MULTILINEARIZED LITTLEWOOD-PALEY OPERATORS

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ABSTRACT. We consider multilinear operators $T = T(f_1, f_2, \dots, f_m)$ of the following form: $T(f_1, f_2, \dots, f_m)(x) = \int_0^\infty ((\varphi_1)_t * f_1)(x)((\varphi_2)_t * f_2)(x) \cdots ((\varphi_m)_t * f_m)(x)\, dt/t$. It is known that under appropriate conditions on φ_j, there exists $C > 0$ such that $\|T(f_1, f_2, \dots, f_m)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2} \cdots \|f_m\|_{p_m}$ for $1 < p_1, p_2, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \le 1$. In this paper, we treat the case without restriction $p \ge 1$. To prove this, we use a recent work on multilinear singular integrals of Grafakos and Torres, and one on Littlewood-Paley's g-functions by S. Sato.

§1. INTRODUCTION

Multilinearized Littlewood-Paley operators were first considered by R. R. Coifman and Y. Meyer in [2]. Since then, many authors treated multilinearized Littlewood-Paley operators. Recently, Grafakos and Torres [4] established multilinear Calderón-Zygmund theory and got new estimates for a class of multilinear Fourier multipliers. Also, deep results are developed on square functions in the Littlewood-Paley theory, Sato [6], etc. In this paper, we report that we can obtain new estimates for multilinearized Littlewood-Paley operators, by using their recent results.

We say that an m-linear operator T is *good* if T is a bounded operator from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ to L^p for $1 < p_1, p_2, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, i.e., there exists $C > 0$ such that

$$\|T(f_1, f_2, \dots, f_m)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2} \cdots \|f_m\|_{p_m}.$$

We consider the following m-linearized Littlewood-Paley type operator

$$T_{\varphi_1, \varphi_2, \dots, \varphi_m}(f_1, f_2, \dots, f_m)(x) = \int_0^\infty ((\varphi_1)_t * f_1)(x)((\varphi_2)_t * f_2)(x) \cdots ((\varphi_m)_t * f_m)(x)\, \frac{dt}{t},$$

In the above and in the sequel, $f_t(x)$ denotes $t^{-n} f(x/t)$. Let $W(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$ be the Gauss kernel, $W_t(x) = t^{-n} W(x/t)$, and $\widetilde{W}_t(x) = t\frac{\partial W_t(x)}{\partial t}$, $\widetilde{W}(x) = \widetilde{W}_1(x)$. Then, $\widehat{W_t}(\xi) = e^{-|t\xi|^2/2}$, $\widehat{\widetilde{W}_t}(\xi) = -|t\xi|^2 e^{-|t\xi|^2/2}$. A consequence of a recent result of Grafakos and Torres [4] is the following.

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Lemma 1. $T_{\widetilde{W},W,\dots,W}$ *is good.*

To prove this, we recall the definition of m-linear Fourier multiplier. An m-linear operator T_σ is said to be an m-linear Fourier multiplier with symbol σ, if T_σ has the following form

$$T_\sigma(f_1,\dots,f_m)(x)=\frac{1}{(2\pi)^{nm}}\int_{\mathbb{R}^{nm}} e^{ix\cdot(\xi_1+\cdots+\xi_m)}\sigma(\xi_1,\dots,\xi_m)\hat{f}_1(\xi)\cdots\hat{f}_m(\xi_m)\,d\xi_1\cdots d\xi_m.$$

Proof of Lemma 1. The symbol $\sigma(\xi_1,\xi_2,\dots,\xi_m)$ of $T_{\widetilde{W},W,\dots,W}$ as an m-linear Fourier multiplier is given by

$$\begin{aligned}\sigma(\xi_1,\xi_2,\dots,\xi_m)&=\int_0^\infty \widehat{\widetilde{W}}(t\xi_1)\widehat{W}(t\xi_2)\cdots\widehat{W}(t\xi_m)\,\frac{dt}{t}\\&=\int_0^\infty -|t\xi_1|^2e^{-|t\xi_1|^2/2}e^{-|t\xi_2|^2/2}\cdots e^{-|t\xi_m|^2/2}\frac{dt}{t}\\&=-|\xi_1|^2\int_0^\infty te^{-|t\xi|^2/2}\,dt=-|\xi_1|^2/|\xi|^2\in C^\infty(\mathbb{R}^{nm}\setminus\{0\}).\end{aligned}$$

This symbol function is of homogeneous of degree 0. Hence, by a theorem of Grafakos and Torres, the conclusion holds true. $\square$

The symbol of our operators $T_{\varphi_1,\varphi_2,\dots,\varphi_m}$ as m-linear Fourier multiplier is given by

$$\int_0^\infty \widehat{\varphi}_1(t\xi_1)\widehat{\varphi}_2(t\xi_2)\cdots\widehat{\varphi}_m(t\xi_m)\,\frac{dt}{t},$$

at least formally, and really if the integrand of the above integral is absolutely integrable. Clearly this symbol function is of homogeneous of degree 0, however, in general, this is not so smooth in $\mathbb{R}^{nm}\setminus\{0\}$. For example, if φ_j is the Poisson kernel $P(x)=c_n(1+|x|^2)^{-\frac{n+1}{2}}$ $(j=2,\dots,m)$, and $\varphi_1(x)=\frac{\partial P_t(x)}{\partial t}\big|_{t=1}$, then the symbol is $-|\xi_1|/(|\xi_1|+|\xi_2|+\cdots+|\xi_m|)$. So, in general we cannot use Grafakos-Torres theorem directly.

Known results on multilinearlized Littlewood-Paley opertors are restricted on the case $p\ge 1$, Coifman and Meyer [2], Yabuta [10]. In this paper, we treat the case $p>1/m$, $1/m<1$ for $m\ge 2$.

§2. Preliminaries and main result

To state our result, we introduce some definitions. We consider the least non-increasing radial majorant of a function ψ defined by

$$H_\psi(x)=\sup_{|y|\ge|x|}|\psi(y)|.$$

We also use two seminorms

$$\begin{aligned}B_\varepsilon(\psi)&:=\int_{|x|>1}|\psi(x)||x|^\varepsilon\,dx\qquad\text{for }\varepsilon>0\ ,\\D_\eta(\psi)&:=\left(\int_{|x|<1}|\psi(x)|^\eta\,dx\right)^{1/\eta}\qquad\text{for }\eta>1\end{aligned}$$

Definition 1. A function $\varphi(x)$ is said to belong to LP if $\varphi \in L^1(\mathbb{R}^n)$, $B_\varepsilon(\varphi) < \infty$ for some $\varepsilon > 0$, $D_\eta(\varphi) < \infty$ for some $\eta > 1$, and $H_\varphi \in L^1(\mathbb{R}^n)$.

A function $\varphi(x)$ is said to belong to LP_0 if $\varphi \in LP$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 0$.

A function $\varphi(x)$ is said to belong to LP_{00} if $\varphi \in LP_0$ and $\int_0^\infty \varphi(sx)s^{n-1}\,ds = 0$ for all non-zero $x \in \mathbb{R}^n$, and $H_\varphi(x)|\log|x|| \in L^1(\mathbb{R}^n)$.

Now we can state our main result.

Theorem 1. *Let $\varphi_j \in LP$, $j = 1, 2, \dots, m$. Suppose one of φ_j belongs to LP_{00}. Then, $T_{\varphi_1,\varphi_2,\dots,\varphi_m}$ is good.*

Corollary 1. *Let $1 < p_1 < \infty$. Let $\varphi_1 \in LP_{00}$ and $\varphi_j \in LP$, $j = 2, \dots, m$. Let $b(\xi)$ be a bounded function of homogeneous of degree 0 and S_b be the Fourier multiplier defined by $\widehat{S_b f}(\xi) = b(\xi)\hat{f}(\xi)$. Suppose S_b is bounded on $L^{p_1}(\mathbb{R}^n)$ and $\psi = S_b\varphi_1$ is well-defined as an $L^1(\mathbb{R}^n)$-function. Then, for $1 < p_j < \infty$ $(j = 2, \dots, m)$, $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$ there exists $C > 0$ such that*

$$\|T_{\psi,\varphi_2,\dots,\varphi_m}(f_1, f_2, \dots, f_m)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2}\cdots\|f_m\|_{p_m}.$$

Example. Let

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}$$

be the Poisson kernel. Put $\psi(x) = \frac{\partial P(x)}{\partial x_j}$, where $P = P_1$, and $\varphi_1(x) = \frac{\partial P_t(x)}{\partial t}\big|_{t=1}$. We can check that $\varphi_1 \in LP_{00}$, in particular, the condition:

$$(\dagger) \qquad \int_0^\infty \varphi_1(tx)t^{n-1}\,dt = 0 \quad \text{for all non-zero } x \in \mathbb{R}^n.$$

So, for $\varphi_j \in LP$ $(j = 2, \dots, m)$ we see that $T_{\varphi_1,\varphi_2,\dots,\varphi_m}$ is good. On the other hand, we can see that ψ does not satisfy the condition $(\dagger)$. Now, let $b(\xi) = -i\xi_j/|\xi|$ for $\xi \ne 0$. Then S_b is the Riesz transform R_j. We observe that $\psi = R_j\varphi_1$. Therefore, although we cannot apply Theorem 1 directly to $T_{\psi,\varphi_2,\dots,\varphi_m}$, by the L^p boundedness of the Riesz transform and by Corollary 1 we can see that $T_{\psi,\varphi_2,\dots,\varphi_m}$ is also good.

§3. Proofs of Theorem 1 and Corollary 1

We first note that if $|\varphi(x)| \le C(1 + |x|)^{-n-\varepsilon}$ $(x \in \mathbb{R}^n)$ for some $C > 0$ and $\varepsilon > 0$, then $\varphi \in LP$.

To prove our theorem, we prepare some lemmas.

Lemma 2. *Let $\varphi_j \in LP$, $j = 1, 2, \dots, m$. Suppose two of φ_j belong to LP_0. Then, $T_{\varphi_1,\varphi_2,\dots,\varphi_m}$ is good.*

Proof. We may assume $\varphi_1, \varphi_2 \in LP_0$. From the assumption $H_{\varphi_j} \in L^1(\mathbb{R}^n)$ it follows

$$|(\varphi_j)_t * f_j(x)| \le CM(f_j)(x),$$

where $M(f)$ denotes the Hardy-Littlewood maximal function (see, for example, Stein-Weiss [9]). Hence, we have by the Cauchy-Schwarz inequality

$$\begin{aligned}
&|T_{\varphi_1,\varphi_2,\dots,\varphi_m}(f_1, f_2, \dots, f_m)(x)| \\
&\le C\left(\int_0^\infty \left|((\varphi_1)_t * f_1)(x)\right|^2 \frac{dt}{t}\right)^{\frac12} \left(\int_0^\infty \left|((\varphi_2)_t * f_2)(x)\right|^2 \frac{dt}{t}\right)^{\frac12} M(f_3)(x)\cdots M(f_m)(x).
\end{aligned}$$

By Theorem 1 of S. Sato [6, p. 200], the first two terms in the right hand side of the above inequality is bounded for every $L^r(\mathbb{R}^n)$ $(1 < r < \infty)$. As is well-known, maximal functions are bounded for every $L^r(\mathbb{R}^n)$ $(1 < r < \infty)$. Hence, using Hölder's inequality, we have the conclusion. $\square$

Lemma 3. *Let $\psi \in L^1(\mathbb{R}^n)$ satisfy $D_\eta(\psi) < +\infty$ for some $\eta > 1$. Put*

$$\Psi(x) = -\int_1^\infty \psi\Big(\frac{x}{s}\Big)\frac{ds}{s^{n+1}} = -\int_0^1 \psi(sx)s^{n-1}\,ds.$$

Then it holds

$$D_\eta(\Psi) \le \frac{\eta}{n(\eta-1)} D_\eta(\psi).$$

Proof. Using Minkowski's inequality we have

$$\begin{aligned} D_\eta(\Psi) &= \left(\int_{|x|<1}\Big|\int_0^1 \psi(sx)s^{n-1}\,ds\Big|^\eta dx\right)^{\frac{1}{\eta}} \le \int_0^1 \left(\int_{|x|<1} |\psi(sx)|^\eta\,dx\right)^{\frac{1}{\eta}} s^{n-1}\,ds \\ &= \int_0^1 \left(\int_{|x|<s} |\psi(y)|^\eta\,dy\right)^{\frac{1}{\eta}} s^{-n/\eta}s^{n-1}\,ds \le D_\eta(\psi)\int_0^1 s^{n(1-1/\eta)-1}\,ds \\ &= \frac{\eta}{n(\eta-1)} D_\eta(\psi). \qquad \square \end{aligned}$$

Lemma 4. *Let $\psi \in L^1(\mathbb{R}^n)$ satisfy $B_\varepsilon(\psi) < \infty$ for some $\varepsilon > 0$ and $\int_0^\infty \psi(sx)s^{n-1}\,ds = 0$ for all $x \neq 0$. Let Ψ be as in Lemma 3. Then,*

$$B_\varepsilon(\Psi) \le \frac{1}{\varepsilon} B_\varepsilon(\psi).$$

Proof. Since $\int_0^\infty \psi(sx)s^{n-1}\,ds = 0$, we have $\Psi(x) = \int_1^\infty \psi(sx)s^{n-1}\,ds$. Hence

$$B_\varepsilon(\Psi) = \int_{|x|>1}\Big|\int_1^\infty \psi(sx)s^{n-1}\,ds\Big||x|^\varepsilon\,dx.$$

Thus

$$\begin{aligned} B_\varepsilon(\Psi) &\le \int_1^\infty \left(\int_{|x|>1} |\psi(sx)||x|^\varepsilon\,dx\right)s^{n-1}\,ds = \int_1^\infty \left(\int_{|y|>s} |\psi(y)||y|^\varepsilon\,dy\right)s^{-n-\varepsilon}s^{n-1}\,ds \\ &\le B_\varepsilon(\psi)\int_1^\infty s^{-1-\varepsilon}\,ds = \frac{1}{\varepsilon}B_\varepsilon(\psi). \qquad \square \end{aligned}$$

Lemma 5. *Suppose $\int_{\mathbb{R}^n} H_\psi(x)|\log|x||\,dx < \infty$ and $\int_0^\infty \psi(sx)s^{n-1}\,ds = 0$ for all $x \neq 0$. Let Ψ be as in Lemma 3. Then*

$$\int_{\mathbb{R}^n} H_\Psi(x)\,dx \le \int_{\mathbb{R}^n} H_\psi(x)|\log|x||\,dx.$$

Proof. Since $|\Psi(x)| = \left|\int_0^1 \psi(sx)s^{n-1}\,ds\right| \le \int_0^1 H_\psi(sx)s^{n-1}\,ds$, we have $H_\Psi(x) \le \int_0^1 H_\psi(sx)s^{n-1}\,ds$. So, we have, using Fubini's theorem twice,

$$\begin{aligned} \int_{|x|\le 1} H_\Psi(x)\,dx &\le \int_{|x|\le 1}\int_0^1 H_\psi(sx)s^{n-1}\,dsdx = \int_0^1 \left(\int_{|x|\le 1} H_\psi(sx)\,dx\right)s^{n-1}\,ds \\ &= \int_0^1 \left(\int_{|y|\le s} H_\psi(y)\,dy\right)\frac{ds}{s} = \int_{|y|\le 1} H_\psi(y)\left(\int_{|y|}^1 \frac{ds}{s}\right)dy \\ &= \int_{|y|\le 1} H_\psi(y)|\log|y||\,dy. \end{aligned}$$

Next, since $\int_0^\infty \psi(sx)s^{n-1}\,ds = 0$, we have

$$\Psi(x) = \int_1^\infty \psi(sx)s^{n-1}\,ds, \text{ and hence } \quad |\Psi(x)| \le \int_1^\infty H_\psi(sx)s^{n-1}\,ds.$$

So,

$$H_\Psi(x) \le \sup_{|y|\ge|x|} \int_1^\infty H_\psi(sy)s^{n-1}\,ds = \int_1^\infty H_\psi(sx)s^{n-1}\,ds.$$

Thus, using Fubini's theorem twice, we have

$$\begin{aligned}\int_{|x|\ge 1} H_\Psi(x)\,dx &\le \int_{|x|\ge 1}\int_1^\infty H_\psi(sx)s^{n-1}\,dsdx = \int_1^\infty \left(\int_{|x|\ge 1} H_\psi(sx)\,dx\right) s^{n-1}\,ds \\ &= \int_1^\infty \left(\int_{|y|\ge s} H_\psi(y)\,dy\right)\frac{ds}{s} = \int_{|y|\ge 1} H_\psi(y)\left(\int_1^{|y|}\frac{ds}{s}\right)dy \\ &= \int_{|y|\ge 1} H_\psi(y)\log|y|\,dy. \qquad \square\end{aligned}$$

Now we proceed to the proof of our Theorem 1. We may assume $\varphi_1 \in LP_{00}$, and $\int_{\mathbb{R}^n}\varphi_j(x)\,dx = 1$, $j = 2, \dots, m$. We decompose $T_{\varphi_1,\dots,\varphi_m}$ as follows.

$$\begin{aligned}&T_{\varphi_1,\dots,\varphi_m}(f_1,\dots,f_m) \\ &\quad = \int_0^\infty ((\varphi_1)_t * f_1)\big((\varphi_2 - W)_t * f_2\big)((\varphi_3)_t * f_3)\cdots((\varphi_m)_t * f_m)\,\frac{dt}{t} \\ &\quad\quad + \int_0^\infty ((\varphi_1)_t * f_1)(W_t * f_2)((\varphi_3)_t * f_3)\cdots((\varphi_m)_t * f_m)\,\frac{dt}{t} \\ &\quad =: T_1 + T_2.\end{aligned}$$

Since $\int W(x)\,dx = \int \varphi_2(x)\,dx = 1$, we see easily $\varphi_2 - W \in LP_0$. Hence, by Lemma 2, we see that T_1 is good. So, we have only to show T_2 is good. Repeating this procedure, we may assume $\varphi_j(x) = W(x)$, $j = 2, \dots, m$. Set $\psi = \varphi_1$ and Ψ be as in Lemma 3. Then, by Lemmas 3, 4 and 5, we have $\Psi \in LP$. We see also that $t\frac{\partial \Psi_t}{\partial t} = \psi_t$. Noting that $\lim_{t\to\infty}\Psi_t * f_1(x) = 0$, $\lim_{t\to\infty} W_t * f_j(x) = 0$, $j = 2, \dots, m$, $\lim_{t\to 0}\Psi_t * f_1(x) = f_1(x)\int \Psi(x)\,dx$, $\lim_{t\to 0} W_t * f_j(x) = f_j(x)$, $j = 2, \dots, m$, we have, by integration by parts,

$$\begin{aligned}&T_{\varphi_1,\dots,\varphi_m}(f_1,\dots,f_m) \\ &\quad = -\Big(\int \Psi\,dx\Big) f_1(x)f_2(x)\cdots f_m(x) - \int_0^\infty (\Psi_t * f_1)(x)\frac{\partial}{\partial t}\prod_{j=2}^m (W_t * f_j)(x)\,dt.\end{aligned}$$

The first term of the right hand side of the above is clearly good. So, to prove our theorem, we have only to treat the following one.

$$T_{\Psi,\widetilde{W},W,\dots,W}(f_1,f_2,\dots,f_m) = \int_0^\infty (\Psi_t * f_1)(x)(\widetilde{W}_t * f_2)(W_t * f_3)\cdots(W_t * f_m)\,\frac{dt}{t}.$$

If $\int \Psi(x)\,dx = 0$, then it follows that $\Psi \in LP_0$, and hence we can apply Lemma 2. Hence, we may assume $\int \Psi(x)\,dx = 1$. We decompose $T_{\Psi,\widetilde{W},W,\dots,W}(f_1,f_2,\dots,f_m)$ as before.

$$\begin{aligned}&T_{\Psi,\widetilde{W},W,\dots,W}(f_1,f_2,\dots,f_m) \\ &\quad = \int_0^\infty ((\Psi - W)_t * f_1)(x)(\widetilde{W}_t * f_2)(W_t * f_3)\cdots(W_t * f_m)\,\frac{dt}{t} \\ &\quad\quad + \int_0^\infty (W_t * f_1)(x)(\widetilde{W}_t * f_2)(W_t * f_3)\cdots(W_t * f_m)\,\frac{dt}{t}.\end{aligned}$$

We see by Lemma 2 that the first term is good, and by Lemma 1 that the last term is also good. This completes the proof of Theorem 1. □

Proof of Corollary 1. We note that $\psi_t * f(x) = (\varphi_1)_t * (S_b f)(x)$ (this can be seen by taking the Fourier transform and using the homogeneity of $b(\xi)$), and hence we have $T_{\psi,\varphi_2,\dots,\varphi_m}(f_1, f_2, \dots, f_m)(x) = T_{\varphi_1,\varphi_2,\dots,\varphi_m}(S_b f_1, f_2, \dots, f_m)(x)$. Therefore applying Theorem 1 and using L^{p_1} boundedness of S_b, we get the conclusion. □

Remark 1. We can also get weighted versions. If $w(x) \in A_{p_{\min}}$ in Theorem 1, then there exists $C > 0$ such that

$$\|T_{\varphi_1,\varphi_2,\dots,\varphi_m}(f_1, f_2, \dots, f_m)\|_{L^p(w)} \le C\|f_1\|_{L^{p_1}(w)}\|f_2\|_{L^{p_2}(w)} \cdots \|f_m\|_{L^{p_m}(w)},$$

where A_p is the Muckenhoupt weight class and $p_{\min} = \min(p_1, p_2, \dots, p_m)$.

Remark 2. If, in Theorem 1, $|\varphi_j(x)| \le C(1 + |x|)^{-n-\varepsilon}$ $(x \in \mathbb{R}^n)$ for some $C > 0$ and $\varepsilon > 0$ $(j = 1, \dots, m)$, and each φ_j satisfies further for some $\gamma > 0$

$$\int |\varphi_j(x - y) - \varphi_j(x)|\, dx \le C|y|^\gamma, \qquad y \in \mathbb{R}^n,$$

and if one of φ_j satisfies $\int \varphi_j(x)\, dx = 0$ and $\int_0^\infty \varphi_j(sx)s^{n-1}\, ds = 0$ for all non-zero $x \in \mathbb{R}^n$, then, it holds

$$\lambda|\{x \in \mathbb{R}^n; |T_{\varphi_1,\varphi_2,\dots,\varphi_m}(f_1, f_2, \dots, f_m)(x)| > \lambda\}|^{\frac{1}{p}} \le C\|f_1\|_{p_1}\|f_2\|_{p_2} \cdots \|f_m\|_{p_m}, \quad \lambda > 0$$

provided one of $p_j = 1$.

Recently, D. Fan and S. Sato showed the following: Suppose $\psi \in L^1$ satisfies $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$, $D_\eta(\psi) < \infty$ for some $\eta > 1$, $B_\epsilon(\psi) < \infty$ for some $\epsilon > 0$, $\int_{|x|\ge 1} H_\psi(x)\, dx < \infty$ and $\sup_{|x|\ge 1} H_\psi(x) < \infty$ (note that this follows from the condition $H_\psi \in L^1(\mathbb{R}^n)$). Then the Littlewood-Paley function

$$g_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2\, dt/t\right)^{1/2}$$

is bounded on L^p for all $2 \le p < \infty$ (see [3, Corollary 3]). As an application of this, we can give somewhat weak assertion assuming weaker condition on φ_1 in Theorem 1.

Remark 3. Let $\varphi_j \in LP$ for $j = 2, 3, \dots, m$, and let $\varphi_1 \in L^1$ satisfy $\int_{\mathbb{R}^n} \varphi_1(x)\, dx = 0$, $D_\eta(\varphi_1) < \infty$ for some $\eta > 1$, $\int_0^\infty \varphi_1(tx)t^{n-1}\, dt = 0$ for all non-zero $x \in \mathbb{R}^n$, $B_\epsilon(\varphi_1) < \infty$ for some $\epsilon > 0$ and $\int_{|x|\ge 1/2} H_{\varphi_1}(x)\log(2|x|)\, dx < \infty$. (Note that the last two conditions are always satisfied if φ_1 is supported in $\{|x| \le 1/2\}$.) Then we have

$$\|T_{\varphi_1,\varphi_2,\dots,\varphi_m}(f_1, f_2, \dots, f_m)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2} \dots \|f_m\|_{p_m}$$

for $2 \le p_1 < \infty$, $1 < p_2, p_3, \dots, p_m < \infty$, $1/p = 1/p_1 + 1/p_2 + \dots + 1/p_m$.

Arguing as in the proof of Theorem 1, we can prove this as follows. First, we may assume $\varphi_j = W$ for $j = 2, 3, \dots, m$. To see this, we note that the conditions $\sup_{|x|\ge 1} H_{\varphi_1}(x) < \infty$ and $\int_{|x|\ge 1} H_{\varphi_1}(x)\, dx < \infty$ follow from our last assumption on φ_1, and so g_{φ_1} is bounded on L^p, $2 \le p < \infty$. Next, after integration by parts we find that to get the result it suffices to prove the $L^{p_1} \times \dots \times L^{p_m} - L^p$ boundedness of $T_{\widetilde{\Psi},\widetilde{W},W\dots,W}$ with $\widetilde{\Psi} = \Psi - cW$, where Ψ is as in the proof of Theorem 1, $c = \int_{\mathbb{R}^n} \Psi(x)\, dx$ and p, $p_1, \dots, p_m$ are as above. Now, Lemmas 3 and 4 imply $D_\eta(\Psi) < \infty$ and $B_\epsilon(\Psi) < \infty$, respectively. Furthermore, from the proof of Lemma 5 we see that

$$\int_{|x|\geq 1/2} H_\Psi(x)\,dx \leq \int_{|x|\geq 1/2} H_{\varphi_1}(x)\log(2|x|)\,dx < \infty.$$

Therefore we have $\int_{|x|\geq 1} H_\Psi(x)\,dx < \infty$ and $\sup_{|x|\geq 1} H_\Psi(x) < \infty$. It is easy to see that $\widetilde{\Psi}$ satisfies the same conditions. Since we also have $\int_{\mathbb{R}^n} \widetilde{\Psi}(x)\,dx = 0$, $g_{\widetilde{\Psi}}$ is bounded on L^p, $2 \leq p < \infty$. So, arguing as in the proof of Lemma 2, we get the conclusion.

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