CAUCHY NETS IN THE CONSTRUCTIVE THEORY OF APARTNESS SPACES

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Abstract. A notion of Cauchy net is introduced into the constructive theory of apartness spaces. It is shown that for a sequence in a metric space this notion is equivalent to the standard metric notion of Cauchy sequence. Applications of this notion are then given, culminating in a generalisation of Bishop's Lemma on locatedness.

1 Introduction Axioms for a constructive theory of apartness between sets were introduced in [12], where the particular example of a uniform space was discussed in detail. In the present paper we discuss Cauchy and convergent sequences in the framework of that theory.

By *constructive mathematics* we mean mathematics developed with intuitionistic logic [2, 3, 4, 5]. Thus all our work is fully compatible with classical mathematics and can be interpreted *mutatis mutandis* within intuitionistic mathematics, recursive mathematics, and, as far as we know, any other framework for computable mathematics (such as Weihrauch's TTE theory [17]). Background material on constructive mathematics can be found in [1, 3, 5, 15]. Further work on the constructive theory of apartness is presented in [6, 7, 8, 11]. Intuitionistic topology, which uses not only intuitionistic logic but also Brouwer's principles, is discussed in [14, 16].

In their classical presentation [9], Cameron, Hocking, and Naimpally use nearness, rather than apartness, as the primary relation between subsets of a set X. They call a sequence $s = (x_n)_{n=1}^{\infty}$ in X a Cauchy sequence if for all infinite sets A, B of positive integers, the sets $s(A), s(B)$ are near each other, where, for example,

$$s(A) = \{x_n : n \in A\}.$$

We shall call such a sequence a **CHN–Cauchy sequence**.[1] Bearing in mind that the canonical nearness relation between subsets S, T of a metric space (X, ρ) is defined by

$$S \text{ is near } T \text{ if and only if } \forall_{\varepsilon>0}\exists_{s\in S}\exists_{t\in T}\ (\rho(s,t) < \varepsilon),$$

we easily show that in a metric space, a metrically Cauchy sequence—that is, a Cauchy sequence in the normal, elementary sense—is a CHN–Cauchy sequence; the converse is proved in [9] by an indirect, and therefore nonconstructive, argument. The following Brouwerian example shows that we cannot prove that converse constructively.

Let $(x_n)_{n=0}^{\infty}$ be a binary sequence with at most one term equal to 1. Such a sequence has a much stronger property than that required of a CHN–Cauchy sequence: if each of two sets A, B of positive integers has at least two distinct elements, then $s(A)$ is near $s(B)$. However, if (x_n) is a metrically Cauchy sequence, then there exists N such that $|x_m - x_n| < 1$ for all $m, n \geq N$; by testing $x_0, \dots, x_N$, we can determine that either $x_n = 0$ for all n or else there exists $n \leq N$ such that $x_n = 1$. Thus the statement

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[1] **C**ameron, **H**ocking, **N**aimpally

Every binary CHN sequence is a metrically Cauchy sequence

entails the **Limited Principle of Omniscience (LPO)**:

$$\forall_{\mathbf{a}\in\{0,1\}^{\mathbf{N}}} \left(\forall_n \left(a_n = 0\right) \vee \exists_n \left(a_n = 1\right)\right),$$

which is false in the recursive model of constructive mathematics (it implies the decidability of the Halting Problem).

Now let X be any apartness space: that is, a set with a set–set apartness relation $\bowtie$ satisfying the axioms presented at the beginning of Section 3 below. Note that the set–set apartness induces a binary relation $\neq$ on X given by

$$x \neq y \Leftrightarrow \{x\} \bowtie \{y\}.$$

This relation is an **inequality** on X : that is, it has the properties

$$x \neq y \Rightarrow \neg\left(x = y\right),$$
$$x \neq y \Rightarrow y \neq x.$$

Classically this inequality would be the denial inequality, where $x \neq y$ if and only if $\neg\left(x = y\right)$; but constructively this need not be the case.

An appropriate constructive notion of a Cauchy sequence in X will be classically equivalent to that of a CHN–Cauchy sequence, and in the case where X is a metric space will be constructively equivalent to the metric notion. The following definitions introduce such a notion, in the more general setting of nets.

By a **directed set** we mean a nonempty set L with both a binary inequality relation $\neq$ and a partial order $\succcurlyeq$ such that for all $x, y \in L$ there exists $z \in L$ with $z \succcurlyeq x$ and $z \succcurlyeq y$. A **net** in X is a mapping $\lambda \rightsquigarrow x_\lambda$ of L into X that is **strongly extensional** in the sense that $x_\lambda \neq x_{\lambda'}$ entails $\lambda \neq \lambda'$; we denote such a net by $\left(x_\lambda\right)_{\lambda\in L}$. In the case where L is the set $\mathbf{N}^+$ of positive integers (with the standard inequality and partial order), we call the net a **sequence** and denote it by $\left(x_n\right)_{n=1}^{\infty}$ or simply $\left(x_n\right)$. By a **Cauchy net** in X we mean a net $s = \left(x_\lambda\right)_{\lambda\in L}$ such that for all subsets A, B of L with $s\left(A\right) \bowtie s\left(B\right)$, there exists λ_0 such that if $\lambda \in A$ for some $\lambda \succcurlyeq \lambda_0$, then

$$B \subset \sim\{\lambda : \lambda \succcurlyeq \lambda_0\} = \{\lambda' \in L : \forall_{\lambda\succcurlyeq\lambda_0} \left(\lambda' \neq \lambda\right)\}.$$

It is easy to show that in a metric space, a Cauchy sequence in this sense is a CHN–Cauchy sequence, and every metrically Cauchy sequence is a Cauchy sequence. Our first aim is to prove the following.

Theorem 1 *Every Cauchy sequence in a metric apartness space is a metrically Cauchy sequence.*

This aim will be fulfilled in Section 2. In Sections 3 and 4 we examine some fundamental properties of Cauchy and convergent sequences in a not–necessarily–metric apartness space; in particular, we show that an important lemma due to Bishop extends from the metric space context to provide a characterisation of weakly located subsets in a first countable apartness space.

2 Cauchy implies metrically Cauchy Our proof of Theorem 1 requires some preliminary technical lemmas.

Lemma 2 *Let $s = (x_n)_{n=1}^{\infty}$ be a Cauchy sequence in a metric apartness space X. Then for all real numbers α, β with $0 < \alpha < \beta$, and for each positive integer m,*

$$\exists_{k>m}\ (\rho(x_m, x_k) > \alpha) \vee \forall_{k \geq m}\ (\rho(x_m, x_k) < \beta)\,.$$

Proof. We may assume, for convenience, that $m = 1$. Define

$$A = \left\{ n : \rho(x_1, x_n) < \frac{\alpha+\beta}{2} \right\},$$
$$B = \left\{ n : \rho(x_1, x_n) > \frac{\alpha+3\beta}{4} \right\}.$$

Then

$$\forall_{m \in A} \forall_{n \in B} \left(\rho\left(x_m, x_n\right) > \frac{\beta - \alpha}{4} \right),$$

so $s\left(A\right) \bowtie s\left(B\right)$. Since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence (in the sense of the definition given above), we can choose N so that if $\nu \in A$ for some $\nu \geq N$, then $B \subset \{1, \ldots, N-1\}$. Either $\rho(x_N, x_1) > \alpha$ or else $\rho(x_N, x_1) < \frac{\alpha+\beta}{2}$. In the latter case, $N \in A$, so $B \subset \{1, \ldots, N-1\}$ and therefore

$$\rho(x_n, x_1) \leq \frac{\alpha+3\beta}{4} < \beta$$

for all $n \geq N$. By testing $x_2, \ldots, x_{N-1}$, we can now determine that either $\rho(x_1, x_k) > \alpha$ for some $k \leq N-1$, or else $\rho(x_1, x_k) < \beta$ for all $k \leq N-1$ and therefore for all k. Q.E.D.

Lemma 3 *Let $s = (x_n)_{n=1}^{\infty}$ be a Cauchy sequence in a metric apartness space X. Then for all real numbers α, β with $0 < \alpha < \beta$,*

- $\triangleright$ *either there exists m such that $\rho\left(x_n, x_m\right) < \beta$ for all $n > m$*
- $\triangleright$ *or else there exists a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $\rho(x_{n_{k+1}}, x_{n_k}) > \alpha$ for each k.*

Proof. Setting $n_1 = 1$ and using Lemma 2, we may assume that there exists $n_2 > n_1$ such that $\rho\left(x_{n_2}, x_{n_1}\right) > \alpha$; we then set $\lambda_1 = 0$. Now suppose that for some $k > 1$ we have constructed a certain nonnegative integer n_k. By Lemma 2, either there exists $n_{k+1} > n_k$ such that $\rho(x_{n_{k+1}}, x_{n_k}) > \alpha$, or else $\rho(x_n, x_{n_k}) < \beta$ for all $n \geq n_k$. In the former case set $\lambda_k = 0$ and $A_k = B_k = \emptyset$. In the latter case set $\lambda_k = 1$, $A_k = \{n_k\}, B_k = \{n_{k-1}\}$, and for every $j > k$,

$$n_j = n_k,\ A_j = \emptyset,\ B_j = \emptyset.$$

This completes the inductive construction of an increasing binary sequence $(\lambda_k)_{k=1}^{\infty}$, an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers, and sequences $(A_k)_{k=1}^{\infty}$, $(B_k)_{k=1}^{\infty}$ of subsets of $\mathbf{N}^{+}$. Let $A = \bigcup_{k=1}^{\infty} A_k$ and $B = \bigcup_{k=1}^{\infty} B_k$.Then $s\left(A\right) \bowtie s\left(B\right)$: for if $i \in A$ and $j \in B$, then there exists k such that $A_k = \{n_k\}, B_k = \{n_{k-1}\}$, and $\rho(x_{n_k}, x_{n_{k-1}}) > \alpha$. Choose N such that if $\nu \in A$ for some $\nu \geq N$, then $B \subset \{1, \ldots, N-1\}$. If $\lambda_N = 1$, then there exists $k \leq N$ such that $\rho\left(x_n, x_{n_k}\right) < \beta$ for all $n \geq n_k$. In the case $\lambda_N = 0$, suppose that $\lambda_m = 1 - \lambda_{m-1}$ for some $m > N$. Then $A = \{n_m\}$ and $B = \{n_{m-1}\}$, where $n_m \geq m > N$ and $n_{m-1} \geq m-1 \geq N$. This is impossible, by our choice of N. Hence $\lambda_m = 0$ for all $m > N$ and therefore for all m. Thus $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers such that $\rho(x_{n_{k+1}}, x_{n_k}) > \alpha$ for each k. Q.E.D

Lemma 4 *Let $s = (x_n)_{n=1}^{\infty}$ be a Cauchy sequence in a metric apartness space X, let $\alpha > 0$, and suppose that there exists a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $\rho(x_{n_{k+1}}, x_{n_k}) > \alpha$ for each k. Then* LPO *holds.*

PROOF. Let $(\lambda_k)_{k=1}^{\infty}$be an increasing binary sequence. If $\lambda_k = 0$, set $A_k = B_k = \emptyset$; if $\lambda_k = 1 - \lambda_{k-1}$, set $A_k = \{n_k\}$, $B_k = \{n_{k+1}\}$, and $A_j = B_j = \emptyset$ for every $j > k$. Let $A = \bigcup_{k=1}^{\infty} A_k$ and $B = \bigcup_{k=1}^{\infty} B_k$. Then, as in the preceding proof, we have $s(A) \bowtie s(B)$. So we can choose N such that if $\nu \in A$ for some $\nu \geq N$, then $B \subset \{1, \ldots, N-1\}$. If $\lambda_N = 1$, then there is nothing to prove. In the case $\lambda_N = 0$, suppose that $\lambda_m = 1 - \lambda_{m-1}$ for some $m > N$. Then $A = \{n_m\}$ and $B = \{n_{m+1}\}$, where $n_m \geq m > N$ and $n_{m+1} \geq m + 1 > N$. This contradicts our choice of N. Hence $\lambda_m = 0$ for all $m > N$ and therefore for all m. Q.E.D.

The following result is proved in [8].

Lemma 5 *Assume LPO. Let $(a_n), (b_n)$ be sequences in a metric space X, and r a positive number such that $\rho(a_n, b_n) \geq r$ for each n. Then there exists a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $\rho(a_{n_j}, b_{n_k}) \geq r/5$ for all j and k.*

We now give the **Proof of Theorem 1**. Let $s = (x_n)_{n=1}^{\infty}$ be a Cauchy sequence in a metric apartness space X. Given $\varepsilon > 0$, we see from Lemma 3 that either $\rho(x_m, x_n) < \varepsilon$ for all sufficiently large m and n, or else, as we may assume, there exists a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $\rho\left(x_{n_k}, x_{n_{k+1}}\right) > \varepsilon/4$ for each k. It remains to rule out this latter alternative. Taking

$$a_k = x_{n_k},\ b_k = x_{n_{k+1}},\ \text{and } r = \frac{\varepsilon}{4}$$

in Lemma 5, we construct a strictly increasing sequence $(k_j)_{j=1}^{\infty}$ of positive integers such that

$$\rho\left(a_{k_i}, b_{k_j}\right) \geq \frac{\varepsilon}{20}$$

for all i, j. Then

$$A = \left\{n_{k_j} : j \geq 1\right\} \text{ and } B = \left\{n_{k_j+1} : j \geq 1\right\}$$

are infinite sets of positive integers such that $s(A) \bowtie s(B)$. But this is impossible, since, as we observed toward the beginning of this section, a Cauchy sequence is a CHN–Cauchy sequence. Q.E.D.

3 Applications of the Cauchy property In the remainder of the paper, X will be a set that carries a binary relation $\bowtie$ of **set–set apartness** between subsets of X. We will specify the properties required of $\bowtie$ shortly; but first we define the corresponding **point–set apartness** on X by

$$x \bowtie S \Leftrightarrow \{x\} \bowtie S,$$

and the corresponding point–point **inequality** on X by

$$x \neq y \Leftrightarrow \{x\} \bowtie \{y\}.$$

Note that, as a consequence of axiom **B5** below, the inequality is cotransitive:

$$x \neq y \Rightarrow \forall_{z \in X} (x \neq z \vee z \neq y).$$

A subset S of X has three important types of complement:

- the **logical complement** $\neg S = \{x \in X : \forall y \in S \neg (x = y)\}$;
- the **complement** $\sim S = \{x \in X : \forall y \in S \ (x \neq y)\}$;
- and the **apartness complement** $-S = \{x \in X : x \bowtie S\}$.

Naturally we write $A \sim S$ and $A - S$ rather than $A \cap \sim S$ and $A \cap -S$, respectively.

We assume that the relations $\neq$ and $\bowtie$ satisfy the following axiom and therefore provide X with the structure of an **apartness space:**

B1 $X \bowtie \emptyset$

B2 $S \bowtie T \Longrightarrow S \cap T = \emptyset$

B3 $R \bowtie (S \cup T) \Longleftrightarrow R \bowtie S \wedge R \bowtie T$

B4 $x \bowtie S \wedge -S \subset \sim T \Longrightarrow x \bowtie T$

B5 $x \bowtie S \Longrightarrow \forall_{y \in X} (x \neq y \vee y \bowtie S)$

B6 $S \bowtie T \Longrightarrow T \bowtie S$

B7 $S \bowtie T \Longrightarrow \forall_{x \in X} \exists_{R \subset X} (x \in -R \wedge (S - R \neq \emptyset \Longrightarrow \neg R \bowtie T))$

This system differs somewhat from the one presented in [12], in that it eliminates one of the axioms in favour of our definition of point–set apartness, it has a weaker form of axiom **B4**, and it has a stronger conclusion in axiom **B7**. It is easy to show that our current axioms are satisfied by the canonical apartness defined on a metric space (X, ρ) by

$$S \bowtie T \Leftrightarrow \exists_{r>0} \forall_{s \in S} \forall_{t \in T} (\rho(s,t) \geq r) .$$

They also hold for the canonical apartness in a uniform space. For all but one of the axioms this was proved in [12]; we relegate the verification of the outstanding case, our strong form of **B7**, to an appendix.

Apartness complements form the basis for a topology, the **apartness topology**, on X [6]. This topology is Hausdorff. For if $x \neq y$, then, applying **B7** with $S = \{x\}$ and $T = \{y\}$, we obtain U such that $x \in -U$ and $y \bowtie \neg U$. Applying **B7** once more, with $S = \{y\}$ and $T = \neg U$, we now obtain V such that $y \in -V$ and $\neg V \bowtie \neg U$; whence $-U \bowtie -V$ and therefore $-U \cap -V = \emptyset$.

In view of the definition of the apartness topology, it makes sense to say that a net $(x_\lambda)_{\lambda \in L}$ in X **converges** to the limit l in X if

$$\forall_{U \subset X} \ (l \in -U \Rightarrow \exists_{\lambda_0 \in L} \forall_{\lambda \succcurlyeq \lambda_0} \ (x_\lambda \in -U)) .$$

Proposition 6 *The net $s = (x_\lambda)_{\lambda \in L}$ converges to l in X if and only if*

$$\forall_{B \subset L} \ (l \bowtie s(B) \Rightarrow \exists_{\lambda_0 \in L} \ (B \subset \sim\{\lambda : \lambda \succcurlyeq \lambda_0\})) . \tag{1}$$

Proof. Suppose that s converges to l in X. Let $B \subset L$ and $l \bowtie s(B)$. By **B7**, there exists $U \subset X$ such that $l \in -U$ and $\neg U \bowtie s(B)$; whence

$$s(B) \subset -\neg U \subset \sim\neg U.$$

Choosing λ_0 such that $x_\lambda \in -U$ for all $\lambda \succcurlyeq \lambda_0$, we now see from the strong extensionality of the net s that $B \subset \sim\{\lambda : \lambda \succcurlyeq \lambda_0\}$.

Now suppose, conversely, that (1) holds. Let $l \in -U$, and apply **B7** to obtain $V \subset X$ such that $l \in -V$ and $\neg V \bowtie U$. Defining

$$B = \{\lambda \in L : x_\lambda \in V\},$$

we see that $l \bowtie s(B)$, so there exists $\lambda_0 \in L$ such that

$$B \subset \sim\{\lambda : \lambda \succcurlyeq \lambda_0\}.$$

If $\lambda \succcurlyeq \lambda_0$, then $x_\lambda \in \neg V \subset -U$. Q.E.D.

Proposition 7 *Every convergent net is a Cauchy net.*

PROOF. Let $s = (x_\lambda)_{\lambda \in L}$ be a net in X converging to an element l, and let A, B be subsets of L such that $s(A) \bowtie s(B)$. Then, by axiom **B7**, there exists $U \subset X$ such that $l \in -U$ and such that if $s(A) - U \neq \emptyset$, then $\neg U \bowtie s(B)$. Choose λ_0 in L such that $x_\lambda \in -U$ for all $\lambda \succcurlyeq \lambda_0$. Suppose that for some $\lambda \succcurlyeq \lambda_0$ we have $\lambda \in A$. Then $x_\lambda \in s(A) - U$, so $\neg U \bowtie s(B)$. It follows from the strong extensionality of the net that $B \subset \sim\{\lambda : \lambda \succcurlyeq \lambda_0\}$. Q.E.D.

By a **subnet** of a net $s = (x_\lambda)_{\lambda \in L}$ we mean a net $t = (x_\mu)_{\mu \in M}$ where M is a directed subset of L such that for each $\lambda \in L$ there exists $\mu \in M$ with $\mu \succcurlyeq \lambda$.

Proposition 8 *Let $s = (x_\lambda)_{\lambda \in L}$ be a Cauchy net in X that contains a subnet converging to a limit l in X. Then s converges to l.*

PROOF. Let $(x_\mu)_{\mu \in M}$ be a subnet of s converging to l in X, and let $l \in -U$. Using **B7**, find $V \subset X$ such that $l \in -V$ and $\neg V \bowtie U$; again using **B7**, find $W \subset X$ such that $l \in -W$ and $\neg W \bowtie V$. Let

$$A = \{\lambda \in L : x_\lambda \in -W\},$$
$$B = \{\lambda \in L : x_\lambda \in V\}.$$

Since $s(A) \bowtie s(B)$, there exists $\lambda_0 \in L$ such that if $\lambda \succcurlyeq \lambda_0$ for some $\lambda \in A$, then

$$B \subset \sim\{\lambda : \lambda \succcurlyeq \lambda_0\}.$$

But there exists $\mu_0 \in M$ such that $x_\mu \in -W$ for all $\mu \in M$ with $\mu \succcurlyeq \mu_0$. Choose $\mu_1 \in M$ such that $\mu_1 \succcurlyeq \lambda_0$. Since M is directed, there exists $\mu \in M$ such that $\mu \succcurlyeq \mu_0$ and $\mu \succcurlyeq \mu_1$; whence $x_\mu \in -W$ and $\mu \succcurlyeq \lambda_0$. Thus $B \subset \sim\{\lambda : \lambda \succcurlyeq \lambda_0\}$. It follows that if $\lambda \succcurlyeq \lambda_0$, then $x_\lambda \in \neg V$ and so $x_\lambda \in -U$. Thus s converges to l. Q.E.D.

We define a notion of nearness of points and sets as follows:

$$\mathbf{near}(x, A) \quad \text{if and only if} \quad \forall_{U \subset X}\,(x \in -U \Rightarrow A - U \neq \emptyset).$$

The closure of A in X in the apartness topology on X is the set

$$\overline{A} = \{x \in X : \mathbf{near}(x, A)\}.$$

Proposition 9 *The closure of a subset A of X consists of all points of X that are limits of nets in A.*

PROOF. If $(x_\lambda)_{\lambda \in L}$ is a net in A converging to an element x of X, then

$$\mathbf{near}\,(x, \{x_\lambda : \lambda \in L\})$$

and therefore $\mathbf{near}\,(x, A)$; whence $x \in \overline{A}$.

Conversely, if $x \in \overline{A}$, then $A - U$ is nonempty for each $U \subset X$ with $x \in -U$. Let

$$\begin{aligned} \mathcal{U} &= \{U \subset X : x \in -U\} \\ L &= \{(y, U) : y \in A - U,\, U \in \mathcal{U}\}. \end{aligned}$$

We show that L is directed by the partial order relation $\succcurlyeq$ defined by

$$(y, U) \succcurlyeq (y', U') \quad \text{if and only if} \quad -U \subset -U'.$$

Given $(y, U), (y', U')$ in L, we have $x \in -U \cap -U'$. It follows from axiom **B3** that $U \cup U' \in \mathcal{U}$. Moreover, for each $z \in A - (U \cup U')$ we have $(z, U \cup U') \succcurlyeq (y, U)$ and $(z, U \cup U') \succcurlyeq (y', U')$. This completes the verification that L is directed. Now let $(y_\lambda)_{\lambda \in L}$ be the net in A defined by the mapping $(y, U) \rightsquigarrow y$, and let $U \subset X$ be such that $x \in -U$. Since $\mathbf{near}\,(x, A)$ and $U \in \mathcal{U}$, there exists $y \in A - U$; let $\lambda_0 = (y, U)$. For each $\lambda = (z, V) \succcurlyeq \lambda_0$ we have $x \in -V \subset -U$ and $z \in A - V$; whence $y_\lambda \in -V \subset -U$. Thus $(y_\lambda)_{\lambda \in L}$ converges to x. Q.E.D.

The somewhat elaborate construction of the directed index set L in the above proof is occasioned by our need to avoid using the full axiom of choice, which implies the law of excluded middle [10].

In view of Proposition 9, the following results can now be proved using familiar classical arguments.

Proposition 10 *Every complete subspace of an apartness space is nearly closed.*

Proposition 11 *Every nearly closed subspace of a complete apartness space is complete.*

4 Weak locatedness and Bishop's Lemma

We conclude our paper with an important application of completeness in the context of weak locatedness. A subset S of our apartness space X is said to be **weakly located** if

$$\forall_{x \in X}\, \forall_{U \subset X}\, \left(x \in -U \Rightarrow (x \in -S \vee S - U \neq \emptyset)\right).$$

If (X, ρ) is a metric space and the distance

$$\rho(x, S) = \inf\{\rho(x, s) : s \in S\}$$

exists for each x in S then we say that S is **located**. In that case, S is weakly located. However, Exercise 7.3.2 on page 381 of [15] shows that the proposition

Every weakly located subset of $\mathbf{R}$ *is located*

entails LPO and so is essentially nonconstructive. Since locatedness is a fundamental property in constructive analysis, it is a significant problem—currently unsolved—to find an appropriate generalisation of it in the context of a general apartness space.

One of the most useful results about locatedness in metric spaces is **Bishop's Lemma** ([2], page 177, Lemma 7; [3], page 92, Lemma (3.8)):

If S is a complete located subset of a metric space (X, ρ), then for each $x \in X$ there exists $y \in S$ such that if $\rho(x, y) > 0$, then $\rho(x, S) > 0$.

We conclude our paper by showing that Bishop's Lemma can be lifted to the context of an apartness space, with locatedness replaced by weak locatedness. First, we observe that if the apartness topology on an apartness space X is first countable—that is, if every element of X has a countable base of neighourhoods in the apartness topology—then for each $x \in X$ there exists a sequence $(U_n)_{n=1}^{\infty}$ of subsets of X with the following properties:

- ▷ $x \in -U_n$ for each n,
- ▷ for each $U \subset X$ with $x \in -U$, there exists n such that $x \in -U_n \subset -U$;
- ▷ $-U_1 \supset -U_2 \supset -U_3 \supset \cdots$.

Proposition 12 *Let X be an apartness space, and S a nonempty subset of S. If*

$$\forall_{x \in X} \exists_{y \in S} (x \neq y \Rightarrow x \bowtie S), \tag{2}$$

then S is weakly located. Conversely, if X is first countable and S is complete and weakly located, then (2) holds.

PROOF. Supposing that (2) holds, and given $x \in X$, choose $y \in S$ as in (2). If $x \in -U$, then by axiom **B5**, either $x \neq y$ and therefore $x \bowtie S$, or else $y \bowtie U$ and therefore $S - U \neq \emptyset$. Thus S is weakly located.

Now suppose, conversely, that X is first countable and that S is complete and weakly located. Fix an element b of S. Given $x \in X$, choose a sequence $(U_n)_{n=1}^{\infty}$ of subsets of X with the properties immediately preceding this proposition, and construct an increasing binary sequence $(\lambda_n)_{n=1}^{\infty}$ such that

$$\begin{aligned} \lambda_n = 0 &\quad \Rightarrow \quad S - U_n \neq \emptyset, \\ \lambda_n = 1 &\quad \Rightarrow \quad x \bowtie S. \end{aligned}$$

If $\lambda_1 = 1$, then

$$x \neq b \Rightarrow x \bowtie S.$$

So without loss of generality we may assume that $\lambda_1 = 0$. For each n, if $\lambda_n = 0$, choose $y_n \in S - U_n$; and if $\lambda_n = 1$, set $y_n = y_{n-1}$. To show that the sequence $s = (y_n)_{n=1}^{\infty}$ obtained in this way is a Cauchy sequence, let A and B be subsets of $\mathbf{N}^+$ such that $s(A) \bowtie s(B)$. By axiom **B7**, there exists V such that

$$x \in -V \wedge (s(A) - V \neq \emptyset \Rightarrow \neg V \bowtie s(B)).$$

Pick N such that $-U_N \subset -V$, and suppose that $\nu \geq N$ for some $\nu \in A$. We claim that $B \subset \{1, \ldots, N-1\}$. If $\lambda_N = 0$, then $y_\nu \in s(A)$ and $y_\nu \in -U_\nu \subset -U_N \subset -V$, so $\neg V \bowtie s(B)$. It follows that if $n \in B$, then $y_n \notin -V$: for if $y_n \in -V$, then

$$y_n \in -V \cap -\neg V = -(V \cup \neg V) \subset \neg(V \cup \neg V) = \emptyset,$$

which is absurd. Hence $y_n \notin -U_N$ and therefore $n < N$. In the case where $\lambda_N = 1$, if $n \in B$ and $n \geq N$, then $y_\nu = y_N = y_n$, which is absurd since $s(A) \bowtie s(B)$; so again $n < N$.

Since S is complete, (y_n) converges to a limit $y \in S$. Suppose that $x \neq y$. Then, since the apartness topology on X is Hausdorff, there exists m such that $y \bowtie -U_m$. If $\lambda_m = 1$, then we are finished; so it remains to rule out the possibility that $\lambda_m = 0$. If $\lambda_n = 1 - \lambda_{n-1}$ for some $n > m$, then $y = y_{n-1} \in -U_{n-1} \subset -U_m$, which contradicts our choice of m. Thus $\lambda_n = 0$ for all $n > m$ and therefore for all n. It follows that $y_k \in -U_n$ for all $k \geq n$, so the sequence (y_n) converges to x. Since the apartness topology on X is Hausdorff, we conclude that $y = x$, a final contradiction. Q.E.D.

Appendix: Uniform apartness spaces In this appendix, $(X,\mathcal{U})$ will be a uniform space with a nontrivial inequality $\neq$. We require that in addition to the usual classical properties, the uniform structure $\mathcal{U}$ on X satisfy the following two axioms (which are numbered as in [12] and are classically superfluous).

U3 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$ and

$$\forall_{x,y\in X}\left((x,y) \in {\sim}V \vee (x,y) \in U\right),$$

where

$$V^2 = \left\{(x,z) \in X \times X : \exists_{y\in X}\ \left((x,y) \in V \wedge (y,z) \in V\right)\right\}.$$

U4 For all $x, y \in X$,

$$x \neq y \Longrightarrow \exists_{U\in\mathcal{U}}\ \left((x,y) \in {\sim}U\right).$$

We define the apartness relation $\bowtie$ between subsets S, T of X by setting

$$S \bowtie T \Longleftrightarrow \exists_{U\in\mathcal{U}}\ (S \times T \subset {\sim}U).$$

It is shown in [12] that $\bowtie$ satisfies the axioms of a set–set apartness as defined in that paper. We want show that it satisfies the stronger version of the axiom **B7** that we gave at the beginning of the present paper.

Let $S \bowtie T$ in the uniform space X. Choosing $U \in \mathcal{U}$ such that $S \times T \subset {\sim}U$, let $V \in \mathcal{U}$ be as in axiom **U3.** Applying **U3** twice more, choose first a symmetric element W of $\mathcal{U}$ such that $W^2 \subset V$ and

$$\forall_{x,y\in X}\ \left((x,y) \in {\sim}W \vee (x,y) \in V\right),$$

and then an element E of $\mathcal{U}$ such that $E^2 \subset W$ and

$$\forall_{x,y\in X}\ \left((x,y) \in {\sim}E \vee (x,y) \in W\right).$$

A simple modification of the proof of Lemma 17 of [12] shows that

$$x \in -{\sim}E[x] \subset \neg{\sim}E[x] \subset W[x],$$

where, for example,

$$E\left[x\right] = \left\{y \in X : (x,y) \in E\right\}.$$

Writing $R = {\sim}E[x]$, suppose that $S - R \neq \emptyset$. It is enough to prove that $W[x] \times T \subset {\sim}E$: for then $W\left[x\right] \bowtie T$ and therefore $\neg R \bowtie T$. Accordingly, let $(z,t) \in W[x] \times T$; then $(x,z) \in W$. Either $(z,t) \in {\sim}E$ or $(z,t) \in W$. In the first case we are done. In the second we get $(x,t) \in W^2 \subset V$. Choosing y in $S - R$, we see that

$$\{y\} \times T \subset S \times T \subset {\sim}U \tag{3}$$

and $y \in W\left[x\right]$. From the latter it follows by the symmetry of W that $(y,x) \in W \subset V$. Hence $(y,t) \in V^2 \subset U$, which contradicts (3). Hence the case $(z,t) \in W$ is ruled out. This completes the verification of our axiom **B7** for uniform spaces.

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