A NOTE ON THE FIRST HOMOLOGY OF THE GROUP OF POLYNOMIAL AUTOMORPHISMS OF THE COORDINATE SPACE

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Abstract. We consider the group of all polynomial automorphisms of the coordinate space and its subgroups, and compute the first homologies of their groups. Our main result is that the first homology group of the group of all polynomial automorphisms of the coordinate plane is isomorphic to $\mathbf{K}^*$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and $\mathbf{K}^* = \mathbf{K} - \{0\}$. This fact relates mostly to the topology of transversely algebraic foliations.

§ 1. Introduction

In this note we shall study the structure of the group of polynomial automorphisms of the coordinate space $\mathbf{K}^n$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) and its subgroups and compute the first homology groups of these groups, where the first homology group of a group G is defined by the quotient group of G by its commutator subgroup.

There are many results on commutators of the groups of automorphisms preserving a geometric structure such as volume structure, symplectic structure, foliated structure and so on (for example, see [A-F1],[A-F2],[B],[F],[F-I1],[F-I2],[T1],[T2], $\cdots$). In those cases, the first homology groups are not necessarily trivial. Then the calculation of the first homology is the next problem. The first homology groups are interesting for us because they are expected relating mostly to the fundamental geometric structures.

Our main result is as follows.

Theorem(Theorem 8). *Let G_2 be the group of all polynomial automorphisms of the coordinate plane $\mathbf{K}^2$. Then $H_1(G_2) \cong \mathbf{K}^*$, where $\mathbf{K}^* = \mathbf{K} - \{0\}$.*

Let G_n^δ be the group of polynomial automorphisms of the coordinate space $\mathbf{K}^n$ with the discrete topology and BG_n^δ denote the classifying space, by which transversely algebraic foliated $\mathbf{K}^n$-bundles are classified . Since BG_n^δ has the homotopy type of an Eilenberg-Maclane space $K(G_n, 1)$, we have the following as a corollary of Theorem.

Corollary. $H_1(BG_2^\delta) \cong \mathbf{K}^*$.

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§ 2. First homologies of groups of polynomial automorphisms

Let G_n be the group of all polynomial automorphisms of the coordinate space $\mathbf{K}^n$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. By definition, an element g of G_n is a polynomial mapping

$$(x_1, \cdots, x_n) \mapsto g(x, \cdots, x_n) = (X_1(x, \cdots, x_n), \cdots, X_n(x, \cdots, x_n))$$

from the coordinate space to itself which is bijective and has polynomial inverse. Then we note that the Jacobian determinant $\det(J(g))$ of an element g of G_n is constant.

Let $A_n (\subset G_n)$ be the group of all afine transformations. Let $E_n (\subset G_n)$ be the group of all polynomial automorphisms which are *elementary* mappings of the form

$$\begin{aligned} & e(x_1, \cdots, x_n) \\ = \; & (\alpha_1 x_1 + p_1(x_2, \cdots, x_n), \, \alpha_2 x_2 + p_2(x_3, \cdots, x_n), \\ & \cdots, \alpha_{n-1} x_n + p_{n-1}(x_n), \, \alpha_n x_n + \beta) \end{aligned}$$

for some constants $\alpha_i (i = 1, 2, \cdots, n)$ and β with $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n \neq 0$, and for some polynomial functions $p_i(x_{i+1}, \cdots, x_n)\, (i = 1, 2, \cdots, n-1)$. Let $\hat{G}_n$ denote the subgroup of G_n which is generated by A_n and E_n.

First we consider the first homology of A_n. Then we have the following.

Proposition 1. *$H_1(A_n) \cong \mathbf{K}^*$ for $n \geq 2$, where $\mathbf{K}^* = \mathbf{K} - \{0\}$ is the multiplicative group.*

We define a map $J : A_n \to GL(n, \mathbf{K})$ by $d(f) = J(f)$ for any $f \in A_n$, which is an epimorphism. Then we have the following proposition.

Proposition 2. $\ker J = [\ker J, \, A_n]$.

Proof. Any $f \in \ker J$ is of the form $f(x_1, \cdots, x_n) = (x_1 + b_1, \cdots, x_n + b_n)$. We put

$$f_1(x_1, \cdots, x_n) = (x_1, \, x_2 + b_2, \cdots, x_n + b_n),$$

and

$$f_2(x_1, \cdots, x_n) = (x_1 + b_1, \, x_2, \cdots, x_n).$$

Then we have $f = f_2 \circ f_1$. Put

$$g_1(x_1, \cdots, x_n) = (x_1, \, \frac{1}{2} x_2, \cdots, \frac{1}{2} x_n)$$

and

$$g_2(x_1, \cdots, x_n) = (\frac{1}{2} x_1, \, x_2, \cdots, x_n).$$

Then we have $f_i = f_i^{-1} \circ g_i^{-1} \circ f_i \circ g_i = [f_i^{-1}, \, g_i^{-1}], \, (i = 1, 2)$. Thus f is contained in $[\ker J, \, A_n]$. This completes the proof.

Proof of Proposition 1. From the epimorphism $J : A_n \to GL(n, \mathbf{K})$, we have the following exact sequence

$$\to \ker J / [\ker J, \, A_n] \to H_1(A_n) \to H_1(GL(n, \mathbf{K})) \to 1.$$

Hence we have $H_1(A_n) \cong H_1(GL(n,\mathbf{K})) \cong \mathbf{K}^*$ from Proposition 2. This completes the proof.

Secondly we consider the first homology of E_n. We have the following by easy computations.

Lemma 3. *The commutator subgroup $[E_n, E_n]$ of E_n is the group consisting of elements of the form*

$$\begin{aligned} & f(x_1, \cdots, x_n) \\ = \; & (x_1 + p_1(x_2, \cdots, x_n),\, x_2 + p_2(x_3, \cdots, x_n), \\ & \cdots, x_{n-1} + p_{n-1}(x_n),\, x_n + \beta) \end{aligned}$$

for some constant β and for some polynomial functions $p_i(x_{i+1}, \cdots, x_n)$ $(i = 1, 2, \cdots, n-1)$.

Remark 4. E_n is a solvable group. In fact, the n-th commutator subgroup of E_n is the commutative subgroup consisting of elements of the form $f(x_1, \cdots, x_n) = (x_1 + p_1(x_2, \cdots, x_n), x_2, \cdots, x_n)$ (cf. [S-M]).

Theorem 5. $H_1(E_n) \cong (\mathbf{K}^*)^n$ *for* $n \geq 2$.

Proof. We easily see from Lemma 3 that the quotient group $E_n/[E_n, E_n]$ consists of cosets of the form $[f_\alpha] = f_\alpha[E_n, E_n]$, where f_α is of the form $f(x_1, \cdots, x_n) = (\alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_{n-1} x_n, \alpha_n x_n)$, where $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n \neq 0$. Thus we have $H_1(E_n) \cong \{[f_\alpha]\} \cong \{ \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} \mid \alpha_i \in \mathbf{K}^* \}$. This completes the proof.

Thirdly we consider the first homology of $\hat{G}_n$. Then we have the following.

Theorem 6. $H_1(\hat{G}_n) \cong \mathbf{K}^*$ *for* $n \geq 2$.

We define a map $d : \hat{G}_n \to \mathbf{K}^*$ by $d(f) = \det(J(f))$ for any $f \in \hat{G}_n$, which is an epimorphism. Then we have the following.

Proposition 7. $\ker d = [\ker d, \hat{G}_n]$.

Proof. Any $f \in \ker d$ is represented by elements of E_n and elements of A_n with Jacobian determinant $= 1$.

(1) Case where f is an element of E_n. In this case, f has the form

$$\begin{aligned} & f(x_1, \cdots, x_n) \\ = \; & (\alpha_1 x_1 + p_1(x_2, \cdots, x_n),\, \alpha_2 x_2 + p_2(x_3, \cdots, x_n), \\ & \cdots, \alpha_{n-1} x_{n-1} + p_{n-1}(x_n),\, \alpha_n x_n + \beta) \end{aligned}$$

for some constants $\alpha_i(i=1,2,\cdots,n)$ and β with $\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_n=1$, and for some polynomial functions $p_i(x_{i+1},\cdots,x_n)\,(i=1,2,\cdots,n-1)$. We define $f_0,\ f_i(i=1,\cdots,n-1)$ and f_n in the following :

$$f_0(x_1,\,\cdots,\,x_n)=(\alpha_1x_1,\,\alpha_2x_2,\,\cdots,\,\alpha_{n-1}x_n,\,\alpha_nx_n),$$

$$\begin{aligned} &f_i(x_1,\,\cdots,\,x_n)\\ &=\ (x_1,\,x_2,\cdots,\,x_{i-1},\\ &\qquad x_i+p_i(\frac{x_{i+1}}{\alpha_{i+1}},\,\cdots,\,\frac{x_n}{\alpha_n}),\,x_{i+1},\cdots,\,x_n)\end{aligned}$$

$(i=1,\cdots,n-1)$
and

$$f_n(x_1,\,\cdots,\,x_n)=(x_1,\,x_2,\,\cdots,\,x_{n-1},\,x_n+\beta).$$

Then we have $f=f_n\circ f_{n-1}\circ\cdots\circ f_1\circ f_0$.

(a) f_0 is of the form

$$f_0(\begin{pmatrix} x_1\\ x_2\\ \cdots\\ x_n\end{pmatrix})=\begin{pmatrix}\alpha_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_n\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}.$$

Note that the matrix $\begin{pmatrix}\alpha_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_n\end{pmatrix}$ belongs to $SL(n;\mathbf{K})$. Since $SL(n;\mathbf{K})$ is a simple group, f_0 is expressed as a product of commutators of elements of A_n with Jacobian determinant $=1$ and zero constant term.

(b) Put $g_i(x_1,\,\cdots,\,x_n)=(x_1,\,x_2,\cdots,\,x_{i-1},\,\frac{1}{2}x_i,\,x_{i+1},\cdots,x_n)$, $(i=1,\cdots,n-1)$. Then we have $f_i=[f_i^{-1},g_i^{-1}]\ (i=1,\cdots,n-1)$. Thus each f_i is contained in $[\ker d,\,\hat{G}_n]$.

(c) f_n is an element of $\ker J$ in Proposition 1. Thus f_n is contained in $[\ker d,\,\hat{G}_n]$.

(2) Case where f is an element of A_n. In this case, we can prove by the same argument as in Proposition 2 that f is contained in $[\ker d,\,\hat{G}_n]$.

This completes the proof.

Proof of Theorem 6. From the epimorphism $d:\hat{G}_n\to\mathbf{K}^*$, we have the following exact sequence

$$\to\ker d/[\ker d,\,\hat{G}_n]\ \to\ H_1(\hat{G}_n)\ \to\ \mathbf{K}^*\ \to\ 1.$$

Hence we have $H_1(\hat{G}_n)\cong\mathbf{K}^*$ from Proposition 7. This completes the proof.

We have the following by Theorem of Jung([J]) stating that G_2 is generated by A_2 and E_2.

Theorem 8. $H_1(G_2)\cong\mathbf{K}^*$.

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