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CONTROLLED CONVERGENCE THEOREM FOR BANACH-VALUED HL INTEGRALS

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Abstract. Henstock's strongly variational integral for Banach-valued functions is called the HL integral, which is in the form of Henstock's Lemma. In this paper, we shall prove a controlled convergence theorem for such integrals.

The Henstock integral for Banach-valued functions has been discussed in [1-8, 10, 12, 17-23]. However Henstock's Lemma may not hold for such integral [2, 17-20]. The stronger version (see Definition 1.2) [2, 12], using Henstock's Lemma as a definition of an integral, has richer properties. For example, it has differentiation and measurability properties [2, 4, 6, 21, 23]. On the other hand, the Denjoy-Dunford, Denjoy-Pettis and Denjoy-Bochner integrals have been discussed in [7, 9, 11, 16, 24]. In [24], a controlled convergence theorem is claimed to be true without proof, for the Denjoy-Bochner integral. In this note, following the idea in [14], we shall prove a controlled convergence theorem for the HL integral. We remark that we do not follow the idea in [13, p40], since in [13, p40, line 17], we do not know whether the primitive function is differentiable a.e.

1 HL integral and $AC^*(X)$ In this section, we shall define the HL integral and discuss properties of $AC^*(X)$.

Definition 1.1. Let δ be a positive function on a closed interval $[a,b]$. A division $D = \{([u,v],\xi)\}$ of $[a,b]$ is said to be *Henstock δ-fine* if $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ for every $([u,v],\xi) \in D$.

In the following, we always use partial divisions instead of divisions. $D = \{([u,v],\xi)\}$ is said to be a partial division of $[a,b]$ if $\{[u,v]\}$ is a collection of nonoverlapping subintervals of $[a,b]$. The union of $[u,v]$ in D may not equal to $[a,b]$.

Definition 1.2. Let $(B, \| \ \|)$ denote a Banach space with norm $\| \ \|$. A function $f : [a,b] \to (B, \| \ \|)$ is *HL integrable* on $[a,b]$ if there exists a function $F : [a,b] \to (B, \| \ \|)$ satisfying the following property: for every $\epsilon > 0$, there exists a positive function $\delta(\xi)$ on $[a,b]$ such that if $D = \{([u,v],\xi)\}$ is a Henstock δ-fine partial division of $[a,b]$, we have

$$(D)\sum \|f(\xi)(v-u) - F(u,v)\| < \epsilon$$

where $F(u,v) = F(v) - F(u)$.

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Henceforth, a Banach-valued function shall be referred to as a function with values in $(B, \| \|)$.

Definition 1.3. A Banach-valued function F is said to be *absolutely continuous* on $[a, b]$ if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$, with $\sum_i |b_i - a_i| < \eta$ we have

$$\sum \|F(a_i, b_i)\| < \epsilon$$

where $F(a_i, b_i) = F(b_i) - F(a_i)$.

Definition 1.4. Let $X \subset [a, b]$. A Banach valued function F defined on X is said to be $AC(X)$ if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ satisfying $\sum_i |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all i, we have

$$\sum_i \|F(a_i, b_i)\| < \epsilon$$

where the endpoints $a_i, b_i \in X$ for all i.

Definition 1.5. A Banach-valued function F defined on $X \subset [a, b]$ is said to be $AC^*(X)$ if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ satisfying $\sum_i |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all i, we have

$$\sum_i \omega(F; [a_i, b_i]) < \epsilon$$

where ω denotes the oscillation of F over $[a_i, b_i]$, i.e.,

$$\omega(F; [a_i, b_i]) = \sup\{\|F(x, y)\|; x, y \in [a_i, b_i]\}.$$

Definition 1.6. A Banach-valued function F is said to be ACG^* on X if X is the union of a sequence of closed sets $\{X_i\}$ such that on each X_i, F is $AC^*(X_i)$.

Following ideas in [13, pp27-28], we can prove

Lemma 1.7. Let X be a closed set in $[a, b]$ and $(a, b) \setminus X$ be the union of (c_k, d_k) for $k = 1, 2, ...$ Suppose a Banach-valued function F is continuous on $[a, b]$. Then the following statements are equivalent:

(i) F is $AC^*(X)$

(ii) F is $AC(X)$ and $\sum_{k=1}^{\infty} \omega(F; [c_k, d_k]) < \infty$

(iii) Definition 1.4 holds with a_i or b_i belonging to X for every i.

To justify that X is closed in Definition 1.6, we shall prove the following lemma.

Lemma 1.8. Let $X \subset [a, b]$. If F is $AC^*(X)$ and continuous on $[a, b]$, then F is $AC^*(\overline{X})$, where $\overline{X}$ is the closure of X.

Proof. Suppose F is $AC^*(X)$. Then for every $\epsilon > 0$, there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in X$ and $\sum_i \mid b_i - a_i \mid < \eta$, we have

$$\sum_i \|F(b_i) - F(a_i)\| < \epsilon.$$

Now, let $\{[c_i, d_i]\}$ be any finite or infinite sequence of non-overlapping intervals with $c_i, d_i \in \overline{X}$ and $\sum_i \mid d_i - c_i \mid < \eta$. For each i, there exist $u_i, v_i \in X$ with $u_i < v_i$ and $\sum_i \mid v_i - u_i \mid < \eta$ such that

$$\|F(u_i) - F(c_i)\| < \frac{\epsilon}{2^i} \qquad \text{and} \qquad \|F(v_i) - F(d_i)\| < \frac{\epsilon}{2^i}.$$

Observe that $\{[u_i, v_i]\}$ may not be non-overlapping intervals. However, we can divide $\{[c_i, d_i]\}$ into two parts, wherein intervals in each part are disjoint, so that we can choose $\{[u_i, v_i]\}$ to be disjoint. Hence we may assume $\{[u_i, v_i]\}$ to be non-overlapping. As a result, we have

$$\begin{aligned}\sum_i \|F(d_i) - F(c_i)\| \quad &\leq \quad \sum_i \|F(d_i) - F(v_i)\| + \sum_i \|F(v_i) - F(u_i)\| \\ &\qquad + \sum_i \|F(u_i) - F(c_i)\| \\ &< \quad \epsilon + \epsilon + \epsilon.\end{aligned}$$

Therefore, F is $AC^*(\overline{X})$. □

Remark 1.9. Similarly, we can prove that if the statement (iii) in Lemma 1.7 holds for X, then it also holds for $\overline{X}$. Hence, when referring to $AC^*(X)$, we may assume that X is closed.

Definition 1.10. A sequence $\{f_n\}$ of Banach-valued functions is said to be *control convergent* to f on $[a, b]$ if the following conditions are satisfied:

(i) $f_n(x) \to f(x)$ a.e. in $[a, b]$ as $n \to \infty$ where each f_n is HL integrable in $[a, b]$;

(ii) the primitives F_n of f_n are ACG^* uniformly in n, i.e., $[a, b]$ is the union of a sequence of closed sets X_i such that on each X_i, the functions F_n are $AC^*(X_i)$ uniformly in n;

(iii) the primitives F_n converge uniformly on $[a, b]$.

2 Properties of HL integral Most of the theorems that we will be using in proving our main theorem shall be discussed in this section.

Lemma 2.1. If $f(x) = 0$ a.e. in $[a, b]$, i.e., for all $x \in [a, b]$ except perhaps on a set X of measure zero, then f is HL integrable to 0 on $[a, b]$.

The proof is standard [13, p6].

Theorem 2.2. If f is HL integrable on $[a, b]$, then its primitive F is continuous on $[a, b]$.

Proof. See [13, p12].

Theorem 2.3. If f is HL integrable on $[a, b]$, then its primitive F is ACG^* on $[a, b]$.

Proof. The proof is standard. However we shall give the detail here.

For every $\epsilon > 0$, there is a function $\delta(\xi) > 0$ such that for any Henstock δ-fine partial division $D = \{[u, v]; \xi\}$ in $[a, b]$, we have

$$(D)\sum \|F(u, v) - f(\xi)(v - u)\| < \epsilon.$$

We may assume that $\delta(\xi) \leq 1$. Let

$$X_{ni} = \{x \in [a, b] : \|f(x)\| \leq n; \frac{1}{n} < \delta(x) \leq \frac{1}{n-1} \text{ and } x \in [a + \frac{i-1}{n}, a + \frac{i}{n})\}$$

for $n = 2, 3, ..., i = 1, 2, ...$ Fix X_{ni} and let $\{[a_k, b_k]\}$ be any finite sequence of non-overlapping intervals with $a_k, b_k \in X_{ni}$ for all k. Then $\{([a_k, b_k], a_k)\}$ is a Henstock δ-fine partial division of $[a, b]$. Furthermore, if $a_k \leq u_k \leq v_k \leq b_k$, then $\{([a_k, u_k], a_k)\}, \{([v_k, b_k], b_k)\}$ are Henstock δ-fine partial divisions of $[a, b]$. Thus,

$$\begin{aligned}
\sum_k \|F(u_k, v_k)\| &\leq \sum_k \|F(a_k, u_k)\| + \sum_k \|F(v_k, b_k)\| + \sum_k \|F(a_k, b_k)\| \\
&\leq 3\epsilon + \sum_k \|f(a_k)(u_k - a_k)\| + \sum_k \|f(b_k)(b_k - v_k)\| \\
&\quad + \sum_k \|f(a_k)(b_k - a_k)\| \\
&\leq 3\epsilon + 3n\sum_k (b_k - a_k).
\end{aligned}$$

Choose $\eta \leq \frac{\epsilon}{3n}$ and $\sum_k (b_k - a_k) < \eta$. Then

$$\sum_k \omega(F; [a_k, b_k]) \leq 3\epsilon + \epsilon.$$

Therefore, F is $AC^*(X_{ni})$ and also $AC^*(\overline{X}_{ni})$. Consequently, F is ACG^* on $[a, b]$. □

Theorem 2.4. If f is HL integrable on $[a,b]$, then its primitive F is differentiable a.e. and $F'(x) = f(x)$ a.e. on $[a,b]$.

Proof. See [13, p21].

Theorem 2.5. Let $(B, \|\|)$ be a Banach space and $f : [a,b] \to (B, \|\|)$. Suppose there exists a function $F : [a,b] \to B$ which is continuous and ACG^* on $[a,b]$ such that $F'(x) = f(x)$ a.e. in $[a,b]$. Then f is HL integrable on $[a,b]$ with primitive F.

Proof. See [13, p31].

The following is a special version of Egoroff's theorem for Banach-valued functions.

Lemma 2.6. If $f_n(x) \to f(x)$ a.e. in $[a,b]$ as $n \to \infty$ where each f_n is HL integrable then for every $\eta > 0$ there exists an open set G with $|G| < \eta$ such that f_n converges uniformly to f on $[a,b] \setminus G$.

Theorem 2.7. Suppose
(i) $f_n(x) \to f(x)$ a.e. in $[a,b]$ as $n \to \infty$ where each f_n is HL integrable on $[a,b]$
(ii) the primitives F_n of f_n are uniformly absolutely continuous.
Then for every $\epsilon > 0$ there exists a positive integer N such that for every partial partition $D = \{[u,v]\}$ of $[a,b]$ we have

$$(D)\sum \|F_n(u,v) - F_m(u,v)\| < \epsilon$$

whenever $n, m \geq N$.

Proof. See [13, pp 37 - 38].

Theorem 2.8. Suppose
(i) $f_n(x) \to f(x)$ a.e. in $[a,b]$ as $n \to \infty$, where each f_n is HL integrable on $[a,b]$;
(ii) the primitives F_n of f_n are uniformly absolutely continuous.
Then f is HL integrable on $[a,b]$ and

$$\int_a^b f_n \longrightarrow \int_a^b f \qquad \text{as} \quad n \to \infty.$$

Proof. See [13, p38].

Theorem 2.9. Let $\{f_n\}$ be a sequence of Banach-valued functions on $[a,b]$ which is control convergent to f on $[a,b]$. Then for each X_i and for every $\epsilon > 0$, there exists a positive integer N such that for every partial partition $D = \{[u,v]\}$ of $[a,b]$ with $u, v \in X_i$, we have

$$(D)\sum \omega(F_n - F_m; [u,v]) < \epsilon$$

whenever $n, m \geq N$.

Proof. Fix X_i and let $X = X_i$. Assume that $a, b \in X$. Define $G_n(x) = F_n(x)$ when $x \in X$ and linear elsewhere in $[a,b]$. More precisely, let $(a,b) \setminus X = \bigcup_k (a_k, b_k)$ and define

$$G_n(x) = \begin{cases} F_n(x) & \text{if } x \in X \\ \dfrac{b_k - x}{b_k - a_k} F_n(a_k) + \dfrac{x - a_k}{b_k - a_k} F_n(b_k) & \text{if } x \in (a_k, b_k), k = 1, 2, \ldots \end{cases}$$

Observe that if $\{[u_i, v_i]\}$ is a finite or infinite sequence of non-overlapping intervals contained in (a_k, b_k), then

$$\begin{aligned} \sum_i \|G_n(u_i, v_i)\| &= \sum_i \|(\frac{b_k - v_i}{b_k - a_k} F_n(a_k) + \frac{v_i - a_k}{b_k - a_k} F_n(b_k)) \\ &\qquad -(\frac{b_k - u_i}{b_k - a_k} F_n(a_k) + \frac{u_i - a_k}{b_k - a_k} F_n(b_k))\| \\ &= \frac{1}{b_k - a_k} \sum_i \|(v_i - u_i)(F_n(b_k) - F_n(a_k))\| \\ &= \frac{\|F_n(b_k) - F_n(a_k)\|}{b_k - a_k} \sum_i |v_i - u_i|. \end{aligned} \tag{2.1}$$

On the other hand, $\sum_k \omega(F_n; [a_k, b_k])$ converges uniformly in n, due to the fact that F_n is $AC^*(X)$ uniformly in n. Hence, by (2.1), we need only to consider the first finite number of intervals $[a_k, b_k], k = 1, 2, ..., m$. It is clear from (2.1) that the functions G_n are absolutely continuous on each $[a_k, b_k]$. Consequently, the functions G_n are uniformly absolutely continuous on $[a, b]$ in view of the fact that $G_n(x) = F_n(x)$ on X.

Now, define

$$g_n(x) = \begin{cases} f_n(x) & \text{if } x \in X \\ \dfrac{F_n(b_k) - F_n(a_k)}{b_k - a_k} & \text{if } x \in (a_k, b_k). \end{cases}$$

Then g_n converges a.e. on $[a, b]$ and $g_n(x) \to f(x)$ a.e. on X. We shall now use Definition 1.2 to prove that each g_n is HL integrable on $[a, b]$ and the primitive of g_n is G_n. First, note that if $\xi \in (a_k, b_k)$, we can choose $\delta(\xi) > 0$ such that whenever $([u, v], \xi)$ is δ-fine, we have $[u, v] \subset (a_k, b_k)$. By linearity of G_n on (a_k, b_k) and definition of g_n, we have

$$\|g_n(\xi)(v - u) - G_n(u, v)\| = 0.$$

Secondly if $\xi \in X$, we consider interval-point pairs of the form $([u, \xi], \xi)$ or $([\xi, v], \xi)$. For the case $u, v \in X$, we observe that

$$\|g_n(\xi)(\xi - u) - G_n(u, \xi)\| = \|f_n(\xi)(\xi - u) - F_n(u, \xi)\|.$$

Similarly for the case $([\xi, v], \xi)$. Hence we need only to consider the case when $u, v \notin X$. Now suppose $u \in (a_k, b_k)$ for some k. Then

$$\begin{aligned} &\|g_n(\xi)(\xi - u) - G_n(u, \xi)\| \\ \leq\ &\|f_n(\xi)(\xi - b_k) - G_n(b_k, \xi)\| + \|f_n(\xi)(b_k - u) - G_n(u, b_k)\| \\ =\ &\|f_n(\xi)(\xi - b_k) - F_n(b_k, \xi)\| + \|f_n(\xi)(b_k - u) - G_n(u, b_k)\|. \end{aligned}$$

Therefore finally we need only to consider

$$\|f_n(\xi)(b_k - u) - G_n(u, b_k)\|.$$

Let $X_q = \{\xi \in X; q-1 \leq \|f_n(\xi)\| < q\}$, $q = 1, 2, \ldots$. Let q be fixed. Given $\epsilon > 0$, we first choose ℓ such that

$$\sum_{k=\ell}^{\infty} |b_k - a_k| < \frac{\epsilon}{q \cdot 2^q} \text{ and } \sum_{k=\ell}^{\infty} \omega(F_n; [a_k, b_k]) < \frac{\epsilon}{2^q}.$$

Let $\xi \in X_q$ and $\xi \neq a_k$, b_k for all k. Now we choose $\delta(\xi) > 0$ such that when $[a_k, b_k] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$, we have $k \geq \ell$. Hence, if $D = \{([u, \xi], \xi)\}$ is a δ-fine partial division with $\xi \in X_q$, we have

$$\begin{aligned} & (D)\sum \|f_n(\xi)(b_k - u) - G_n(u, b_k)\| \\ \leq & (D)\sum \|f_n(\xi)(b_k - u)\| + (D)\sum \|G_n(u, b_k)\| \\ < & \frac{\epsilon}{2^q} + \frac{\epsilon}{2^q}. \end{aligned}$$

When $\xi \in X_q$ and $\xi = a_p$ or b_p for some p, in view of the continuity of G_n at ξ, we can choose $\delta(\xi) > 0$ such that when $([u, v], \xi)$ in δ-fine, we have

$$\begin{aligned} & \|g_n(\xi)(v - u) - G_n(u, v)\| \\ = & \|f_n(\xi)(v - u) - G_n(u, v)\| \\ \leq & \|f_n(\xi)(v - u)\| + \|G_n(u, v)\| \\ < & \epsilon/2^p + \epsilon/2^p. \end{aligned}$$

From the above analysis, g_n is HL integrable on $[a, b]$ with primitive G_n. By Theorem 2.7 we get the required result, without oscillation. To get the required result with oscillation, we observe that, for each n, there exist $p_k, q_k \in [a_k, b_k]$ such that

$$\omega(F_n - F; [a_k, b_k]) = \|(F_n - F)(p_k, q_k)\|.$$

However, p_k and q_k depend on n. Now we do some adjustment. Let $c_k, d_k \in (a_k, b_k)$ with $c_k < d_k$ and fixed, independent of n. Define $H_n(x) = F_n(x) - F(x)$ if $x \in X$ and linearly on $[a_k, c_k]$, $[c_k, d_k]$ and $[d_k, b_k]$ with $H_n(a_k) = (F_n - F)(a_k)$; $H_n(c_k) = (F_n - F)(b_k)$; $H_n(d_k) = (F_n - F)(p_k, q_k) + H_n(c_k)$ and $H_n(b_k) = H_n(c_k)$. Hence $\|H_n(d_k) - H_n(c_k)\| = w(F_n - F; [a_k, b_k])$, and the oscillation of H_n over $[a_k, b_k]$ is equal to that of $F_n - F$ over $[a_k, b_k]$.

As in the proof of the first part with $G_n(x)$ replaced by $H_n(x)$ and $\{[a_k, b_k]\}$ replaced by $\{[a_k, c_k], [c_k, d_k], [d_k, b_k]\}$, by Theorem 2.7, given any $\epsilon > 0$, there exists a positive integer N such that for any partial partition $D = \{[u, v]\}$ of $[a, b]$, we have

$$(D)\sum \|H_n(u, v) - H_m(u, v)\| < \epsilon$$

whenever $n, m \geq N$. Note that the limit of the sequence $H_n(x)$ exists as $n \to \infty$ for each x and in view of (iii) of Definition 1.10, it is zero. Thus, the above inequality implies that

$$(D)\sum \|H_n(u,v)\| < \epsilon \tag{2.2}$$

whenever $n \geq N$. Observe that if $u, v \in X$ and $[p,q] \subset (u,v)$ with $p, q \notin X$, we can divide $[p,q]$ into three sub-intervals, where two of them are in $\bigcup_k [a_k, b_k]$ and another with endpoints in X, namely $[p,q] = [p,s] \cup [s,t] \cup [t,q]$, where $p \in [a_i, b_i]$, $q \in [a_j, b_j]$ and $s, t \in X$. Then

$$\begin{aligned}
\|(F_n - F)(p,q)\| &\leq \|(F_n - F)(p,s)\| + \|(F_n - F)(s,t)\| + \|(F_n - F)(t,q)\| \\
&\leq w(F_n - F;[a_i,b_i]) + \|(F_n - F)(s,t)\| + w(F_n - F;[a_j,b_j]) \\
&= \|H_n(c_i,d_i)\| + \|H_n(s,t)\| + \|H_n(c_j,d_j)\|
\end{aligned}$$

Hence, by (2.2), for any partial partition $D = \{[u,v]\}$ of $[a,b]$ and any $[p,q] \subset [u,v]$, we have

$$(D)\sum \|(F_n - F)(p,q)\| < \epsilon$$

whenever $n \geq N$. Note that $[p,q]$ is any subinterval of $[u,v]$. Thus

$$(D)\sum w(F_n - F;[u,v]) \leq \epsilon$$

whenever $n \geq N$. Consequently

$$(D)\sum w(F_n - F_m;[u,v]) \leq 2\epsilon$$

whenever $n, m \geq N$.

3 Main Result Theorem 3.1. Controlled Convergence Theorem

If a sequence of Banach-valued functions $\{f_n\}$ is control convergent to f on $[a,b]$, then f is also HL integrable on $[a,b]$ and

$$\int_a^b f_n(x)dx \longrightarrow \int_a^b f(x)dx \qquad \text{as} \quad n \to \infty.$$

Proof. In view of Lemma 2.1, we may assume $f_n(x) \to f(x)$ everywhere in $[a,b]$ as $n \to \infty$. Since each f_n is HL integrable on $[a,b]$, with primitive F_n, then given $\epsilon > 0$ there exists $\delta_n(\xi) > 0$ such that for any Henstock δ_n-fine partial division $D = \{[u,v];\xi\}$ of $[a,b]$, we have

$$(D)\sum \|f_n(\xi)(v-u) - F_n(u,v)\| < \epsilon 2^{-n}. \tag{3.1}$$

Since $f_n(x) \to f(x)$, there exists a positive integer $m = m(\epsilon, \xi)$ such that

$$\|f_m(\xi) - f(\xi)\| < \epsilon. \tag{3.2}$$

By the hypothesis, we also have

$$\lim_{n\to\infty} F_n(u,v) = F(u,v) \qquad \text{exists} \tag{3.3}$$

for any sub-interval $[u, v]$ of $[a, b]$.

From the definition of control convergence, $[a, b]$ is the union of a sequence of closed sets X_i such that on each X_i, the functions F_n are $AC^*(X_i)$ uniformly in n. By Theorem 2.9, it follows that, for each i, there exists a positive integer $N(i)$ such that for any partial partition $D = \{[u, v]\}$ of $[a, b]$ with $u, v \in X_i$, we have

$$(D) \sum w(F_n - F; [u, v]) < \epsilon$$

whenever $n \geq N(i)$.

Hence, for each i, there exists a subsequence $\{F_{n(i,j)}\}_{j=1}^{\infty}$ of $\{F_n\}_{n=1}^{\infty}$ such that

$$(D) \sum w(F_{n(i,j)} - F; [u, v]) < \epsilon 2^{-i-j} \tag{3.4}$$

for any partial partition $D = \{[u, v]\}$ of $[a, b]$ with $u, v \in X_i$. We may assume that for each $i > 1$, $\{F_{n(i,j)}\}_{j=1}^{\infty}$ is a subsequence of $\{F_{n(i-1,j)}\}_{j=1}^{\infty}$. From now onwards, $n(j, j)$ is denoted by $m(j)$, and we only consider subsequences $\{f_{m(j)}\}$ and $\{F_{m(j)}\}$. Now we shall define $\delta(\xi)$ on $[a, b]$. If $\xi \in Y_i = X_i \backslash (X_1 \cup X_2 \cup \cdots \cup X_{i-1})$, where $X_0 = \emptyset$, then we choose $m(j) > m(i)$ such that $\|f_{m(j)}(\xi) - f(\xi)\| < \epsilon$. Note that $m(j)$ depends on ξ. We denote $m(j)$ by $m(\xi)$. Define $\delta(\xi) = \delta_{m(\xi)}(\xi)$. Let $D = \{([u, v]; \xi)\}$ be any Henstock δ-fine partial division of $[a, b]$, we shall prove that

$$(D) \sum \|f(\xi)(v - u) - F(u, v)\| < \epsilon(b - a) + 2\epsilon. \tag{3.5}$$

First

$$\begin{aligned}(D) \sum \|f(\xi)(v - u) - F(u, v)\| \quad &\leq \quad (D) \sum \|f(\xi) - f_{m(\xi)}(\xi)\|(v - u) \\ &\quad + (D) \sum \|f_{m(\xi)}(\xi)(v - u) - F_{m(\xi)}(u, v)\| \\ &\quad + (D) \sum \|F_{m(\xi)}(u, v) - F(u, v)\|\end{aligned}$$

The first sum on the right side of the above inequality is less than $\epsilon(b - a)$. The second sum can be written as

$$\sum_{j=1}^{\infty} (D_j) \sum \|f_{m(\xi)}(\xi)(v - u) - F_{m(\xi)}(u, v)\|,$$

where $D_j = \{([u, v], \xi)\}$ is a subset of D and each ξ in D_j induces the same $m(j)$ i.e. $m(\xi) = m(j)$ for all ξ in D_j. Hence the second sum is less than

$$\epsilon \sum_{j=1}^{\infty} 2^{-m(j)}. \qquad \text{by (3.1)}$$

Consequently it is less than ϵ. Now we shall handle the third sum. For convenience, we may assume that $a, b \in X_i$, for all i. For any $([u, v], \xi)$ in D, $[u, v] = [u, \xi] \cup [\xi, v]$. Suppose $\xi \in Y_i = X_i \backslash (X_1 \cup X_2 \cup \cdots \cup X_{i-1})$. Then either $u \in X_i$ or $[u, \xi]$ lies in an interval with endpoints in X_i. On the other hand, the third sum can be written as

$$\sum_{i} \sum_{j} \sum_{\xi \in X_i, m(\xi)=m(j)} \|F_{m(\xi)}(u,v) - F(u,v)\|.$$

Recall that $m(\xi) = m(j) = n(j,j) > n(i,i)$. Thus $j > i$. Hence $\{n(j,k)\}_{k=1}^{\infty}$ is a subsequence of $\{n(i,k)\}_{k=1}^{\infty}$. So $m(j) = n(j,j) = n(i,k(j))$ for some $k(j)$. Hence, by (3.4),

$$\sum_{\xi \in X_i, m(\xi)=m(j)} \|F_{m(\xi)}(u,v) - F(u,v)\| \leq \epsilon 2^{-i-k(j)}.$$

Note that $\{n(j+1,k)\}_{k=1}^{\infty}$ is a subsequence of $\{n(j,k)\}_{k=1}^{\infty}$. We may choose $\{n(j+1,k)\}_{k=1}^{\infty}$ such that $k(j)$ is strictly increasing. Thus the third sum is less than ϵ. Consequently, (3.5) holds. With (3.5) and (3.3), the proof is complete.

Remark. In general, a Banach-valued function F which is ACG^* may not be differentiable a.e. From the result of proof, we know that F is differentiable a.e. if it satisfies the conditions of Theorem 3.1, however the ideas in [13, p40] does not work for proving the above theorem, since in the proof, we use the result "F is differentiable a.e.".

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