma)})$.
Let $f = \exp(-\frac{a_i^2}{2N}(\hat\sigma_N^2 - \sigma^2)) + \frac{a_i^2}{2N}(\hat\sigma_N^2 - \sigma^2) - 1$ and $\mathrm{E}(f) = I + II$, where $I = \int_C f dP$ and $II = \int_{\bar C} f dP$. Then we have

$$I = \int_C \frac{1}{2}\left(\frac{a_i^2(\hat\sigma_N^2 - \sigma^2)}{2N}\right)^2 \exp(\Delta_N) dP = (\frac{a_i^4}{8n_a^3})\int_C (\frac{n_a}{N})^3 N(\hat\sigma_N^2 - \sigma^2)^2 \exp(\Delta_N) dP,$$

where $|\Delta_N| \leq \dfrac{a_i^2|\hat{\sigma}_N^2 - \sigma^2|}{2N} \leq \dfrac{a_i^2\delta}{2m}$ on C. Also $(\dfrac{n_a}{N})^3 N(\hat{\sigma}_N^2 - \sigma^2)^2 \exp(\Delta_N)$ is uniformly integrable on C and converges in law to $2\sigma^4\chi^2_{[1]}$. Therefore $I = O(n_a^{-3})$. For II,

$$II = \int_{\bar{C}} f dP \leq \int_{\bar{C}} \{\exp(\frac{a_i^2\sigma^2}{2N}) + \frac{a_i^2|\hat{\sigma}_N^2 - \sigma^2|}{2m}\} dP$$

$$\leq \exp(\frac{a_i^2\sigma^2}{2m})\Pr(\bar{C}) + \frac{a_i^2}{2m}\int_{\bar{C}} |\hat{\sigma}_N^2 - \sigma^2| dP.$$

By Hölder's inequality , we have for $1/r + 1/s = 1$ with $r, s > 0$,

$$\int_{\bar{C}} |\hat{\sigma}_N^2 - \sigma^2| dP \leq K^{1/r}\{\Pr(\bar{C})\}^{1/s},$$

where $K = \mathrm{E}|\hat{\sigma}_N^2 - \sigma^2|^r < \infty$ for $r > 1$. Then we have $II = O(n_a^{-(m-p)/(2\gamma s)})$. From the assumption, there exists $s > 1$ such that $(m-p)/(2\gamma s) > 2$. Thus $II = o(n_a^{-2})$.

Proposition 3.2. If $\frac{1}{2}(m-p) > 2\gamma$, we have

$$\mathrm{E}L(\hat{\beta}_N, \beta) = \frac{1}{8}(\sum_{i=1}^{p} a_i^4 b_i \sigma^4) n_a^{-2} + \mathrm{E}(\frac{A\sigma^2}{2N}) + o(n_a^{-2}).$$

Proof. Let $Y_i = \dfrac{a_i^2\sigma^2}{2N}$, then

$$\mathrm{E}L(\hat{\beta}_N, \mu) = \sum_{i=1}^{p} b_i \mathrm{E}\{\exp(Y_i) - 1\} = \sum_{i=1}^{p} b_i \mathrm{E}\{\exp(Y_i) - Y_i - 1\} + \mathrm{E}(\frac{A\sigma^2}{2N}).$$

We show

$$\mathrm{E}\{\exp(Y_i) - Y_i - 1\} = \frac{1}{8} a_i^4 \sigma^4 n_a^{-2} + o(n_a^{-2}). \tag{3.2}$$

The l.h.s. of (3.2)=$\mathrm{E}\{\frac{1}{2}Y_i^2 \exp(\Delta'_N)\} = \dfrac{a_i^4\sigma^4}{8n_a^2}\mathrm{E}\{(\dfrac{n_a}{N})^2 \exp(\Delta'_N)\}$, where $|\Delta'_N| \leq Y_i \leq \dfrac{a_i^2\sigma^2}{2m}$. Since $\mathrm{E}\{(\dfrac{n_a}{N})^2 \exp(\Delta'_N)\} = I + II$, where

$$I = \int_{N < \epsilon n_a} (\frac{n_a}{N})^2 \exp(\Delta'_N) dP \quad \text{and} \quad II = \int_{N \geq \epsilon n_a} (\frac{n_a}{N})^2 \exp(\Delta'_N) dP,$$

then we have

$$I \leq (\frac{n_a}{m})^2 \exp(\frac{a_i^2\sigma^2}{2m})\Pr(N < \epsilon n_a) = n_a^2 O(n_a^{-(m-p)/(2\gamma)}) = o(1).$$

In $\{N \geq \epsilon n_a\}$, $(\dfrac{n_a}{N})^2 \exp(\Delta'_N)$ is uniformly integrable and converges to 1 in probability. Thus $II = 1 + o(1)$.

4. Minimum risk problem and bounded risk problem

For minimum risk problem, we have $a = \dfrac{2c}{A}$, $L_0 = 2(p-\ell)$, $n_c = (\dfrac{A}{2c})^{1/2}\sigma$, $\alpha = 3$ and $\gamma = \dfrac{1}{2}$.

Theorem 4.1. If $m > p+2$, we have

$$\mathrm{E}(T_c) = n_c + \frac{1}{2}\nu_1 + \ell - \frac{3}{4} + o(1),$$

where $\nu_1 = \sigma^2(\dfrac{3}{2} - \sum_{n=1}^{\infty} n^{-1}\mathrm{E}(S_n - 3n)^+)$ and $S_n = \chi^2_{[n]}$.

From Proposition 3.1, if $m > p+2$, we have

$$R_{T_c} - R_{n_c} = \mathrm{E}\{L(\tilde{\beta}_{N_c}, \beta) + cT_c - 2cn_c\} = c\mathrm{E}\left(\frac{(T_c - n_c)^2}{T_c}\right) + o(c) = \frac{c}{2} + o(c).$$

Theorem 4.2. If $m > p+2$, we have

$$\frac{R_{T_c} - R_{n_c}}{c} = \frac{1}{2} + o(1).$$

R'_{T_c} stands for the risk when we use $\hat{\beta}_{T_c}$. Then we have

$$R'_{T_c} - R_{n_c} = \frac{c}{4A}(\sum_{i=1}^{p} a_i^4 b_i \sigma^2) + c\mathrm{E}\left(\frac{(T_c - n_c)^2}{T_c}\right) + o(c).$$

Theorem 4.3. If $m > p+2$, then

$$\frac{R'_{T_c} - R_{n_c}}{c} = \frac{1}{2} + \frac{1}{4A}(\sum_{i=1}^{p} a_i^4 b_i \sigma^2) + o(1).$$

From the above results, $\hat{\beta}_{N_c}$ is not asymptotically admissible.

Finally we consider the bounded risk problem. Then we have $a = \dfrac{2W}{A}$, $L_0 = p - \ell$, $n_W = (\dfrac{A\sigma^2}{2W})$, $\alpha = 2$ and $\gamma = 1$.

Theorem 4.4. If $m > p+4$, we have

$$\mathrm{E}(T_W) = n_W + \nu_2 + \ell - 2 + o(1),$$

where $\nu_2 = \sigma^2\{\dfrac{3}{2} - \sum_{n=1}^{\infty} n^{-1}\mathrm{E}(S_n - 2n)^+\}$ and $S_n = \chi^2_{[n]}$.

Theorem 4.5. If $m > p+4$, we have

$$\mathrm{E}L(\tilde{\beta}_{T_W}, \beta) = W + \frac{2W^2}{A\sigma^2}(4 - \nu_2 - \ell) + o(W^2).$$

Corollary 4.1. If $m > p+4$ and $\ell \geq 4$, we have

$$\mathrm{E}L(\tilde{\beta}_{T_W}, \beta) \leq W + o(W^2).$$

Theorem 4.6. If $m > p+4$, we have

$$\mathrm{E}L(\hat{\beta}_{T_W}, \beta) = W + \frac{W^2}{A\sigma^2}\{2(4-\nu_2-\ell) + \frac{\sigma^2}{2A}(\sum_{i=1}^{p} a_i^4 b_i)\} + o(W^2).$$

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