CHARACTERIZATION OF AMENABLE REPRESENTATIONS

QIN XU

Received April 27, 2001

ABSTRACT. Let G be a locally compact group. A continuous unitary representation π of G is said to be amenable if there exists a *G-invariant* mean on the space of $B(H_\pi)$. In this paper, we investigate various characterization of amenable representations through the study of the existence and properties of invariant means on spaces of operators. One of our characterizations is an analogue of the *Dixmier criterion* for amenable groups. We also attempt to formulate a fixed point property for the amenable representations.

A locally compact group G is said to be amenable if there exists a left invariant mean on G, *i.e.*, a continuous linear functional m on $L^\infty(G)$ such that $m(\mathbf{1}) = 1 = \|m\|$ and $m({}_xf) = m(f)$ for all $x \in G$ and $f \in L^\infty(G)$. Amenable groups have been studied from various angles. Among many others, one of more recent approaches is through amenable representations.

A continuous unitary representation π of G is said to be amenable if there exists a G-invariant mean on $B(H_\pi)$. This notion is introduced by M. Bekka [1]. The significance of this concept is best demonstrated by the following result of Bekka: G is amenable if and only if every continuous unitary representation of G is amenable. The amenability of representations can be characterized in various ways. For example, Bekka presented the analogues of Reiter's properties, Day's asymptotic invariance properties, and Følner's condition for amenable groups in [1].

In the following, we are also primarily concerned with different ways of characterizing amenable representations. We start with discussing some interesting properties of the means on a space of operators. Then we present various characterizations of amenable representations. At the end, we attempt to formulate a fixed point property to describe the amenability of representations.

Let H be a Hilbert space, $B(H)$ the space of all bounded linear operators on H. For a subspace $\mathcal{A}$ of $B(H)$, let $\mathcal{A}_h = \{T \in \mathcal{A}; T^* = T\}$. The following lemma has its analogue for spaces of functions.

Lemma 1. *Let $\mathcal{A}$ be a conjugate closed subspace of $B(H)$ containing identity I. Let M be a bounded linear functional on $\mathcal{A}$. If M satisfies any two of the following three conditions, it must satisfies the remaining one, and then M is a mean.*

$$(i)\ M(I) = 1. \quad (ii)\ \|M\| = 1. \quad (iii)\ M \geq 0\ \ (i.e.,\ M(T) \geq 0\ if\ T \geq 0).$$

Proof. (i) *and* $(ii) \Longrightarrow (iii)$. Our first observation is that (i) and (ii) ensures $M(S) \in \mathbf{R}$ for all $S \in \mathcal{A}_h$. For any $S \in \mathcal{A}$, assume that $M(S) = \alpha + i\beta,\ \alpha,\ \beta \in \mathbf{R}$. Then, for any $\gamma \in \mathbf{R}$, we have

$$(\beta+\gamma)^2 \ \leq\ |\alpha + i(\beta+\gamma)|^2 \ =\ |M(S+i\gamma I)|^2 \ \leq\ \|S+i\gamma I\|^2 \ \leq\ \|S\|^2 + \gamma^2.$$

1991 *Mathematics Subject Classification.* Primary 43A60, 22D05, 22D10.
Key words and phrases. Invariant means, amenable representations.

Hence we get $2\beta\gamma \leq \|S\|^2 - \beta^2$ for all $\gamma \in \mathbf{R}$. But this can not be true unless $\beta = 0$. Thus $M(S) = \alpha \in \mathbf{R}$. Now, we assume $M(T) < 0$ for some $T \in \mathcal{A}$ and $T \geq 0$. Let $S = \|T\|I - T$. Note that $S \in \mathcal{A}_h$ and $\|S\| \leq \|T\|$. But, (i) and (ii) implies that $\|S\| \geq M(S) = \|T\| - M(T) > \|T\|$, a contradiction. Thus (iii) holds.

(i) *and* $(iii) \Longrightarrow (ii)$. Note that, for any $T \in \mathcal{A}_h$, $\|T\|I \pm T \geq 0$, and hence $M(T) \in \mathbf{R}$ and $|M(T)| \leq \|T\|$. Given $T \in \mathcal{A}$, there must exist an $\alpha \in \mathbf{C}$ with $|\alpha| = 1$ and $T_1, T_2 \in \mathcal{A}_h$ such that $\alpha T = T_1 + iT_2$ and

$$0 \leq |M(T)| = M(\alpha T) = M(T_1 + iT_2) = M(T_1) + iM(T_2).$$

This implies that $M(T_2) = 0$ and $M(T_1) \leq \|T_1\| \leq \|\alpha T\| = \|T\|$, hence $\|M\| \leq 1$. Together with (i), we have $\|M\| = 1$.

(ii) *and* $(iii) \Longrightarrow (i)$. Let α, T_1 and T_2 be as above. Then, $\|T_1\|I - T_1 \geq 0$ implies $|M(T)| = M(T_1) \leq M(I)\|T_1\| \leq M(I)\|T\|$, and thus $M(I) \geq \|M\| = 1$. Together with (ii), we have $M(I) = 1$. □

For $T \in B(H)$, the set $W(T) = \{\langle T\xi, \xi\rangle;\ \xi \in H \text{ and } \|\xi\| = 1\}$ $(\subseteq \mathbf{C})$ is commonly known as the *numerical range* of the operator T. It is well-known that $W(T)$ is always convex but not closed (see Halmos [3, Chapter 17]). If T is hermitian, then $W(T) \subseteq \mathbf{R}$. If T is positive, then $W(T) \subseteq \mathbf{R}^+ = [0, \infty)$. Now, we give another characterization of means.

Lemma 2. *Let $\mathcal{A}$ and M be as in Lemma 1. Then the following statements are equivalent.*

(i) *M is a mean on $\mathcal{A}$.* (ii) $\inf W(S) \leq M(S) \leq \sup W(S)$ *for all $S \in \mathcal{A}_h$.*

Proof. $(ii) \Longrightarrow (i)$. Since $W(I) = \{1\}$, we have $M(I) = 1$. If $S \geq 0$, then $M(S) \geq \inf W(S) \geq 0$. Thus M is a mean by Lemma 1.

$(i) \Longrightarrow (ii)$. For $S \in \mathcal{A}_h$, we denote $\alpha = \sup W(S)$ and consider the operator $(\alpha I - S) \in \mathcal{A}_h$. If $\eta \in H$ and $\|\eta\| = 1$, then

$$\langle(\alpha I - S)\eta, \eta\rangle = \alpha\|\eta\|^2 - \langle S\eta, \eta\rangle = \alpha - \langle S\eta, \eta\rangle \geq 0.$$

So $(\alpha I - S) \geq 0$. By (i), we have $M(\alpha I - S) \geq 0$, *i.e.* $M(S) \leq \alpha = \sup W(S)$. Replacing S by $-S$, we have $-M(S) = M(-S) \leq \sup W(-S) = -\inf W(S)$. Hence (ii) holds. □

Let G be a locally compact group and π a continuous unitary representation of G on a Hilbert space H_π. The Banach space $B(H_\pi)$ forms a left G-module with the natural group action defined by

$$x \cdot_\pi T = \pi(x)\, T\, \pi(x^{-1}), \qquad x \in G,\ T \in B(H_\pi).$$

Under this group action, a subset $\mathcal{A}$ of $B(H_\pi)$ is said to be *G-invariant* if $G \cdot \mathcal{A} = \{x \cdot T;\ x \in G,\ T \in \mathcal{A}\} \subseteq \mathcal{A}$. $\mathcal{A}$ is said to be an *admissible subspace* of $B(H_\pi)$ if it is a norm closed, conjugate closed and G-invariant subspace of $B(H_\pi)$ containing I. A bounded linear functional M on an admissible subspace $\mathcal{A}$ is called a mean on $\mathcal{A}$ if it satisfies $\|M\| = M(I) = 1$. It is called a *G-invariant mean* if it also satisfies $M(x \cdot T) = M(T)$ for all $x \in G$ and $T \in \mathcal{A}$. Now, for an admissible subspace $\mathcal{A}$ of $B(H_\pi)$, we define $J(\mathcal{A}) = \mathit{span}\{T - x \cdot T;\ T \in \mathcal{A},\ x \in G\}$.

Lemma 3. *If $\mathcal{A}$ is an admissible subspace of $B(H_\pi)$, then the following statements are equivalent.*

(i) *There exists a G-invariant mean on $\mathcal{A}$.*

(ii) $\sup W(S) \geq 0$ *for all $S \in J(\mathcal{A})_h$.*

Proof. $(i) \Longrightarrow (ii)$. Let M be a G-invariant mean on $\mathcal{A}$. Then $M(J(\mathcal{A})) = \{0\}$. By Lemma 2, $\sup W(S) \geq M(S) = 0$ for all $S \in J(\mathcal{A})_h$.

$(ii) \Longrightarrow (i)$. Note that $\mathcal{A}_h$ is a real normed linear space, and $J(\mathcal{A})_h$ is a proper real subspace of $\mathcal{A}_h$ because $(-I) \notin J(\mathcal{A})_h$ by (ii). Define a function p by $p(S) = \sup W(S)$ for all $S \in \mathcal{A}_h$. Clearly, p is a subadditive, positively homogeneous, real-valued function on $\mathcal{A}_h$. Define another function $M_\circ$ by $M_\circ(S) = 0$ for all $S \in J(\mathcal{A})_h$. Obviously, we have $M_\circ \leq p$ on $J(\mathcal{A})_h$. By the Hahn-Banach theorem, $M_\circ$ can be extended to a real-valued linear functional M on $\mathcal{A}_h$ satisfying $M(S) \leq p(S)$ for all $S \in \mathcal{A}_h$. Replacing S by $-S$, we get

$$-\|S\| \ \leq \inf W(S) \ \leq \ M(S) \ \leq \sup W(S) \ \leq \ \|S\| \,, \qquad \textit{for all } S \in \mathcal{A}_h \ .$$

So, $|M(S)| \leq \|S\|$, *i.e.,* M is continuous on $\mathcal{A}_h$. In a natural way, M can be (uniquely) extended to a bounded linear functional on $\mathcal{A}$, still denoted by M. Then M is a mean on $\mathcal{A}$ by Lemma 2, and is G-invariant because $M(J(\mathcal{A})) = \{0\}$. □

When $\mathcal{A}$ is an admissible subalgebra of $B(H_\pi)$, we have the following

Lemma 4. *If $\mathcal{A}$ is an admissible subalgebra of $B(H_\pi)$, then the following statements are equivalent.*

(i) *There exists a G-invariant mean on $\mathcal{A}$.*

(ii) $\inf\{\|I - S\|;\ S \in J(\mathcal{A})\} \ = \ 1$.

(iii) $\overline{J(\mathcal{A})} \subsetneq \mathcal{A}$.

Proof. $(i) \Longrightarrow (ii)$. Let M be a G-invariant mean on $\mathcal{A}$. Then we have $M(J(\mathcal{A})) = \{0\}$, and $1 = M(I - S) \leq \|I - S\|$ for all $S \in J(\mathcal{A})$. Moreover $0 \in J(\mathcal{A})$ and $\|I - 0\| = 1$. Thus (ii) holds.

$(ii) \Longrightarrow (iii)$. Trivial, because (ii) implies that $I \notin \overline{J(\mathcal{A})}$.

$(iii) \Longrightarrow (i)$. We choose a $T_\circ \in \mathcal{A} \setminus \overline{J(\mathcal{A})}$. Then $T_\circ \neq 0$. By the Hahn-Banach theorem, there exists an $M \in \mathcal{A}^*$ such that $M(T_\circ) \neq 0$, and $M(\overline{J(\mathcal{A})}) = \{0\}$. For convenience, we define $x \cdot M \in \mathcal{A}^*$ by $x \cdot M(T) = M(x \cdot T)$ for $x \in G$, $T \in \mathcal{A}$. Then $M(\overline{J(\mathcal{A})}) = \{0\}$ yields that $x \cdot M = M$ for all $x \in G$, i.e., M is G-invariant.

Without loss of generality, we may assume that M is hermitian (otherwise, we can replace M by either $(M + \tilde{M})/2$ or $(M - \tilde{M})/2i$ whichever is nonvanishing at $T_\circ$, where $\tilde{M}(T) = \overline{M(T^*)}$). Then M has a unique Jordan decomposition satisfying

(1) $M = M^+ - M^-$ with M^+ and M^- positive;

(2) $\|M\| = \|M^+\| + \|M^-\|$.

(see Takesaki [7, Theorem III.2.1]). Now, for all $x \in G$, $x \cdot M^+$ and $x \cdot M^-$ are both positive, and $\|x \cdot M^{\pm}\| = x \cdot M^{\pm}(I) = M^{\pm}(I) = \|M^{\pm}\|$. Thus, $\|M\| = \|M^+\| + \|M^-\| = \|x \cdot M^+\| + \|x \cdot M^-\|$. By the uniqueness of the decomposition, we have $x \cdot M^{\pm} = M^{\pm}$, for all $x \in G$. Therefore, both M^+ and M^- are G-invariant. Since $M(T_\circ) \neq 0$, one of $M^+(T_\circ)$ and $M^-(T_\circ)$ must be non-zero, say $M^+(T_\circ) \neq 0$. Then $M^+(I) = \|M^+\| > 0$, and so $\hat{M}(T) = M^+(T)/M^+(I)$ $(T \in \mathcal{A})$ defines a G-invariant mean on $\mathcal{A}$. □

As a direct consequence of the Lemma 3 and 4, we have following characterizations for amenable representations.

Theorem 5. *For a continuous unitary representation π, following conditions are equivalent.*

(i) *π is amenable.*

(ii) $\sup W(S) \geq 0$ *for all* $S \in J(B(H_\pi))_h$.

(iii) $\inf_{S \in J(B(H_\pi))} \|I - S\| = 1$.

(iv) $\overline{J(B(H_\pi))} \subsetneq B(H_\pi)$.

We will call the second condition in Theorem 5 the *Dixmier criterion* for amenable representations because it is analogous to the Dixmier criterion for amenable groups.

Now, we consider the space $TC(H_\pi)$ of all trace class operators on H_π equipped with the trace-norm. With the restriction of the group action defined before Lemma 3 to $TC(H_\pi)$, $TC(H_\pi)$ forms a left Banach G-module (see Bekka [1, Lemma 2.1]). We observe that $TC(H_\pi)$ also forms a right Banach $L^1(G)$-module with the action defined as follows: for any $A \in TC(H_\pi)$ and $f \in L^1(G)$,

$$\langle A \cdot f,\, T\rangle = \int_G f(y)\langle y^{-1} \cdot A,\, T\rangle dy\,, \qquad T \in B(H_\pi)\,.$$

By the duality $B(H_\pi) = TC(H_\pi)^*$, $B(H_\pi)$ can be made into a left Banach $L^1(G)$-module with the action given by

$$\langle (f \cdot T)\xi,\, \eta\rangle = \int_G f(y)\langle (y \cdot T)\xi,\, \eta\rangle dy\,, \qquad \xi,\, \eta \in H_\pi,$$

for all $f \in L^1(G)$ and $T \in B(H_\pi)$. This module structure is introduced by Bekka in [1].

Let $L^1(G)_1^+ = \{f \in L^1(G);\, f \geq 0,\, \|f\|_1 = 1\}$. A subset $\mathcal{A}$ of $B(H_\pi)$ is said to be *topologically invariant* (t-invariant) if $L^1(G)_1^+ \cdot \mathcal{A} = \{f \cdot T;\; f \in L^1(G)_1^+,\; T \in \mathcal{A}\} \subseteq \mathcal{A}$. $\mathcal{A}$ is said to be a *topologically admissible* (t-admissible) subspace if it is a norm closed, conjugate-closed, t-invariant subspace of $B(H_\pi)$ containing I. An $M \in \mathcal{A}^*$ is called a *topologically invariant* (t-invariant) mean if it is a mean on $\mathcal{A}$ satisfying $M(f \cdot T) = M(T)$, for all $f \in L^1(G)_1^+, T \in \mathcal{A}$. One can easily check that if $\mathcal{A}$ is closed under the weak operator topology, then $\mathcal{A}$ is t-invariant if and only if it is G-invariant. It is also known that the existence of a t-invariant mean on $B(H_\pi)$ is equivalent to the existence of a G-invariant mean on $B(H_\pi)$ (see Bekka [1]).

Let $\mathcal{A}$ be a t-admissible subspace of $B(H_\pi)$. We put

$$J_t(\mathcal{A}) \;=\; span\{T - f \cdot T\,;\, T \in \mathcal{A},\; f \in L^1(G)_1^+\}\;\; (\subseteq \mathcal{A}).$$

Then we have analogues of Lemma 3 and 4 for the topological invariance case.

Lemma 6. *If $\mathcal{A}$ is a t-admissible subspace of $B(H_\pi)$, then the following statements are equivalent.*

(i) There exists a t-invariant mean on $\mathcal{A}$.

(ii) $\sup W(S) \geq 0$ *for every* $S \in J_t(\mathcal{A})_h$.

Furthermore, if $\mathcal{A}$ is a t-admissible subalgebra, they are also equivalent to:

(iii) $\inf\{\|I - S\|\,;\, S \in J_t(\mathcal{A})\} = 1$.

The proof is similar to that of Lemma 3 and 4. So we omit it. As a direct consequence of Lemma 6, we have

Theorem 7. *For a continuous unitary representation π, following conditions are equivalent.*

(i) π is amenable.

(ii) $\sup W(S) \geq 0$, *for all* $S \in J_t(B(H_\pi))_h$.

(iii) $\inf_{S \in J_t(B(H_\pi))} \|I - S\| = 1$.

(iv) $\overline{J_t(B(H_\pi))} \subsetneqq B(H_\pi)$.

As an application of Lemma 3 and 6, we have the following result, which is an analogue of [2, Proposition 1.1].

Theorem 8. *If $\mathcal{A}$ is a t-admissible [resp. admissible] subspace of $B(H_\pi)$, then $\mathcal{A}$ admits a t-invariant [resp. G-invariant] mean if and only if $\mathcal{A}$ contains a linear subspace $\mathcal{Y}$ with the following properties.*

(1) $f \cdot T - T \in \mathcal{Y}$ *for all* $f \in L^1(G)_1^+$, $T \in \mathcal{A}$.
[*resp.* (1′) $x \cdot T - T \in \mathcal{Y}$ *for all* $x \in G$, $T \in \mathcal{A}$.]
(2) $T \in \mathcal{A}_h$ *and* $\inf W(T) > 0$ *imply that* $T \notin \mathcal{Y}$.

Proof. We will only prove the theorem for t-invariant case. The proof for G-invariant case is similar. Assume that M is a t-invariant mean on $\mathcal{A}$. Let $\mathcal{Y} = ker(M)$. Then $\mathcal{Y}$ satisfies (1) and (2). In fact, the t-invariance of M implies that $M(f \cdot T - T) = 0$ for all $f \in L^1(G)_1^+$ and $T \in \mathcal{A}$. Therefore (1) holds. By Lemma 2, for $T \in \mathcal{A}_h$, we have $\inf W(T) \le M(T) \le \sup W(T)$. Thus $\inf W(T) > 0$ implies that $M(T) > 0$ and hence $T \notin \mathcal{Y} = ker(M)$. Therefore (2) holds.

Conversely, assume (1) and (2) hold for some subspace $\mathcal{Y}$ of $\mathcal{A}$. Then, (1) implies $J_t(\mathcal{A}) \subseteq \mathcal{Y}$, and (2) implies $\inf W(T) \le 0$ for $T \in J_t(\mathcal{A})_h$. Replacing T by $-T$, we get $\sup W(T) \ge 0$. Now, the existence of t-invariant mean on $\mathcal{A}$ follows immediately from Lemma 6. □

By Theorem 8, a continuous unitary representation is amenable if and only if $B(H_\pi)$ contains a linear subspace which satisfies (1) and (2) in Theorem 8. The following result is an analogue of a result of Wong and Riazi [8], which gives a handy candidate of such subspaces.

Theorem 9. *Let $\mathcal{A}$ be a t-admissible subspace of $B(H_\pi)$. Let*

$$\mathcal{Y}_\circ(\mathcal{A}) \;=\; \{T \in \mathcal{A};\; \inf_{f \in L^1(G)_1^+} \|f \cdot T\| \;=\; 0\}.$$

Then $\mathcal{Y}_\circ(\mathcal{A})$ satisfies (1) *and* (2) *in Theorem 8 and is closed under scalar multiplication. Consequently, $\mathcal{A}$ admits a t-invariant mean if $\mathcal{Y}_\circ(\mathcal{A})$ is closed under addition.*

Proof. It is trivial that $\mathcal{Y}_\circ(\mathcal{A})$ is closed under scalar multiplication. Let $f \in L^1(G)_1^+$. For $k \in \mathbf{N}$ (positive integers), we define

$$f^k \;=\; \overbrace{f * f * \,\cdots\, * f}^{k} \;.$$

Since $L^1(G)_1^+ * L^1(G)_1^+ \subseteq L^1(G)_1^+$, we have $f^k \in L^1(G)_1^+$ for all $k \in \mathbf{N}$. For any fixed $n \in \mathbf{N}$, put $F_n = \frac{1}{n}\sum_{k=1}^{n} f^k$. Obviously, $F_n \in L^1(G)_1^+$. If $T \in \mathcal{A}$, then

$$F_n \cdot (f \cdot T - T) = (\frac{1}{n}\sum_{k=1}^{n} f^k) \cdot (f \cdot T - T) = \frac{1}{n}(f^{n+1} \cdot T \;-\; f \cdot T).$$

Thus we have

$$\|F_n \cdot (f \cdot T - T)\| = \frac{1}{n}\,\|f^{n+1} \cdot T - f \cdot T\| \le \frac{2}{n}\,\|T\| \longrightarrow 0 \;\, (as\ n \to \infty).$$

Therefore $(f \cdot T - T) \in \mathcal{Y}_\circ(\mathcal{A})$ for all $f \in L^1(G)_1^+$, $T \in \mathcal{A}$, *i.e.*, (1) holds for $\mathcal{Y}_\circ(\mathcal{A})$.

To show that $\mathcal{Y}_\circ(\mathcal{A})$ satisfies (2), we assume that $T \in \mathcal{A}_h$ and $\inf W(T) = \varepsilon > 0$. Note that, for any $f \in L^1(G)_1^+$ and $\xi, \eta \in H_\pi$, we have $\langle \xi, (f \cdot T)\eta \rangle = \overline{\langle (f \cdot T)\eta, \xi \rangle} = \langle (f \cdot T)\xi, \eta \rangle$, *i.e.*, $f \cdot T \in \mathcal{A}_h$. Moreover,

$$\begin{aligned} \|f \cdot T\| &= \sup_{\xi \in H_\pi,\, \|\xi\|=1} |\langle (f \cdot T)\xi \,,\, \xi \rangle| \\ &= \sup_{\xi \in H_\pi,\, \|\xi\|=1} \Big| \int_G f(y)\, \langle T\pi(y^{-1})\xi \,,\, \pi(y^{-1})\xi \rangle \, dy \,\Big| \\ &\geq \inf_{\xi \in H_\pi,\, \|\xi\|=1} \int_G f(y)\, \langle T\pi(y^{-1})\xi \,,\, \pi(y^{-1})\xi \rangle \, dy \\ &\geq \varepsilon \int_G f(y)\, dy \;=\; \varepsilon. \end{aligned}$$

Thus, by the definition of $\mathcal{Y}_\circ(\mathcal{A})$, $T \notin \mathcal{Y}_\circ(\mathcal{A})$. □

Given any $x \in G$, we define a linear map $U_x : B(H_\pi) \longrightarrow B(H_\pi)$ by $U_x(T) = x \cdot T$ for all $T \in B(H_\pi)$. Let $\Gamma(G) = co\{U_x;\, x \in G\}$ $(\subseteq B(B(H_\pi)))$. An important observation about $\Gamma(G)$ is that $\Gamma(G) \cdot \Gamma(G) \subseteq \Gamma(G)$.

Theorem 10. *Let $\mathcal{A}$ be an admissible subalgebra of $B(H_\pi)$. Let*

$$\mathcal{Y}_1(\mathcal{A}) = \{T \in \mathcal{A};\, \inf_{U \in \Gamma(G)} \|U(T)\| = 0\}.$$

Then $\mathcal{Y}_1(\mathcal{A})$ satisfies $(1)'$ and (2) in Theorem 8 and is closed under scalar multiplication. Consequently, $\mathcal{A}$ admits a G-invariant mean if $\mathcal{Y}_1(\mathcal{A})$ is closed under addition.

Proof. Similar to the proof of Theorem 9, we just need to replace F_n there by

$$U_n \;=\; \frac{1}{n} \sum_{k=1}^{n} {U_x}^k \;=\; \frac{1}{n} \sum_{k=1}^{n} U_{x^k} \,.$$

□

Next, we will formulate a kind of fixed point property for a unitary continuous representation π of G on a Hilbert space H_π. We will show that it is a necessary condition for π to be amenable. For this purpose, we notice that, for any Banach $B(H_\pi)$-bimodule Σ, both Σ^* and Σ^{**} can be regarded as Banach $B(H_\pi)$-bimodules in the canonical way. It is also well-known that the unit ball of $B(\Sigma^{**})$ is W^*OT compact (see Kadison [5]). Now we define a linear map $\Phi : T \in B(H_\pi) \longmapsto \Phi_T \in B(\Sigma^{**})$, where $\Phi_T(m) = T \cdot m$ for all $m \in \Sigma^{**}$. It is readily checked that $\|\Phi_T\| \leq \kappa \|T\|$, where κ is the constant for the module structure (see Johnson [4]). Let

$$\mathcal{P}_\Sigma = \overline{\{\Phi_T;\; T \in TC(H_\pi)_1^+\}}^{W^*OT},$$

where $TC(H_\pi)_1^+ = \{T \in TC(H_\pi);\; T \geq 0,\, \|T\|_1 = 1\}$. Obviously, $\mathcal{P}_\Sigma$ is contained in the ball of radius κ in $B(\Sigma^{**})$, therefore, it is W^*OT compact.

We will say that π has the *fixed point property* if, for any Banach $B(H_\pi)$-bimodule Σ, there exists an operator $\Phi_\circ \in \mathcal{P}_\Sigma$ such that $\Phi_{\pi(x)} \Phi_\circ \Phi_{\pi(x^{-1})} = \Phi_\circ$ for all $x \in G$.

Theorem 11. *If π is amenable, then π has the fixed point property.*

Proof. If π is amenable, by [1, Theorem 3.6], there exists a net $\{S_\alpha\}$ in $TC(H_\pi)_1^+$ such that $\lim_\alpha \|x \cdot S_\alpha - S_\alpha\|_1 = 0$ for all $x \in G$. Let Σ be a Banach $B(H_\pi)$-bimodule. Consider the net $\{\Phi_{S_\alpha}\}$ in $\mathcal{P}_\Sigma$. For all $x \in G$ and $m \in \Sigma^{**}$, we have

$$\begin{aligned} \|\Phi_{\pi(x)} \Phi_{S_\alpha} \Phi_{\pi(x^{-1})}(m) - \Phi_{S_\alpha}(m)\| &= \|\Phi_{(x \cdot S_\alpha - S_\alpha)}(m)\| \;\leq\; \kappa \,\|x \cdot S_\alpha - S_\alpha\| \,\|m\| \\ &\leq\; \kappa \,\|x \cdot S_\alpha - S_\alpha\|_1 \,\|m\| \;\longrightarrow\; 0 \,. \end{aligned}$$

Since $\mathcal{P}_\Sigma$ is W^*OT-compact, we may assume that $\Phi_{S_\alpha} \xrightarrow{W^*OT} \Phi_\circ$ for some $\Phi_\circ \in \mathcal{P}_\Sigma$. This leads to $\Phi_{\pi(x)}\Phi_{S_\alpha}\Phi_{\pi(x^{-1})} \xrightarrow{W^*OT} \Phi_{\pi(x)}\Phi_\circ\Phi_{\pi(x^{-1})}$. On the other hand,

$$\Phi_{\pi(x)}\Phi_{S_\alpha}\Phi_{\pi(x^{-1})} = (\Phi_{\pi(x)}\Phi_{S_\alpha}\Phi_{\pi(x^{-1})} - \Phi_{S_\alpha}) + \Phi_{S_\alpha} \xrightarrow{W^*OT} \Phi_\circ\,, \quad \textit{for all } x \in G\,.$$

Therefore, $\Phi_{\pi(x)}\Phi_\circ\Phi_{\pi(x^{-1})} = \Phi_\circ$ for all $x \in G$. □

This formulation of the fixed point property is motivated by [6, Theorem 5.1]. It would be very interesting to know whether the converse of Theorem 11 is true or not.

This paper forms part of the author's Ph.D dissertation under the supervision of Professor Anthony T. Lau, to whom the author wishes to express his deepest gratitude for his invaluable guidance and continued encouragement.

References

[1] Bekka, M.E.B., *Amenable unitary representations of locally compact groups*, Invent. math. **100** (1990), 383-401.

[2] Emerson, W.R., *Characterization of amenable groups*, Trans. Amer. Math. Soc. **241** (1978), 183-194.

[3] Halmos, P.R., *A Hilbert Space Problem Book*, D. Van Nostrand Company, Inc., Princeton-Toronto-London (1967).

[4] Johnson, B.E., *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972).

[5] Kadison, R.V., *The trace in finite operator algebras*, Proc. Amer. Math. Soc. **12** (1961), 973-977.

[6] Lau, A.T. and Paterson, Alan L.T., *Inner amenable locally compact groups*, Trans. Amer. Math. Soc. **325** (1991), 155-169.

[7] Takesaki, M., *Theory of Operator Algebras I*, Springer-Verlag, Berlin-Heidelberg-New York (1979).

[8] Wong, J.C.S., and Riazi, A., *Characterization of amenable locally compact semigroups*, Pacific J. of Math. **108** (1983), 479-496.

Department of Mathematics
University of Alberta
Edmonton, Alberta, Canada T6G 2G1

Current Address:
455 State Street
LaSalle, Ontario, Canada N9J 3G1
email: qin.xu@eds.com