On R-maps in BCK/BCI-algebras

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ABSTRACT. In this paper, we investigate some properties of R-maps in BCK/BCI-algebras.

1. Introduction.

K. H. Dar and B. Ahmad([DA]) studied R-maps and L-maps for BCK-algebras and obtained some results. In fact early in 1981, H. Huang introduced the notion of R-maps for positive implicative BCK-algebras and gave some interesting results. In this paper, we investigate some properties of R-maps in BCK/BCI-algebras.

2. Introduction.

An algebraic system $(X;*,0)$ is said to be a *BCI-algebra* if it satisfies the following conditions:
(I) $(x*y)*(x*z) \leq z*y$,
(II) $x*(x*y) \leq y$,
(III) $x \leq x$,
(IV) $x \leq y$, $y \leq x$ imply $x = y$,
where $x \leq y$ is defined by $x*y = 0$. A BCI-algebra X is said to be a *BCK-algebra* if $0 \leq x$, for all $x \in X$.

K. Iséki ([Is]) defined the notion of BCI-algebras with condition (S), i.e., for any $a, b \in X$, $A(a,b) := \{x \in X | x*a \leq b\}$ has the greatest element, say $a \circ b$.

Theorem 2.1 ([Is]). *If X is a BCI-algebra with condition (S), then $(X, \circ)$ is a semigroup and 0 is the zero element.*

Theorem 2.2 ([Ho]). *If X is a BCI-algebra with condition (S), then*
(i) $x*(y \circ z) = (x*y)*z$,
(ii) $(x \circ z)*(y \circ z) \leq x*y \leq x \circ y$.

3. Main Results.

Let $(X;*,0)$ be a BCK/BCI-algebra and let $x \in X$. A mapping $R_x : X \to X$ defined by $R_x(y) := y*x$, for all $y \in X$, is called a *right map* of X. The set of all right maps on X is denoted by $\mathbf{R}(X)$. We define a binary operation "$\circledcirc$" on $\mathbf{R}(X)$ as follows:

$$(R_a \circledcirc R_b)(x) := R_a(R_b(x))$$

where $R_a, R_b \in \mathbf{R}(X)$ and $x \in X$.

Proposition 3.1. If $(X;*,0)$ is BCI-algebra with condition (S), then $R_a \circledcirc R_b = R_{a \circ b}$ for any $R_a, R_b \in \mathbf{R}(X)$.

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Proof. By applying Theorem 2.2-(i) we have $(R_a \circledcirc R_b) = R_a(R_b(x)) = R_a(x * b) = (x * b) * a = x * (a \circ b) = R_{a \circ b}(x)$, proving the proposition. □

Since $(X; \circ)$ is a semigroup if X is a BCK/BCI-algebra with condition (S), we can easily see that $(\mathbf{R}(X); \circ)$ is a commutative semigroup. Concerning Theorem 3.12 and 3.13 ([MeJu, pp. 129-130]) the condition "positive implicative" is superfluous and the proof of Theorem 3.13 is incorrect. We restate Theorem 3.12 and give correct proof of Theorem 3.13 as follows:

Theorem 3.2. *If $(x; *, 0)$ is a BCK/BCI-algebra with condition (S), then $(\mathbf{R}(X), \circledcirc)$ is a commutative semigroup with zero element R_0.*

Theorem 3.3. *If $(x; *, 0)$ is a BCK/BCI-algebra with condition (S), then $(X, \circ) \cong (\mathbf{R}(X), \circledcirc)$ as a semigroup.*

Proof. If we define $\phi : (X, \circ) \to (\mathbf{R}(X), \circledcirc)$ by $\phi(x) = R_x$, then ϕ is surjective. Since $\phi(x \circ y) = R_{x \circ y} = R_x \circledcirc R_y = \phi(x) \circledcirc \phi(y)$, it is a semigroup homomorphism. Assume $\phi(x) = \phi(y)$ for some $x \neq y$ in X. Then $R_x = R_y$ and hence $y * x = R_x(y) = R_y(y) = y * y = 0$ and $x * y = R_y(x) = R_x(x) = x * x = 0$. By (IV) we have $x = y$, a contradiction. □

We can restate Theorem 7.9 ([MeJu, pp. 40]) in terms of R-maps as follows:

Proposition 3.4. *Let $(X; *, 0)$ be a BCK-algebra. Assume that there is a binary operation ∇ on X such that $R_a \circledcirc R_b = R_{a \nabla b}$ for any $a, b \in X$. Then X is with condition (S) and ∇ is exactly the operation "$\circ$".*

Combining Proposition 3.4 with Proposition 3.1 we obtain:

Theorem 3.5. *Let $(X; *, 0)$ be a BCK-algebra. Then the following are equivalent:*
(i) *X is with condition (S),*
(ii) *there is a binary operation ∇ on X such that $R_a \circledcirc R_b = R_{a \nabla b}$ for any $a, b \in X$.*

Lemma 3.6 ([MeJu]).*Let $(X; *, 0)$ be a BCK-algebra with condition (S). Then the following are equivalent:*
(a) *X is positive implicative,*
(b) $(x \circ y) * z = (x * z) \circ (y * z)$,
(c) $x \circ y = x \circ (y * x)$.

We define $R_x \leq R_y$ if and only if $R_x(z) \leq R_y(z)$ for all $z \in X$, and $R_x = R_y$ if and only if $R_x \leq R_y$ and $R_y \leq R_x$.

Proposition 3.7. *Let X be a BCK/BCI-algebras and $x, y, z \in X$. Then*
(i) *if $x \leq y$ then $R_y \leq R_x$,*
(ii) *if $R_x \leq R_y$ then $R_x \circledcirc R_z \leq R_y \circledcirc R_z$, for any $z \in X$,*
(iii) *if X is with condition (S) then $R_{(x \circ y) \circ z} \leq R_{(x * y) \circ z}$,*
(iv) *if X is positive implicative with condition (S) then $R_{y \circ (z \circ y)} \leq R_{y \circ z}$.*

Proof. (i). Refer to [DA].
(ii). If $R_x \leq R_y$ then $u * x \leq u * y$ for any $u \in X$ and hence $(u * x) * z \leq (u * y) * z$. Hence $(R_x \circledcirc R_z)(u) \leq (R_y \circledcirc R_z)(u)$, i.e., $R_x \circledcirc R_z \leq R_y \circledcirc R_z$.
(iii). If X is with condition (S), then $x * y \leq x \circ y$. By (i) we have $R_{x \circ y} \leq R_{x * y}$. By applying Proposition 3.1 and (ii) we obtain

$$R_{(x \circ y) \circ z} = R_{x \circ y} \circledcirc R_z \leq R_{x * y} \circledcirc R_z = R_{(x * y) \circ z}.$$

(iv). If X is positive implicative with condition (S), then by (iii) and Lemma 3.6-(c) we obtain:

$$R_{y\circ(z\circ y)} = R_{z\circ y} \circledcirc R_y \leq R_{z*y} \circledcirc R_y = R_{y\circ(z*y)} = R_{y\circ z},$$

proving the proposition. □

Remark. 1. By Proposition 3.7-(iv) we have $x*(y\circ(z\circ y)) \leq x*(y\circ z)$. Since X is a positive implicative BCK-algebra, using Theorem 2.2-(i) we obtain $x*(y\circ(z\circ y)) \leq (x*z)*(y*z)$.
2. Since every BCI-algebra with $(x*y)*y = x*y$ becomes a BCK-algebra, there is no non-trivial positive implicative BCI-algebra.

Using the notion of R-maps we can give very simple proof of Theorem 2.2-(i).

Theorem 3.8. *If X is a BCI-algebra with condition (S) then $x*(y\circ z) = (x*y)*z$.*

Proof. $x*(y\circ z) = R_{y\circ z}(x) = (R_y \circledcirc R_z)(x) = R_y(R_z(x)) = R_y(x*z) = (x*z)*y = (x*y)*z$, completing the proof. □

Lemma 3.9 ([MeJu, pp. 129]). *If $(X;*,0)$ is a positive implicative BCK-algebra with condition (S), then any right map $R_z : (X,\circ) \to (X,\circ)$, $z \in X$, is a semigroup homomorphism.*

Proof. For any $x, y \in X$, we have

$$R_z(x\circ y) = (x\circ y)*z = (x*z)\circ(y*z) = R_z(x)\circ R_z(y),$$

proving the lemma. □

It is known that a BCK-algebra is implicative if and only if it is both commutative and positive implicative. It is also known the useful properties in implicative BCK-algebras with condition (S).

Proposition 3.10 ([MeJu, pp. 45]). *If $(X;*,0)$ is an implicative BCK-algebra with condition (S), then*
(i) $c*(a\wedge b) = (c*a)\circ(c*b)$,
(ii) $c*(a\circ b) = (c*a)\wedge(c*b)$.

Using the notion of R-maps we can restate the Proposition 3.10 as follows:

Proposition 3.11. *If $(X;*,0)$ is an implicative BCK-algebra with condition (S), then*
(i) $R_{a\circ b}(c) = R_a(c)\wedge R_b(c)$,
(ii) $R_{a\wedge b}(c) = R_a(c)\circ R_b(c)$.

Using the Proposition 3.11-(i) we obtain the useful following properties:

Theorem 3.12. *If $(X;*,0)$ is an implicative BCK-algebra with condition (S), then*

$$\begin{aligned}(p\circ q)*(a\circ b) &= \{p*(a\circ b)\}\circ\{q*(a\circ b)\}\cdots\cdots(*)\\ &= \{(p*a)\circ(q*a)\}\wedge\{(p*b)\circ(q*b)\}\cdots\cdots(**)\end{aligned}$$

Proof. Since X is a positive implicative BCK-algebra with condition (S), by Lemma 3.9 $R_{a\circ b}$ is a semigroup homomorphism. Hence

$$R_{a\circ b}(p\circ q) = R_{a\circ b}(p)\circ R_{a\circ b}(q),$$

which means that $(p\circ q)*(a\circ b) = \{p*(a\circ b)\}\circ\{q*(a\circ b)\}$. By applying Proposition 3.11-(i) we obtain

$$(p\circ q)*(a\circ b) = \{(p*a)\circ(q*a)\}\wedge\{(p*b)\circ(q*b)\},$$

proving the theorem. □

Corollary 3.13. *If* $(X;*,0)$ *is an implicative* BCK*-algebra with condition* (S)*, then*
(a) $(a\circ c)*(a\circ b) = (c*a)\wedge[(a*b)\circ(c*b)]$,
(b) $(b*a)\wedge(a*b) = 0$,
(c) $\{a*(a\circ b)\}\circ\{b*(a\circ b)\} = 0$,
(d) $(a\circ c)*(a\circ b) = c*(a\circ b)$,
(e) $(c\circ b)*(a\circ b) = (c*a)*b$.

Proof. (a). Let $p := a, q := c$ in $(**)$. (b). Let $c := b$ in (a). In fact, if X is implicative, then

$$\begin{aligned}(b*a)\wedge(a*b) &= (a*b)*[(a*b)*(b*a)]\\ &= (a*b)*[(a*(b*a))*b]\\ &= (a*b)*(a*b) = 0.\end{aligned}$$

(c). Let $p := a, q := b$ in $(*)$. (d). Let $p := a, q := c$ in $(*)$. (e). Let $p := c, q := b$ in $(*)$. □

Remark. By applying Theorem 2.2-(i) we can see that the condition (d) is equal to the condition (e) in the above Corollary 3.13.

Theorem 3.14. *If* $(X;*,0)$ *is an implicative* BCK*-algebra with condition* (S)*, then*

$$(a\circ b)*(a\wedge b) = (b*a)\circ(a*b) = (a*(a\wedge b))\circ(b*a).$$

Proof. By Proposition 3.11-(ii) we have $R_{a\wedge b}(c) = R_a(c)\circ R_b(c)$. If we put $c := a\circ b$, then

$$\begin{aligned}(a\circ b)*(a\wedge b) &= [(a\circ b)*a]\circ[(a\circ b)*b]\\ &= [(a*a)\circ(b*a)]\circ[(a*b)\circ(b*b)]\\ &= (b*a)\circ(a*b),\end{aligned}$$

since $(x\circ y)*z = (x*z)\circ(y*z)$ holds in any positive implicative BCK-algebra. Similarly,

$$\begin{aligned}(a\circ b)*(a\wedge b) &= [a*(a\wedge b)]\circ[b*(a\wedge b)]\\ &= [a*(b*(b*a))]\circ[b*(b*(b*a))]\\ &= [a*(a\wedge b)]\circ(b*a).\end{aligned}$$

□

Corollary 3.15. *Let $(X;*,0)$ be an implicative BCK-algebra with condition (S). If $b*a=0$ then $(a\circ b)*b=a*b$.*

Proof. If $b*a=0$ then $(a\circ b)*(a\wedge b)=(a\circ b)*(b*(b*a))=(a\circ b)*b$ and $(b*a)\circ(a*b)=a*b$. □

Theorem 3.16. *If $(X;*,0)$ is an implicative BCK-algebra with condition (S), then*

$$(p\wedge q)*(a\wedge b)=[(p\wedge q)*a]\circ[(p\wedge q)*b].$$

Proof. It can be easily obtained from Proposition 3.11-(ii) simply replacing c by $p\wedge q$. □

Corollary 3.17. *If $(X;*,0)$ is an implicative BCK-algebra with condition (S), then*
(a) $(p\wedge a)*(a\wedge b)=(p\wedge a)*b=(a*b)*(a*p)$,
(b) $a*(a*b)\leq b*(b*a)$.

Proof. (a). If we put $q:=a$ in Theorem 3.16, then

$$\begin{aligned}(p\wedge a)*(a\wedge b) &= [(p\wedge a)*a]\circ[(p\wedge a)*b]\\ &= [\{a*(a*p)\}*a]\circ[(p\wedge a)*b]\\ &= 0\circ[(p\wedge a)*b]\\ &= (p\wedge a)*b\\ &= (a*b)*(a*p).\end{aligned}$$

(b). Since $(a*b)*(a*p)\leq p*b$, if we let $p:=b$ in (a), then $(b\wedge a)*(a\wedge b)\leq b*b=0$, hence $b\wedge a\leq a\wedge b$, i.e., $a*(a*b)\leq b*(b*a)$. □

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