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NOSHIRO-TYPE HARMONIC UNIVALENT FUNCTIONS

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Abstract. We define and investigate a family of Noshiro-Type complex-valued harmonic functions of the form $f = h + \bar{g}$, where h and g are analytic in the unit disk Δ. A sufficient coefficient condition for these functions to be univalent and starlike in Δ is determined. This coefficient condition is shown to be also necessary if h has negative and g has positive coefficients. Furthermore, distortion theorem, extreme points, convolution conditions, and convex combinations for this family of harmonic functions are obtained.

1. Introduction. A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = Re(U)$ and $v = Im(V)$. Then $f(z) = h(z) + \overline{g(z)}$ where h and g are, respectively, the analytic functions $(U+V)/2$ and $(U-V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. The mapping $z \to f(z)$ is orientation preserving and locally one-to-one in D if and only if $J_f(z) > 0$ in D. The necessity of this condition is a result of Lewy [6]. See also Clunie and Sheil-Small [3]. The function $f = h + \overline{g}$ is said to be harmonic univalent in D if the mapping $z \to f(z)$ is orientation preserving, harmonic, and one-to-one in D. We call h the analytic part and g the co-analytic part of $f = h + \overline{g}$.

Let H denote the family of functions $f = h + \overline{g}$ that are harmonic, orientation preserving, and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$ with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1}$$

For $0 \le \beta < 1$ we let $K_H(\beta)$ and $S_H(\beta)$, respectively, denote the subclasses of H consisting of functions in H that are convex of order β and starlike of order β. We further let $K_{\overline{H}}(\beta)$ and $S_{\overline{H}}(\beta)$, be the respective subclasses of $K_H(\beta)$ and $S_H(\beta)$ consisting of functions $f = h + \bar{g}$ so that

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \ , \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{2}$$

The classes $K_H(0)$, $S_H(0)$, $K_{\overline{H}}(0)$, and $S_{\overline{H}}(0)$ were studied by Avci and Zlotkiewicz [2], Silverman [8], and Silverman and Silvia [9]. Jahangiri in [4,5] among other results proved the following two theorems.

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Theorem A. Let $f = h + \bar{g}$ be so that h and g are given by (1). Then $f \in S_H(\beta)$ if

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n+\beta}{1-\beta}|b_n| \leq 1, \quad 0 \leq \beta < 1. \tag{3}$$

The condition (3) is shown to be also necessary for $f \in S_{\overline{H}}(\beta)$.

Theorem B. Let $f = h + \bar{g}$ be so that h and g are given by (1). Then $f \in K_H(\beta)$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta}|b_n| \leq 1, \quad 0 \leq \beta < 1. \tag{4}$$

The condition (4) is shown to be also necessary for $f \in K_{\overline{H}}(\beta)$.

Silverman[8] obtained results analogous to Theorems A and B for the special case $b_1 = \alpha = 0$ and Silverman and Silvia [9] improved the results of [8] to the case when b_1 is not necessarily zero. In this paper we define a new class of harmonic functions which contains harmonic convex functions and has an interesting inclusion relation with the class of harmonic starlike functions. We then obtain sufficient and necessary coefficient bounds, distortion theorem, extreme points, convolution conditions, and convex combinations for this class.

For $0 \leq \alpha < 1$ we let $N_H(\alpha)$ denote the class of harmonic functions $f = h + \bar{g}$ where h and g are given by (1) and satisfy the condition

$$Re \frac{\frac{\partial}{\partial\theta} f(z)}{\frac{\partial}{\partial\theta} z} \geq \alpha\ , \quad z = re^{i\theta}\ , \quad z \in \Delta. \tag{5}$$

The subclass of $N_H(\alpha)$ where h and g are given by (2) is denoted by $N_{\overline{H}}(\alpha)$.

We used the letter N in the symbol $N_H(\alpha)$ in the honor of Noshiro [7] who first introduced such characterization (5) for the especial case $\alpha = 0$. Later Al-Amiri [1] generalized Noshiro's results to the case when $0 \leq \alpha < 1$.

2. Coefficient Bounds.

First we prove the sufficient coefficient bounds.

Theorem 1. Let $f = h + \bar{g}$ be so that h and g are given by (1). Furthermore, let

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \ \leq\ 2 - \alpha,\ a_1 = 1,\ 0 \leq \alpha < 1. \tag{6}$$

Then f is orientation preserving, harmonic univalent in Δ, and $f \in N_H(\alpha)$.

Proof. First we note that f is locally univalent and orientation preserving in Δ. This is the case because

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n|r^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \geq 1 - \sum_{n=2}^{\infty} \frac{n}{1-\alpha}|a_n| \\ &\geq \sum_{n=1}^{\infty} \frac{n}{1-\alpha}|b_n| \geq \sum_{n=1}^{\infty} n|b_n| \ > \ \sum_{n=1}^{\infty} n|b_n|r^{n-1} \geq |g'(z)|. \end{aligned}$$

To show that f is univalent in Δ we notice that if $g(z) \equiv 0$, then $f(z)$ is analytic and the univalence of f follows from its starlikeness. If $g(z) \not\equiv 0$, then we show that $f(z_1) \neq f(z_2)$ when $z_1 \neq z_2$. Suppose $z_1,\ z_2 \in \Delta$ so that $z_1 \neq z_2$. Since Δ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in \Delta$ where $0 \leq t \leq 1$. Then we can write

$$f(z_2) - f(z_1) = \int_0^1 \left[(z_2 - z_1)h'(z(t)) + \overline{(z_2 - z_1)g'(z(t))}\right] dt.$$

Dividing the above equation by $z_2 - z_1 \neq 0$ and taking the real parts we obtain

$$\begin{aligned} Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} &= \int_0^1 Re \left[h'(z(t)) + \frac{\overline{z_2 - z_1}}{z_2 - z_1}\overline{g'(z(t))}\right] dt \\ &> \int_0^1 \left[Re h'(z(t)) - |g'(z(t))|\right] dt. \end{aligned} \tag{7}$$

On the other hand

$$\begin{aligned} Re h'(z) - |g'(z)| &\geq Re(h'(z)) - \sum_{n=1}^{\infty} n|b_n| \geq 1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n| \\ &\geq 2 - \alpha - \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \geq 0\ , \quad by\ (6). \end{aligned}$$

This in conjunction with the inequality (7) lead to the univalence of f.

Now we show that $f \in N_H(\alpha)$. Letting $\omega(z) = (\frac{\partial}{\partial\theta} f(z))/(\frac{\partial}{\partial\theta} z)$, it suffices to show that $|1 - \alpha + \omega| \geq |1 + \alpha - \omega|$ for $0 \leq \alpha < 1$. Or equivalently, we need to show that

$$|(1-\alpha)z + zh'(z) - \overline{zg'(z)}| - |(1+\alpha)z - zh'(z) + \overline{zg'(z)}| \geq 0.$$

Differentiating h and g and substituting in the above inequality we obtain

$$\begin{aligned} &|(1-\alpha)z + zh'(z) - \overline{zg'(z)}| - |(1+\alpha)z - zh'(z) + \overline{zg'(z)}| \\ &= \ |(2-\alpha)z + \textstyle\sum_{n=2}^{\infty} na_n z^n - \overline{\sum_{n=1}^{\infty} nb_n z^n}| \ - \ |\alpha z - \sum_{n=2}^{\infty} na_n z^n + \overline{\sum_{n=1}^{\infty} nb_n z^n}| \\ &\geq (2-\alpha)|z| - \textstyle\sum_{n=2}^{\infty} n|a_n||z|^n - \sum_{n=1}^{\infty} n|b_n||z^n| - \alpha|z| - \sum_{n=2}^{\infty} n|a_n||z|^n - \sum_{n=1}^{\infty} n|b_n||z|^n \\ &= \ 2\{(1-\alpha)|z| - \textstyle\sum_{n=2}^{\infty} n|a_n||z|^n - \sum_{n=1}^{\infty} n|b_n||z|^n\} \\ &\geq 2|z|\{2 - \alpha - \textstyle\sum_{n=1}^{\infty} n(|a_n| + |b_n|)\}. \end{aligned}$$

This last expression is non-negative by the hypothesis and so the proof is complete.

The restriction in Theorem 1 placed on the moduli of the coefficients of $f = h + \bar{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent. Our next theorem establishes that such coefficient bounds can not be improved.

Theorem 2. Let $f = h + \bar{g}$ be so that h and g are given by (2). Then $f \in N_{\overline{H}}(\alpha)$ if and only if $\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 2 - \alpha$, where $a_1 = 1$, and $0 \leq \alpha < 1$.

Proof. The "*if*" part follows from Theorem 1 upon noting that $N_{\overline{H}}(\alpha) \subset N_H(\alpha)$. For the "*only if*" part, let $f \in N_{\overline{H}}(\alpha)$. Then for $z = re^{i\theta}$, $0 \leq r < 1$, and θ real we have

$$\begin{aligned} Re\frac{\frac{\partial}{\partial\theta}f(z)}{\frac{\partial}{\partial\theta}z} &= Re\left\{h'(z) - \frac{1}{z}\overline{zg'(z)}\right\} \\ &= Re\left\{1 - \sum_{n=2}^{\infty} n|a_n|r^{n-1}e^{i(n-1)\theta} - \sum_{n=1}^{\infty} n|b_n|r^{n-1}e^{-i(n+1)\theta}\right\} \\ &= 1 - \sum_{n=2}^{\infty} n|a_n|r^{n-1}cos(n-1)\theta - \sum_{n=1}^{\infty} n|b_n|r^{n-1}cos(n+1)\theta \\ &= 2 - \sum_{n=1}^{\infty} n\left[|a_n|cos(n-1)\theta + |b_n|cos(n+1)\theta\right]r^{n-1} \geq \alpha. \end{aligned}$$

The above inequality must hold for all $z \in \Delta$. In particular, letting $z = r \to 1$ lead to the required condition $2 - \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \geq \alpha$.

3. Inclusion Relations, Distortion Theorem, and Extreme Points. In the next two theorems we determine inclusion relations between classes $K_{\overline{H}}(\beta)$, $S_{\overline{H}}(\beta)$, and $N_{\overline{H}}(\alpha)$.

Theorem 3. If $0 \leq \alpha \leq \beta < 1$ then $K_{\overline{H}}(\beta) \subset N_{\overline{H}}(\alpha)$. This inclusion is proper.

Proof. Let $f = h + \bar{g} \in K_{\overline{H}}(\beta)$ where h and g are given by (2). Then, by Theorem B and the condition $0 \leq \alpha \leq \beta < 1$, we have

$$\begin{aligned} |b_1| + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) &\leq (1+\beta)|b_1| + \sum_{n=2}^{\infty}[n(n-\beta)|a_n| + n(n+\beta)|b_n|] \\ &\leq 1 - \beta \leq 1 - \alpha. \end{aligned}$$

Therefore, $f = h + \bar{g} \in N_{\overline{H}}(\alpha)$, by Theorem 2.

To show that the inclusion is proper, consider the function

$$f(z) = z - \frac{1-\alpha}{4}z^2 + \frac{1-\alpha}{4}\bar{z}^2.$$

We observe that

$$2(|a_2| + |b_2|) = 2(\frac{1-\alpha}{4} + \frac{1-\alpha}{4}) = 1 - \alpha$$

and so, $f = h + \bar{g} \in N_{\overline{H}}(\alpha)$. On the other hand

$$\frac{2(2-\beta)}{1-\beta}|a_2| + \frac{2(2+\beta)}{1-\beta}|b_2| = \frac{2(2-\beta)}{1-\beta}\frac{1-\alpha}{4} + \frac{2(2+\beta)}{1-\beta}\frac{1-\alpha}{4} = \frac{2(1-\alpha)}{1-\beta)} > 2.$$

Thus, by Theorem B, $f = h + \bar{g} \notin K_{\overline{H}}(\beta)$.

Theorem 4. If $0 \leq \alpha, \beta < 1$ and $0 \leq \beta \leq \frac{\alpha}{2-\alpha}$ then we have the proper inclusion relation $N_{\overline{H}}(\alpha) \subset S_{\overline{H}}(\beta)$.

Proof. Let $f \in N_{\bar{H}}(\alpha)$. Then by the hypotheses of the theorem we have

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty}\frac{n+\beta}{1-\beta}|b_n| \ \leq \ \sum_{n=2}^{\infty}\frac{n}{1-\alpha}|a_n| + \sum_{n=1}^{\infty}\frac{n}{1-\alpha}|b_n| \ \leq \ 1.$$

Therefore, by Theorem A, $f \in S_{\overline{H}}(\beta)$. Now consider the function

$$f(z) = z - \frac{1-\beta}{2(2-\beta)}z^2 + \frac{1-\beta}{2(2+\beta)}\bar{z}^2.$$

We see that $f \in S_{\overline{H}}(\alpha)$ since

$$\frac{2-\beta}{1-\beta}\left[\frac{1-\beta}{2(2-\beta)}\right] + \frac{2+\beta}{1-\beta}\left[\frac{1-\beta}{2(2+\beta)}\right] = 1.$$

On the other hand

$$\frac{2}{1-\alpha}\left[\frac{1-\beta}{2(2-\beta)}\right] + \frac{2}{1-\alpha}\left[\frac{1-\beta}{2(2+\beta)}\right] = \frac{1-\beta}{1-\alpha}\left(\frac{4}{4-\beta^2}\right) > 1.$$

Therefore, by Theorem 2, $f \notin N_{\overline{H}}(\alpha)$, and so the inclusion is proper.

Corollary. From Theorems A and 2, it follows that $N_{\overline{H}}(0) \equiv S_{\overline{H}}(0)$.

Theorem 5. Let $f = h + \bar{g}$ be so that h and g are given by (2). If $f \in N_{\overline{H}}(\alpha)$ then

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{2}(1-\alpha-|b_1|)r^2, \quad |z| = r < 1,$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{2}(1-\alpha-|b_1|)r^2, \quad |z| = r < 1.$$

Proof. We prove the left hand inequality only. The proof for the right hand inequality is similar and we omit it. Let $f = h + \bar{g} \in N_{\overline{H}}(\alpha)$, and take the absolute value of f. Using the fact that $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq (1-\alpha-|b_1|)$, we obtain

$$\begin{aligned}
|f(z)| = \left| z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\bar{z}^n \right| &\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 &\geq (1-|b_1|)r - \frac{1}{2}\sum_{n=2}^{\infty}n(|a_n|+|b_n|)r^2 \\
\geq (1-|b_1|)r - \frac{1}{2}(1-\alpha-|b_1|)r^2, &\quad |z| = r < 1.
\end{aligned}$$

Equality occurs for $f = h + \bar{g}$ if the coefficients of h and g are so that $a_1 = 1,\ a_n \equiv 0$ $(n = 2, 3, \ldots),\ b_1 = b_1,\ b_2 = \frac{1}{2}(1-\alpha-|b_1|)$, and $b_n \equiv 0$ $(n = 3, 4, \ldots)$.

As a consequence of the left hand inequality in Theorem 5 we have

Corollary. If $f = h + \bar{g} \in N_{\overline{H}}(\alpha)$ then $\left\{w : |w| < \frac{1}{2}(1+\alpha-|b_1|)\right\} \subset f(\Delta)$.

Now we determine the extreme points of closed convex hulls of $N_H(\alpha)$ denoted by *clco* $N_{\overline{H}}(\alpha)$.

Theorem 6. $f \in$ *clco* $N_{\overline{H}}(\alpha)$ if and only if $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$, where $h_1(z) = z,\ h_n(z) = z - \frac{1-\alpha}{n}z^n$ $(n = 2, 3, \ldots),\ g_n(z) = z + \frac{1-\alpha}{n}\bar{z}^n$ $(n = 1, 2, 3, \ldots),\ \sum_{n=1}^{\infty}(X_n + Y_n) = 1,\ X_n \geq 0$, and $Y_n \geq 0$. In particular, the extreme points of $N_{\bar{H}}(\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. First write

$$f(z)=\sum_{n=1}^{\infty}(X_nh_n+Y_ng_n)=\sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{1-\alpha}{n}X_nz^n+\sum_{n=2}^{\infty}\frac{1-\alpha}{n}Y_n\bar{z}^n$$
$$=z-\sum_{n=2}^{\infty}\frac{1-\alpha}{n}X_nz^n+\sum_{n=2}^{\infty}\frac{1-\alpha}{n}Y_n\bar{z}^n.$$

Then

$$\sum_{n=2}^{\infty}\frac{n}{1-\alpha}(|a_n|)+\sum_{n=1}^{\infty}\frac{n}{1-\alpha}(|b_n|)=\sum_{n=2}^{\infty}\frac{n}{1-\alpha}\left(\frac{1-\alpha}{n}X_n\right)+\sum_{n=1}^{\infty}\frac{n}{1-\alpha}\left(\frac{1-\alpha}{n}Y_n\right)$$
$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\leq 1.$$

Therefore $f \in$ *clco* $N_{\overline{H}}(\alpha)$. Conversely, suppose that $f\in clcoN_{\overline{H}}(\alpha)$. Set $X_n=\frac{n}{1-\alpha}|a_n|$ $(n=2,3,\ \ldots)$, $Y_n=\frac{n}{1-\alpha}|b_n|$ $(n=1,2,3,\ \ldots)$, where $\sum_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^n=z-\sum_{n=2}^{\infty}\frac{1-\alpha}{n}X_nz^n+\sum_{n=1}^{\infty}\frac{1-\alpha}{n}Y_n\bar{z}^n$$
$$=z+\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n=\sum_{n=1}^{\infty}(X_nh_n+Y_ng_n).$$

From Theorem 2 it is easily seen that $N_{\overline{H}}(\alpha)$ is convex and closed and so *clco* $N_{\overline{H}}(\alpha)\equiv N_{\overline{H}}(\alpha)$. In other words, the statement of Theorem 6 is actually for $N_{\overline{H}}(\alpha)$.

4. Convolutions and Convex Combinations. In the next theorem we examine the convolution properties of the class $N_{\overline{H}}(\alpha)$.

Define the convolution of two harmonic functions $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^n$ and $F(z)=z-\sum_{n=2}^{\infty}|A_n|z^n+\sum_{n=1}^{\infty}|B_n|\bar{z}^n$ by

$$(f*F)(z)=z-\sum_{n=2}^{\infty}|a_nA_n|z^n+\sum_{n=1}^{\infty}|b_nB_n|\bar{z}^n. \tag{8}$$

Theorem 7. For $0\leq\beta\leq\alpha<1$, suppose that $f\in N_{\overline{H}}(\alpha)$ and $F\in N_{\overline{H}}(\beta)$. Then $f*F\in N_{\overline{H}}(\alpha)\subset N_{\overline{H}}(\beta)$.

Proof. For $f\in N_{\overline{H}}(\alpha)$ and $F\in N_{\overline{H}}(\beta)$, let $f*F$ be given by the above definition (8). Since $|A_n|\leq 1$ and $|B_n|\leq 1$, we can write

$$\sum_{n=2}^{\infty}n|a_n||A_n|+\sum_{n=1}^{\infty}n|b_n||B_n|\leq\sum_{n=2}^{\infty}n|a_n|+\sum_{n=1}^{\infty}n|b_n|.$$

The right hand side of the above inequality is bounded by $1-\alpha$ because $f\in N_{\overline{H}}(\alpha)$. Therefore $f*F\in N_{\overline{H}}(\alpha)\subset N_{\overline{H}}(\beta)$.

Finally, we determine the convex combination properties of the members of $N_{\overline{H}}(\alpha)$.

Theorem 8. The class $N_{\overline{H}}(\alpha)$ is closed under convex combination.

Proof. For $i = 1, 2, 3, \ldots$ suppose that $f_i(z) \in N_{\overline{H}}(\alpha)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i_n}| z^n + \sum_{n=1}^{\infty} |b_{i_n}| \bar{z}^n.$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \bar{z}^n.$$

Since, $\sum_{n=2}^{\infty} n|a_{i_n}| + \sum_{n=1}^{\infty} n|b_{i_n}| \le 1 - \alpha \le 1$, from the above equation we obtain

$$\sum_{n=2}^{\infty} n \left| \sum_{i=1}^{\infty} t_i |a_{i_n}| \right| + \sum_{n=1}^{\infty} n \left| \sum_{i=1}^{\infty} t_i |b_{i_n}| \right| = \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} n|a_{i_n}| + \sum_{n=1}^{\infty} n|b_{i_n}| \right\} \le \sum_{i=1}^{\infty} t_i = 1,$$

and so $\sum_{i=1}^{\infty} t_i f_i(z) \in N_{\overline{H}}(\alpha)$.

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