ON THE PARASPECTRUM AND THE CONTINUITY OF THE SPECTRUM IN ALGEBRA OF OPERATORS

Slaviša V. Djordjević

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Abstract. In this paper some conditions are given for the continuity of the spectrum using the paraspectrum of operators. Also, Luecke's theorem for G_1-operators is given as a simple consequence of those conditions.

1. Introduction

Let X be a complex infinite–dimensional Banach space and let $B(X)$ denotes a Banach algebra of all bounded operators on X. If $T \in B(X)$, then $\sigma(T)$ denotes the spectrum of T. For $A, B \in B(X)$ we define the $*$-prominance of A by B, $* \in \{\alpha, \beta, \gamma\}$, by

$$\begin{aligned} \mathrm{prom}_\alpha(A;B) &= \{\lambda \notin \sigma(A) \,:\, \|(A-\lambda)^{-1}\| \cdot \|A-B\| \geq 1\}\,; \\ \mathrm{prom}_\beta(A;B) &= \{\lambda \notin \sigma(A) \,:\, \|(A-\lambda)(A-B)\| \geq 1\}\,; \\ \mathrm{prom}_\gamma(A;B) &= \{\lambda \notin \sigma(A) \,:\, \|A-B\| \geq d(\lambda, \sigma(A))\}\,. \end{aligned}$$

The $*$-paraspectrum of A by B is the set

$$\sigma_*(A;B) = \mathrm{prom}_*(A;B) \cup \sigma(A)\,, \qquad * \in \{\alpha, \beta, \gamma\}\,.$$

It has been introduced in [**3**] in the case where X is a Hilbert space.

An operator $A \in B(X)$ is a G_1-operator if A satisfies the growth condition [**4**]

$$\|(A-\lambda)^{-1}\| \leq \frac{1}{d(\lambda, \sigma(A))}\,, \qquad \lambda \notin \sigma(A)\,.$$

The continuity of spectra for G_1-operators on a Hilbert space has been discussed by several authors [2,3,4,6]. To discuss it for arbitrary operators on a Banach space, we need the distances d_1 and d_2 among compact subsets in the complex plane. Let M and N be a compact subsets in the complex plane. We define the distances $d_1(M,N)$ and $d_2(M,N)$ between M and N by

$$\begin{aligned} d_1(M,N) &= \sup_{n \in N} \inf_{m \in M} |m-n| = \sup_{n \in N} \mathrm{dist}\ (n, M) \\ d_2(M,N) &= \sup_{m \in M} \inf_{n \in N} |m-n| = \sup_{m \in M} \mathrm{dist}\ (m, N)\,. \end{aligned}$$

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It is well-known that the distance $d(M,N)$ define by

$$d(M,N) = \max\{d_1(M,N), d_2(M,N)\}$$

is the Hausdorff distance between compact subsets M and N.

A mapping p, defined on $B(X)$ whose values are compact subset of $\mathbb{C}$, is said to be upper (lower) semi-continuous at A, provided that if $A_n \to A$ then

$$d_1(p(A), p(A_n)) \to 0 \quad (d_2(p(A), p(A_n)) \to 0), \quad n \to \infty.$$

If p is both upper and lower semi-continuous at A, then it is said to be continuous at A and in this case $\lim p(A_n) = p(A)$.

In this paper we consider the spectral variation inequality

$$d_i(\sigma(A), \sigma(B)) \leq \|A - B\|, \quad i = 1, 2 \tag{1.i.}$$

and we discuss a continuity of the spectrum of A using the $*$-paraspectrum of A by B.

2. Variation of spectrum

Directly from the definition of the $*$-paraspectrum follows that $\sigma(A) \subset \sigma_*(A;B)$, $* \in \{\alpha, \beta, \gamma\}$, for every $B \in B(X)$. Also, by [**3**] we get $\sigma(B) \subset \sigma_\alpha(A;B)$ and $\sigma_\gamma(A;B) \subset \sigma_\beta(A;B) \subset \sigma_\alpha(A;B)$ for every $A, B \in B(X)$.

If (τ_n) is a sequence of compact subsets of $\mathbb{C}$, then its limit inferior is

$$\liminf \tau_n = \{\lambda \in \mathbb{C} : \text{ there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda\}$$

and its limit superior is

$$\limsup \tau_n = \{\lambda \in \mathbb{C} : \text{ there are } \lambda_{n_k} \in \tau_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}.$$

It is well-known that a mapping p which maps $B(X)$ into the family of compact subset of $\mathbb{C}$ is upper (lower) semi-continuous at A if for every sequence $\{A_n\}$ in $B(X)$ such that $A_n \to A$ holds

$$\limsup p(A_n) \subset p(A) \quad (p(A) \subset \liminf p(A_n)).$$

Theorem 1. *Let $A \in B(X)$ and let $\{A_n\}$ be a sequence in $B(X)$ such that $A_n \to A$. Then the next conditions are equivalent:*

(1) $\lim \sigma(A_n) = \sigma(A)$;

(2) $\bigcap_{n=1}^{\infty} \sigma_\alpha(A; A_n) \subset \liminf \sigma(A_n)$;

(3) $\bigcap_{n=1}^{\infty} \sigma_\alpha(A_n; A) \subset \liminf \sigma(A_n)$.

Proof. (1) $\Rightarrow$ (2) Let $\lim \sigma(A_n) = \sigma(A)$ and suppose that (2) is not true. Then there exists a $\lambda \in \left(\bigcap_{n=1}^{\infty} \sigma_\alpha(A; A_n)\right) \setminus (\liminf \sigma(A_n))$. For this λ we get:

(i) $\lambda \in \sigma_\alpha(A; A_n)$, for every $n \in \mathbb{N}$;

(ii) $\lambda \notin \liminf \sigma(A_n)$ and so $\lambda \notin \sigma(A)$ by (1);

By (i) and (ii) it follows $\lambda \in \mathrm{prom}_\alpha(A; A_n)$, i.e.

$$\|(A - \lambda)^{-1}\|^{-1} \leq \|A - A_n\|, \text{ for every } n \in \mathbb{N}.$$

If $n \to \infty$, then $\|(A-\lambda)^{-1}\|^{-1} = 0$. Hence it is a contradiction.

(2) $\Rightarrow$ (3) Let the condition (2) holds and suppose that (3) does not hold. Then there exists a $\lambda \in \left(\bigcap_{n=1}^{\infty} \sigma_\alpha(A_n; A)\right) \setminus (\liminf \sigma(A_n))$. For this λ we get:

(i) $\lambda \in \sigma_\alpha(A_n; A)$, for every $n \in \mathbb{N}$;

(ii) there exists a $n_0 \in \mathbb{N}$ such that $\lambda \notin \sigma(A_n)$ for every $n \geq n_0$.

¿From (i) and (ii) it follows that $\lambda \in \mathrm{prom}_\alpha(A_n, A)$, i.e.

$$(*) \qquad \|(A_n-\lambda)^{-1}\|^{-1} \leq \|A_n - A\| \to 0\,, \quad n \to \infty\,.$$

Suppose that $\lambda \in \sigma(A)$. Then $\lambda \in \sigma_\alpha(A; A_n)$, for every $n \in \mathbb{N}$, i.e. $\lambda \in \bigcap_{n=1}^{\infty} \sigma_\alpha(A; A_n) \subset \liminf \sigma(A_n)$ and this is a contradiction. Hence $\lambda \notin \sigma(A)$.

Since $A_n - \lambda \to A - \lambda$ and $\lambda \notin \sigma(A)$ it follows that $(A_n-\lambda)^{-1} \to (A-\lambda)^{-1}$ (by the continuity of the function $T \mapsto T^{-1}$ [1, Theorem 50.7]). But, by $(*)$, we get that $\|(A_n-\lambda)^{-1}\| \to \infty$, $n \to \infty$, i.e. $(A_n-\lambda)^{-1}$ converges to a noninvertible operator. Hence it is a contradiction.

(3) $\Rightarrow$ (1) Suppose that $\bigcap_{n=1}^{\infty} \sigma_\alpha(A_n; A) \subset \liminf \sigma(A_n)$. Let $\lambda \in \sigma(A)$. Then $\lambda \in \sigma_\alpha(A_n, A)$ for every $n \in \mathbb{N}$ [**3**], i.e.

$$\lambda \in \bigcap_{n=1}^{\infty} \sigma_\alpha(A_n; A) \subset \liminf \sigma(A_n)\,.$$

Hence we have $\sigma(A) \subset \liminf \sigma(A_n)$.

Now, since σ is always upper semi-continuous [**5**, Theorem 1], it follows $\lim \sigma(A_n) = \sigma(A)$. □

Next necessary and sufficient conditions for the continuity of spectrum by means of α–paraspectrum is an easy consequence of the previous theorem.

Corollary 2. *Let $A \in B(X)$. Then the spectrum is continuous at A if and only if for every sequence $\{A_n\}$ such that $A_n \to A$ one of the following equivalent conditions is satisfied:*

(1) $\bigcap_{n=1}^{\infty} \sigma_\alpha(A; A_n) \subset \liminf \sigma(A_n)$*;*

(2) $\bigcap_{n=1}^{\infty} \sigma_\alpha(A_n; A) \subset \liminf \sigma(A_n)$.

Theorem 3. *If for $A, B \in B(X)$ is $\sigma_\gamma(A; B) = \sigma_\alpha(A; B)$, then the spectral variation inequality* (1.1) *holds for A and B.*

Proof. Let $\sigma_\gamma(A; B) = \sigma_\alpha(A; B)$. Since

$$d_1(\sigma(A), \sigma(B)) = \sup_{\lambda \in \sigma(B)} \inf_{\mu \in \sigma(B)} |\lambda - \mu|$$

and $\sigma(B) \subset \sigma_\alpha(A; B) = \sigma_\gamma(A; B)$ we have that

$$d_1(\sigma(A), \sigma(B)) \leq \|A - B\|\,, \ \text{ for every } \lambda \in \sigma(B)\,,$$

we have that the spectral variation inequality (1.1) holds for A and B. □

Corollary 4. *If for $A \in B(X)$ $\sigma_\gamma(B;A) = \sigma_\alpha(B;A)$ holds that for every $B \in B(X)$, then the spectrum is continuous at A.*

Proof. Let $\{A_n\}$ be a sequence in $B(X)$ such that $A_n \to A$. Since $d_1(\sigma(A_n), \sigma(A)) = d_2(\sigma(A), \sigma(A_n))$, Theorem 3 implies

$$d_2(\sigma(A), \sigma(A_n)) \leq \|A - A_n\| \to 0\,,$$

i.e. the spectrum is lower semi–continuous at A. Then it follows from [**5**, Theorem 1] that the spectrum is continuous at A. □

Remark. Recall that $\sigma_\gamma(A;B) = \sigma_\alpha(A;B)$ for every $A \in B(X)$ is not a necessary condition for the continuity of the spectrum at B. An example can be constructed by using [**3**, Example 4 (1)].

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be the matrix acting on a two–dimensional Hilbert space and $B = 2A + A^*$. Then by [**3**, Example 4] it follows $\sigma_\gamma(A;B) \neq \sigma_\alpha(A;B)$. Since $\sigma(B)$ is totally disconnected, the spectrum is continuous at B by [**5**, Theorem 3]. □

It is well-known that the spectrum is a continuous function on the set of G_1-operators [3,4]. Now we can get it as an easy consequence of Theorem 3 and Corollary 4.

Corollary 5. *If $A_n \in B(X)$ are G_1-operators and $A_n \to A$, then* $\lim \sigma(A_n) = \sigma(A)$.

Proof. By [**3**, Theorem 3] we have $\sigma_\gamma(A_n;A) = \sigma_\alpha(A_n;A)$ for every $n \in \mathbb{N}$ and by Corollary 4 we have $\lim \sigma(A_n) = \sigma(A)$. □

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University of Niš, Faculty of Science, Department of Mathematics
Ćirila i Metodija 2, 18000 Niš, Yugoslavia
E-mail : slavdj@ptt.yu
slavdj@pmf.pmf.ni.ac.yu