PENTAGONAL EQUATIONS FOR OPERATORS ASSOCIATED WITH INCLUSIONS OF C^*-ALGEBRAS

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Abstract. We introduce a notion of a pentagonal equation for an adjointable operator on a Hilbert C^*-module in full generality. We call a unitary operator on a Hilbert C^*-module a multiplicative unitary operator (MUO) when it satisfies the pentagonal equation. We give a sufficient condition for the existence of an MUO associated with a general inclusion of C^*-algebras. Then we study an MUO when the inclusion is of index-finite type in the sense of Watatani. We also give an explicit formula for the MUO when the inclusion arises from a crossed product of a C^*-algebra by a finite group.

1. Introduction

A pentagonal equation (PE) first appeared in the duality theory for locally compact groups. The Kac-Takesaki operator in the theory satisfies a PE (cf. [9], [30]). S. Baaj and G. Skandalis [2] called a unitary operator on a Hilbert space a multiplicative unitary (MU) when it satisfies a PE. V. F. R. Jones initiated a study for inclusions of von Neumann algebras (see [11], [13], [14]). MU's appeared in the related theory; M. Enock and R. Nest [8] constructed an MU from an irreducible regular depth 2 inclusion of factors. MU's also appeared in several situations. For example, we refer the reader to [5], [6], [12] [17] and [26]. As for measured groupoids, T. Yamanouchi [36] constructed an analogue of the Kac-Takesaki operator. But this operator does not satisfy a PE. J. M. Vallin [33] showed that it satisfies an equation which is a generalization of a PE. He called a unitary operator a pseudo-multiplicative unitary (PMU) when it satisfies this generalized PE. Vallin defined the generalized PE using the Connes-Sauvageot's relative tensor products of Hilbert spaces. M. Enock and J. M. Vallin [10] constructed a PMU from a regular depth 2 inclusion of von Neumann algebras. The base algebra of the PMU they studied is a (not necessarily commutative) von Neumann algebra. Recently several authors study quantum groupoids. For example, we refer the reader to [3], [7], [18], [20], [28] and [34]. Quantum groupoids are related to inclusions of von Neumann algebras and PMU's. In particular, PMU's in finite-dimension were studied by G. Böhm and K. Szlachányi [3] and by J. M. Vallin [34]. They studied the PMU from the viewpoint of multiplicative isometries. Yamanouchi [37] also studied a partial isometry which satisfies a PE. When we deal with PMU's in the theory of C^*-algebras, it is useful to formulate a generalization of a PE in the framework of Hilbert C^*-modules. As for the usefulness of Hilbert C^*-modules, for example, we refer the reader to the works of M. A. Rieffel [25], E. C. Lance [16], B. Blackadar [4] and Y. Watatani [35]. The author [22] defined a PMU on a Hilbert C^*-module using interior tensor products. The base algebra of the PMU defined there is a commutative C^*-algebra. (When a PMU is

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defined on a tensor product of A-modules, we will call A a base algebra. See Definition 3.1.) An analogue of the Kac-Takesaki operator for a topological groupoid G becomes a PMU in the sense of [22]. Moreover, if G is a measured groupoid, that is, if it has a quasi-invariant measure, then the operator constructed in [22] induces the fundamental isometric isomorphism W_G studied by Vallin [33]. As for inclusions of C^*-algebras, Y. Watatani [36] initiated a theory of indices for C^*-subalgebras. It is interesting to study PMU's arising from inclusions of C^*-algebras in the framework of Watatani's index theory. Following the idea of Enock and Vallin [10], the author [23] constructed a PMU in the sense of [22] from an inclusion of finite-dimensional C^*-algebras in the framework of Watatani's index theory when the inclusion satisfies certain conditions. There we had to assume a condition that implies a commutativity of the base algebra.

In this paper, we will study a PE in full generality. Therefore, in the following, we will not distinguish a PE from a generalization of a PE and we will not distinguish an MU from a PMU. We will give a definition of a PE in full generality in the framework of Hilbert C^*-module. Especially, we will remove the assumption of the commutativity of the base algebra, which was assumed in [22] and [23]. We will call a unitary operator on a Hilbert C^*-module a multiplicative unitary operator (MUO) when it satisfies this newly defined PE. We will study an MUO for a general inclusion of C^*-algebras. Then we will construct an MUO from an inclusion of C^*-algebras in the framework of Watatani's index theory when the inclusion satisfies certain conditions. This construction generalizes that of [23]. We will remove the assumption in [23] which implies the commutativity of the base algebra. We meet several difficulties in defining a PE in the framework of Hilbert C^*-modules. For example, we do not have in general the following objects; a flip on an interior tensor product of Hilbert C^*-modules, a tensor product $I \otimes x$ as operator on an interior tensor product of Hilbert C^*-modules for an adjointable operator x and a modular involution on a Hilbert C^*-module. Therefore our definition of a PE is different from the usual definition of a PE though they are equivalent in special cases. When the base algebra is $\mathbb{C}$, the MUO defined in this paper coincides with the MU defined by Baaj and Skandalis [2] modulo the flip. When the base algebra is commutative, the MUO coincides with the PMU studied in [22] and [23] modulo the flip. Note that we cannot define a flip when the base algebra is not commutative.

The paper is organized as follows: Section 2 is a preliminary section. In Section 3, we introduce a notion of a pentagonal equation for an adjointable operator on a Hilbert C^*-module in full generality (Definition 3.1). We explain the relation between the MUO's defined here and the usual MU's and PMU's. In Section 4, we introduce a notion of a coproduct for a Hilbert C^*-module. We construct such a coproduct from an MUO and a fixed vector with a certain property. Then we define a C^*-algebra associated with the coproduct. We study examples arising from a finite groupoid, an r-discrete groupoid and a compact groupoid. Other examples are studied in Section 7. In Section 5, we study a general inclusion of C^*-algebras $A_0 \subset A_1$. We do not need to assume that A_0 and A_1 are unital. We suppose that there exists a Hilbert A_0-module E_1 and a $*$-homomorphism ϕ_1 of A_1 to $\mathcal{L}_{A_0}(E_1)$. Then we give a sufficient condition for the existence of an MUO associated with the inclusion (Theorem 5.3). In Section 6, we study an MUO when the inclusion is of index-finite type in the sense of Watatani [35]. We show that there exists an MUO when the inclusion satisfies two conditions (P1) and (P2) (Corollary 6.2). As an application, we show that there exists an MUO when A_0 is finite-dimensional and the inclusion satisfies (P1) (Corollary 6.3). The condition (P1) corresponds to the condition that $A_0 \subset A_1$ is of depth 2. In Section 7, we study the inclusion $A_0 \subset A_0 \rtimes_\alpha G$, where A_0 is a unital C^*-algebra and $A_0 \rtimes_\alpha G$ is the crossed product of A_0 by a finite group G. We give an explicit formula for the MUO associated with the inclusion (Theorem 7.3).

2. Preliminaries

First, we recall some definitions and notations on Hilbert C^*-modules. For details, we refer the reader to [16]. Let A be a C^*-algebra. A Hilbert A-module is a right A-module E with an A-valued inner product $< \cdot , \cdot >$ such that E is complete with respect to the norm $\|\xi\| = \| < \xi, \xi > \|^{1/2}$. Note that the inner product is linear in its second variable. A Hilbert A-module E is said to be full if the closure of the linear span of $\{< \xi, \eta >; \xi, \eta \in E\}$ is all of A. Let E and F be Hilbert A-modules. We denote by $\mathcal{L}_A(E, F)$ the set of bounded adjointable operators from E to F and we denote by $\mathcal{K}_A(E, F)$ the closure of the linear span of $\{\theta_{\xi,\eta}; \xi \in F, \eta \in E\}$, where $\theta_{\xi,\eta}$ is the element of $\mathcal{L}_A(E, F)$ defined by $\theta_{\xi,\eta}(\zeta) = \xi < \eta, \zeta >$ for $\zeta \in E$. We abbreviate $\mathcal{L}_A(E, E)$ and $\mathcal{K}_A(E, E)$ to $\mathcal{L}_A(E)$ and $\mathcal{K}_A(E)$ respectively. We denote by I_E the identity operator on E. We often omit the subscript E for simplicity. A unitary operator U of E to F is an adjointable operator such that $U^*U = I_E$ and $UU^* = I_F$.

Let A and B be C^*-algebras. Suppose that E is a Hilbert A-module and that F is a Hilbert B-module. Let ϕ be a $*$-homomorphism of A to $\mathcal{L}_B(F)$. Then we can define the interior tensor product $E \otimes_\phi F$ [16, Chapter 4]. For $\xi \in E$ and $\eta \in F$, we denote by $\xi \otimes_\phi \eta$ the corresponding element of $E \otimes_\phi F$. We often omit the subscript ϕ, writing $\xi \otimes \eta = \xi \otimes_\phi \eta$ for simplicity. We have $\xi a \otimes \eta = \xi \otimes \phi(a)\eta$ for every $a \in A$. Note that $E \otimes_\phi F$ is a Hilbert B-module with a B-valued inner product such that

$$< \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 > = < \eta_1, \phi(< \xi_1, \xi_2 >)\eta_2 >$$

for ξ_1, $\xi_2 \in E$ and η_1, $\eta_2 \in F$. For $x \in \mathcal{L}_A(E)$, define an element $x \otimes_\phi I_F$ of $\mathcal{L}_B(E \otimes_\phi F)$ by $(x \otimes_\phi I_F)(\xi \otimes \eta) = (x\xi) \otimes \eta$ [16, Chapter 4]. Let E_i be a Hilbert A_i-module for $i = 1, 2, 3$ and let ϕ_i be a $*$-homomorphism of A_{i-1} to $\mathcal{L}_{A_i}(E_i)$ for $i = 2, 3$. Define a $*$-homomorphism $\phi_2 \otimes_{\phi_3} \iota$ of A_1 to $\mathcal{L}_{A_3}(E_2 \otimes_{\phi_3} E_3)$ by $(\phi_2 \otimes_{\phi_3} \iota)(a) = \phi_2(a) \otimes_{\phi_3} I_{E_3}$ for $a \in A_1$. We often omit the subscript ϕ_3, writing $\phi_2 \otimes \iota = \phi_2 \otimes_{\phi_3} \iota$ for simplicity. Then we have

$$(E_1 \otimes_{\phi_2} E_2) \otimes_{\phi_3} E_3 = E_1 \otimes_{\phi_2 \otimes \iota} (E_2 \otimes_{\phi_3} E_3).$$

We denote the above tensor product by $E_1 \otimes_{\phi_2} E_2 \otimes_{\phi_3} E_3$.

For $i = 1, 2$, let E_i be a Hilbert A-module, let F_i be a Hilbert B-module and let ϕ_i be a $*$-homomorphism of A to $\mathcal{L}_B(F_i)$. We denote by $\mathcal{L}_B((F_1, \phi_1), (F_2, \phi_2))$ the set of elements x of $\mathcal{L}_B(F_1, F_2)$ such that $x\phi_1(a) = \phi_2(a)x$ for all $a \in A$. We abbreviate $\mathcal{L}_B((F_1, \phi_1), (F_1, \phi_1))$ to $\mathcal{L}_B(F_1, \phi_1)$. We define $\mathcal{K}_B((F_1, \phi_1), (F_2, \phi_2))$ and $\mathcal{K}_B(F_1, \phi_1)$ similarly. The following proposition is useful in later arguments. As for the notation in the following proposition, we often omit the subscript ϕ_1, writing $x \otimes y = x \otimes_{\phi_1} y$ for simplicity.

Proposition 2.1 ([22])**.** *For $x \in \mathcal{L}_A(E_1, E_2)$ and $y \in \mathcal{L}_B((F_1, \phi_1), (F_2, \phi_2))$, there exists an element $x \otimes_{\phi_1} y$ of $\mathcal{L}_B(E_1 \otimes_{\phi_1} F_1, E_2 \otimes_{\phi_2} F_2)$ such that $(x \otimes_{\phi_1} y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ for $\xi \in E_1$ and $\eta \in F_1$.*

3. Pentagonal equations for operators on Hilbert C^*-modules

Let A be a C^*-algebra, let E be a Hilbert A-module and let ϕ and ψ be $*$-homomorphisms of A to $\mathcal{L}_A(E)$. We assume that ϕ and ψ commute, that is, $\phi(a)\psi(b) = \psi(b)\phi(a)$ for all a, $b \in A$. We define a $*$-homomorphism $\iota \otimes_\phi \psi$ of A to $\mathcal{L}_A(E \otimes_\phi E)$ by $(\iota \otimes_\phi \psi)(a) = I \otimes_\phi \psi(a)$ and define a $*$-homomorphism $\iota \otimes_\psi \phi$ of A to $\mathcal{L}_A(E \otimes_\psi E)$ by $(\iota \otimes_\psi \phi)(a) = I \otimes_\psi \phi(a)$. We often omit the subscripts ϕ and ψ, writing $\iota \otimes \psi = \iota \otimes_\phi \psi$ and $\iota \otimes \phi = \iota \otimes_\psi \phi$ for simplicity. Let W be an operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. We assume that W satisfies the following equations;

$$W(\iota \otimes_\psi \phi)(a) = (\phi \otimes_\phi \iota)(a)W, \tag{3.1}$$

$$W(\psi \otimes_\psi \iota)(a) = (\iota \otimes_\phi \psi)(a)W, \tag{3.2}$$

$$W(\phi \otimes_\psi \iota)(a) = (\psi \otimes_\phi \iota)(a)W \tag{3.3}$$

for all $a \in A$. Then, by Proposition 2.1, we can define following operators;

$$\begin{aligned} W \otimes_\psi I &\in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\phi E \otimes_\psi E), \\ I \otimes_{\phi\otimes\iota} W &\in \mathcal{L}_A(E \otimes_\phi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E), \\ W \otimes_\phi I &\in \mathcal{L}_A(E \otimes_\psi E \otimes_\phi E, E \otimes_\phi E \otimes_\phi E), \\ I \otimes_{\psi\otimes\iota} W &\in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)), \\ I \otimes_{\iota\otimes\phi} W &\in \mathcal{L}_A(E \otimes_{\iota\otimes\phi} (E \otimes_\psi E), E \otimes_\phi E \otimes_\phi E). \end{aligned}$$

Since ϕ and ψ commute, there exists an isomorphism Σ_{12} of $E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\psi E)$ as Hilbert A-modules such that

$$\Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3)$$

for $x_i \in E$ $(i = 1, 2, 3)$. Then we can define a pentagonal equation.

Definition 3.1. Let W be an element of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. Assume that W satisfies the equations (3.1), (3.2) and (3.3). An operator W is said to be multiplicative if it satisfies the pentagonal equation

$$(W \otimes_\phi I)(I \otimes_{\phi\otimes\iota} W)(W \otimes_\psi I) = (I \otimes_{\iota\otimes\phi} W)\Sigma_{12}(I \otimes_{\psi\otimes\iota} W). \tag{3.4}$$

The algebra A is called the base algebra of the multiplicative operator W.

Example 3.2. Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $\mathcal{L}_{\mathbb{C}}(E) = \mathcal{L}(H)$ is the C^*-albebra of bounded linear operators on H. Let $\phi = \psi = id$, where $id(\lambda) = \lambda I_H$ for $\lambda \in \mathbb{C}$. Then $E \otimes_{id} E$ is the usual tensor product $H \otimes H$. Let $\Sigma \in \mathcal{L}(H \otimes H)$ be the flip, that is, $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. Let W be an element of $\mathcal{L}(H \otimes H)$. Then the pentagonal equation (3.4) has the following form:

$$(W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W). \tag{3.5}$$

Define an operator $\widetilde{W}$ by $\widetilde{W} = W\Sigma$. Then W satisfies the pentagonal equation (3.5) if and only if $\widetilde{W}$ satisfies the usual pentagonal equation ;

$$\widetilde{W}_{12}\widetilde{W}_{13}\widetilde{W}_{23} = \widetilde{W}_{23}\widetilde{W}_{13}. \tag{3.6}$$

Example 3.3. In Example 3.2, if $W = \Sigma$, then the equation (3.5) is the Yang-Baxter equation for the flip (cf. [15]);

$$(\Sigma \otimes I)(I \otimes \Sigma)(\Sigma \otimes I) = (I \otimes \Sigma)(\Sigma \otimes I)(I \otimes \Sigma).$$

Example 3.4. Let G be a locally compact Hausdorff group and ν be a right Haar measure on G. Set $H = L^2(G, \nu)$. Define an operator W on $H \otimes H$ by $(W\xi)(g, h) = \xi(h, gh)$ for $\xi \in C_c(G \times G)$ and g, $h \in G$. Then W satisfies the pentagonal equation (3.5). The operator $\widetilde{W}$ in Example 3.2 is given by $(\widetilde{W}\xi)(g, h) = \xi(gh, h)$, which is the Kac-Takesaki operator and satisfies the usual pentagonal equation (3.6).

Suppose that $A = C$ is a commutative C^*-algebra. Let E be a Hilbert C-module and ϕ be a $*$-homomorphism of C to $\mathcal{L}_C(E)$. Define a $*$-homomorphism ψ of C to $\mathcal{L}_C(E)$ by $\psi(c)\xi = \xi c$ for $\xi \in E$ and $c \in C$. In this situation, we have defined in [22] a generalized pentagonal equation and we have called a unitary operator pseudo-multiplicative if it satisfies the generalized pentagonal equation. We will describe the relation between the pentagonal equation (3.4) defined in this paper and the generalized pentagonal equation defined in [22]. We wrote $E \otimes_C E$ for $E \otimes_\psi E$ in [22]. Let $\widetilde{W}$ be a unitary operator in $\mathcal{L}_C(E \otimes_\psi E, E \otimes_\phi E)$. Suppose that $\widetilde{W}$ satisfies the following equation;

$$\widetilde{W}(\iota \otimes_\psi \phi)(c) = (\iota \otimes_\phi \phi)(c)\widetilde{W}, \tag{3.7}$$

$$\widetilde{W}(\phi \otimes_\psi \iota)(c) = (\phi \otimes_\phi \iota)(c)\widetilde{W} \tag{3.8}$$

for $c \in C$. There exists an isomorphism σ_1 of $E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\psi E)$ such that $\sigma_1(\xi \otimes (\eta \otimes \zeta)) = \eta \otimes (\xi \otimes \zeta))$ and there exists an isomorphism σ_2 of $E \otimes_\psi E \otimes_\phi E$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\phi E)$ such that $\sigma_2(\xi \otimes \eta \otimes \zeta) = \eta \otimes (\xi \otimes \zeta)$. We define an operator $\widetilde{W}_{13}$ in $\mathcal{L}_C(E \otimes_{\iota\otimes\psi} (E \otimes_\phi E), E \otimes_\psi E \otimes_\phi E)$ by $\widetilde{W}_{13} = \sigma_2^*(I \otimes_{\iota\otimes\phi} \widetilde{W})\sigma_1$. In [22], the generalized pentagonal equation was defined as follows;

$$(\widetilde{W} \otimes_\phi I)\widetilde{W}_{13}(I \otimes_{\iota\otimes\psi} \widetilde{W}) = (I \otimes_{\phi\otimes\iota} \widetilde{W})(\widetilde{W} \otimes_\psi I). \tag{3.9}$$

There exists a flip Σ_ψ in $\mathcal{L}_C(E \otimes_\psi E)$ such that $\Sigma_\psi(\xi \otimes \eta) = \eta \otimes \xi$. Then we have the following;

Proposition 3.5. *Let W be an element of $\mathcal{L}_C(E \otimes_\psi E, E \otimes_\phi E)$. Set $\widetilde{W} = W\Sigma_\psi$. Then W satisfies the equation* (3.4) *if and only if $\widetilde{W}$ satisfies the equation* (3.9).

Proof. Note that we have $\psi \otimes_\phi \iota = \iota \otimes_\phi \phi$ and $\psi \otimes_\psi \iota = \iota \otimes_\psi \psi$. Then the equations (3.1) and (3.8) are equivalent and the equations (3.3) and (3.7) are equivalent. The equation (3.2) is always satisfied in this situation. Then we have

$$\begin{aligned}
&(\widetilde{W} \otimes_\phi I)\widetilde{W}_{13}(I \otimes_{\iota\otimes\psi} \widetilde{W}) \\
&= (W \otimes_\phi I)(I \otimes_{\phi\otimes\iota} W)(W \otimes_\psi I)(I \otimes_{\iota\otimes\psi} \Sigma_\psi)(\Sigma_\psi \otimes_\psi I)(I \otimes_{\iota\otimes\psi} \Sigma_\psi), \\
&(I \otimes_{\phi\otimes\iota} \widetilde{W})(\widetilde{W} \otimes_\psi I) \\
&= (I \otimes_{\iota\otimes\phi} W)\Sigma_{12}(I \otimes_{\psi\otimes\iota} W)(\Sigma_\psi \otimes_\psi I)(I \otimes_{\psi\otimes\iota} \Sigma_\psi)(\Sigma_\psi \otimes_\psi I).
\end{aligned}$$

Then the assertion follows from the following equation

$$(I \otimes_{\iota\otimes\psi} \Sigma_\psi)(\Sigma_\psi \otimes_\psi I)(I \otimes_{\iota\otimes\psi} \Sigma_\psi) = (\Sigma_\psi \otimes_\psi I)(I \otimes_{\psi\otimes\iota} \Sigma_\psi)(\Sigma_\psi \otimes_\psi I).$$

□

Example 3.6. Let G be a second countable locally compact Hausdorff groupoid. We denote by s (resp. r) the source (resp. range) map of G. We denote by $G^{(0)}$ the unit space of G and by $G^{(2)}$ the set of composable pairs. We set $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Let $\{\lambda_u; u \in G^{(0)}\}$ be a right Haar system of G. As for groupoids and groupoid C^*-algebras, see Renault [24]. (See also [19] and [22] for notations and definitions used here.) For an arbitrary topological space X, we denote by $C_c(X)$ the set of complex-valued continuous functions on X with compact supports and by $C_0(X)$ the commutative C^*-algebra of continuous functions on X vanishing at infinity with the supremum norm $\|\cdot\|_\infty$. Let C be the commutative C^*-algebra $C_0(G^{(0)})$ and let $\widetilde{E}$ be the linear space $C_c(G)$. Then $\widetilde{E}$ is a right C-module with the right C-action defined by $(\xi c)(x) = \xi(x)c(s(x))$ for $\xi \in \widetilde{E}$, $c \in C$ and $x \in G$. We define a C-valued inner product of $\widetilde{E}$ by

$$< \xi, \eta > (u) = \int_G \overline{\xi(x)}\eta(x)\, d\lambda_u(x)$$

for ξ, $\eta \in \widetilde{E}$ and $u \in G^{(0)}$. We denote by E the completion of $\widetilde{E}$ by the norm $\|\xi\| = \| < \xi, \xi > \|^{1/2}$. Then E is a full right Hilbert C-module. Define non-degenerate injective $*$-homomorphisms ϕ and ψ of C to $\mathcal{L}_C(E)$ by $(\phi(c)\xi)(x) = c(r(x))\xi(x)$ and $\psi(c)\xi = \xi c$

respectively for $c \in C$, $\xi \in \widetilde{E}$ and $x \in G$. Set $G^2(ss) = \{(x,y) \in G^2; s(x) = s(y)\}$. We define C-valued inner products of $C_c(G^2(ss))$ and $C_c(G^{(2)})$ by

$$< f_1, g_1 > (u) = \iint_{G^2(ss)} \overline{f_1(x,y)} g_1(x,y)\, d\lambda_u(x) d\lambda_u(y),$$
$$< f_2, g_2 > (u) = \iint_{G^{(2)}} \overline{f_2(x,y)} g_2(x,y)\, d\lambda_{r(y)}(x) d\lambda_u(y)$$

respectively for $u \in G^{(0)}$, f_1, $g_1 \in C_c(G^2(ss))$ and f_2, $g_2 \in C_c(G^{(2)})$. Then $C_c(G^2(ss))$ and $C_c(G^{(2)})$ are dense pre-Hilbert C-submodules of $E \otimes_\psi E$ and $E \otimes_\phi E$ respectively. Define a unitary operator W in $\mathcal{L}_C(E \otimes_\psi E, E \otimes_\phi E)$ by $(W\xi)(x,y) = \xi(y, xy)$ for $\xi \in C_c(G^2(ss))$, $(x,y) \in G^{(2)}$. Set $\widetilde{W} = W\Sigma_\psi$. We have $(\widetilde{W}\xi)(x,y) = \xi(xy, y)$ for $\xi \in C_c(G^2(ss))$, $(x,y) \in G^{(2)}$. It follows from [22] that $\widetilde{W}$ satisfies the equation (3.9). By Proposition 3.5, W satisfies the pentagonal equation (3.4).

When G is a measured groupoid, that is, when there exists a quasi-invariant measure on $G^{(0)}$, we discussed in [22] the relation between the operator $\widetilde{W}$ constructed above and the fundamental operator studied by Yamanouchi in [37, §2] and by Vallin in [33, §3]. For the convenience to the reader, we will briefly describe the relation between them. As for the following arguments, compare with Yamanouchi [36, §§1,2]. Note that he started from a left Haar system of G and that the inner products in [36] is linear in the first variable. On the other hand, we have started from a right Haar system and the inner products here are linear in the second variable. Let μ be a quasi-invariant measure on $G^{(0)}$ [24, Definition 1.3.2]. We suppose that the support of μ is $G^{(0)}$. We define a measure ν on G by $\nu = \int \lambda_u \, d\mu(u)$ and define a complex-valued inner product $< \cdot, \cdot >_\mu$ on E by

$$< \xi, \eta >_\mu = \int_{G^{(0)}} < \xi, \eta > (u)\, d\mu(u)$$

for ξ, $\eta \in E$. We denote by $\mu(E)$ the completion of E by the norm induced from $< \cdot, \cdot >_\mu$. Then we have $\mu(E) = L^2(G, \nu)$. Let λ^u $(u \in G^{(0)})$ and ν^{-1} be the images of λ_u and ν, respectively, by the inverse map $x \mapsto x^{-1}$. Set $\Delta = d\nu/d\nu^{-1}$ and $G^2(rr) = \{(x,y) \in G^2; r(x) = r(y)\}$. We define measures ν_1, ν_2, ν_3 and ν_4 on $G^2(ss)$, $G^{(2)}$, $G^2(rr)$ and $G^{(2)}$ respectively by

$$\int_{G^2(ss)} f_1(x,y)\, d\nu_1(x,y) = \iiint f_1(x,y)\, d\lambda_u(x) d\lambda_u(y) d\mu(u),$$
$$\int_{G^{(2)}} f_2(x,y)\, d\nu_2(x,y) = \iiint f_2(x,y)\, d\lambda_{r(y)}(x) d\lambda_u(y) d\mu(u)$$
$$\int_{G^2(rr)} f_3(x,y)\, d\nu_3(x,y) = \iiint f_3(x,y)\, d\lambda^u(x) d\lambda^u(y) d\mu(u)$$
$$\int_{G^{(2)}} f_4(x,y)\, d\nu_4(x,y) = \iiint f_4(x,y) \Delta(x)^{-1}\, d\lambda_u(x) d\lambda^u(y) d\mu(u)$$

for $f_1 \in C_c(G^2(ss))$, $f_2 \in C_c(G^{(2)})$, $f_3 \in C_c(G^2(rr))$ and $f_4 \in C_c(G^{(2)})$. We set $\mathcal{Z} = L^\infty(G^{(0)}, \mu)$. By extending the right action and the left action ϕ of C on E, we have actions of $\mathcal{Z}$ on $\mu(E)$. That is, we have $(f_1 \xi f_2)(x) = f_1(r(x))\xi(x) f_2(s(x))$ for f_1, $f_2 \in \mathcal{Z}$, $\xi \in \mu(E)$ and $x \in G$. Then $\mu(E) = L^2(G, \nu)$ is a $\mathcal{Z}$-bimodule. We denote by $L^2(G, \nu)_\mathcal{Z} \otimes L^2(G, \nu)_\mathcal{Z}$ the relative tensor product of the right $\mathcal{Z}$-module $L^2(G, \mu)$ with itself. (cf. [27]). We denote other relative tensor products of right or left $\mathcal{Z}$-modules $L^2(G,\nu)$ similarly. Then we have

$$\begin{aligned} L^2(G,\nu)_{\mathcal{Z}} \otimes L^2(G,\nu)_{\mathcal{Z}} &= L^2(G^2(ss),\nu_1) = \mu(E \otimes_C E), \\ L^2(G,\nu)_{\mathcal{Z}} \otimes_{\mathcal{Z}} L^2(G,\nu) &= L^2(G^{(2)},\nu_2) = \mu(E \otimes_\phi E), \\ {}_{\mathcal{Z}}L^2(G,\nu^{-1}) \otimes_{\mathcal{Z}} L^2(G,\nu^{-1}) &= L^2(G^2(rr),\nu_3), \\ L^2(G,\nu^{-1})_{\mathcal{Z}} \otimes_{\mathcal{Z}} L^2(G,\nu^{-1}) &= L^2(G^{(2)},\nu_4). \end{aligned}$$

Therefore we can extend the above operator $\widetilde{W}$ to the unitary operator $\overline{W}$ of $L^2(G,\nu)_{\mathcal{Z}} \otimes L^2(G,\nu)_{\mathcal{Z}}$ onto $L^2(G,\nu)_{\mathcal{Z}} \otimes_{\mathcal{Z}} L^2(G,\nu)$. Define a unitary operator

$$j_1 : L^2(G,\nu)_{\mathcal{Z}} \otimes L^2(G,\nu)_{\mathcal{Z}} \longrightarrow {}_{\mathcal{Z}}L^2(G,\nu^{-1}) \otimes_{\mathcal{Z}} L^2(G,\nu^{-1})$$

by $(j_1\xi)(x,y) = \xi(y^{-1},x^{-1})$ for $(x,y) \in G^2(rr)$, and define a unitary operator

$$j_2 : L^2(G,\nu)_{\mathcal{Z}} \otimes_{\mathcal{Z}} L^2(G,\nu) \longrightarrow L^2(G,\nu^{-1})_{\mathcal{Z}} \otimes_{\mathcal{Z}} L^2(G,\nu^{-1})$$

by $(j_2\xi)(x,y) = \xi(y^{-1},x^{-1})$ for $(x,y) \in G^{(2)}$. Then we have $(j_2\overline{W}j_1^*\xi)(x,y) = \xi(x,xy)$ for $(x,y) \in G^{(2)}$. Therefore $j_2\overline{W}j_1^*$ coincides with the fundamental unitary operator W in [36, §2] and coincides with the fundamental isometric isomorphism W_G in [33, §3].

4. Coproducts for Hilbert C^*-modules

It is known that MU's and PMU's give coproducts in several situations (cf. [2], [10], [21], [22], [32], [33], [36]). In this section, we construct a coproduct of a Hilbert C^*-module from an MUO and a fixed vector with a certain property. Then we define a C^*-algebra associated with the coproduct. There are several notions of Hopf C^*-algebras. For example, we refer the reader to [1], [2], [3] and [31]. The author [22] also suggested a notion of a Hopf C^*-algebra on a Hilbert C^*-module. But there are several difficulties in defining a notion of a Hopf C^*-algebra on a Hilbert C^*-module. To find a more natural notion of a Hopf C^*-algebra on a Hilbert C^*-module, we have to study more closely C^*-algebras associated with MUO's. The results in this section will be useful for the study in that direction.

First we introduce a notion of coproducts for Hilbert C^*-modules. We denote by E a Hilbert A-module and by ϕ a $*$-homomorphism of A to $\mathcal{L}_A(E)$.

Definition 4.1. Let δ be an operator in $\mathcal{L}_A(E, E \otimes_\phi E)$. We say that δ is a coproduct of (E,ϕ) if δ satisfies the following equations;

$$\delta\phi(a) = (\phi \otimes \iota)(a)\delta \qquad \text{for all } a \in A, \tag{4.10}$$

$$(\delta \otimes I_E)\delta = (I_E \otimes \delta)\delta. \tag{4.11}$$

Suppose that δ is a coproduct for E. For ξ, $\eta \in E$, we define a product $\xi\eta$ in E by $\xi\eta = \delta^*(\xi \otimes_\phi \eta)$. It follows from (4.11) that this product is associative. Then E is a right A-algebra with this product. Note that we have $\|\xi\eta\| \le \|\delta\|\|\xi\|\|\eta\|$. Therefore, if $\|\delta\| \le 1$, then E is a Banach algebra.

Let ψ be a $*$-homomorphism of A to $\mathcal{L}_A(E)$ such that ϕ and ψ commute and let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be a multiplicative unitary operator. For an element ξ_0 of E, we say that ξ_0 has the property (E1) if it satisfies the following conditions;

(i) $\|\xi_0\| = 1$.
(ii) $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$.
(iii) For every $\xi \in E$, there exists an element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \pi_{\xi_0}(\xi)\zeta > = < W(\xi_0 \otimes_\psi \eta), \xi \otimes_\phi \zeta >$$

for every η, $\zeta \in E$.

Fix an element ξ_0 with the property (E1). Define an operator $\delta = \delta_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\phi E)$ by $\delta(\eta) = W(\xi_0 \otimes_\psi \eta)$. Then we have $\|\delta\| \leq 1$ and $\delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta$. Since W satisfies the pentagonal equation, we can show that δ is a coproduct of (E, ϕ). We denote by $\xi \bullet \eta$ the product of ξ and η associated with δ. Then we have $\pi_{\xi_0}(\xi)\eta = \xi \bullet \eta$. Moreover the map π_{ξ_0} of E to $\mathcal{L}_A(E)$ is a representation of the Banach algebra $(E, \bullet)$. We denote by $B(\xi_0)$ the closure of the set consisting of elements of the form $\pi_{\xi_0}(\xi)$ with $\xi \in E$. Then $B(\xi_0)$ is a Banach subalgebra of $\mathcal{L}_A(E)$. We denote by $C^*(B(\xi_0))$ the C^*-subalgebra of $\mathcal{L}_A(E)$ generated by $B(\xi_0)$.

For an element ξ_0 of E, we say that ξ_0 has the property (E2) if it satisfies the following conditions;

(i) $\|\xi_0\| = 1$.
(ii) $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$.
(iii) For every $\xi \in E$, there exists an element $\widehat{\pi}_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \widehat{\pi}_{\xi_0}(\xi)\zeta > = < W^*(\xi_0 \otimes_\phi \eta), \xi \otimes_\psi \zeta >$$

for every η, $\zeta \in E$.

Fix an element ξ_0 with the property (E2). Define an operator $\widehat{\delta} = \widehat{\delta}_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\psi E)$ by $\widehat{\delta}(\eta) = W^*(\xi_0 \otimes_\phi \eta)$. Since W satisfies the pentagonal equation, we can show that $\widehat{\delta}$ is a coproduct of (E, ψ). We denote by $\xi \diamond \eta$ the product of ξ and η associated with $\widehat{\delta}$. Then we have $\widehat{\pi}_{\xi_0}(\xi)\eta = \xi \diamond \eta$. Moreover the map $\widehat{\pi}_{\xi_0}$ of E to $\mathcal{L}_A(E)$ is a representation of the Banach algebra $(E, \diamond)$. We denote by $\widehat{B}(\xi_0)$ the closure of the set consisting of elements of the form $\widehat{\pi}_{\xi_0}(\xi)$ with $\xi \in E$. Then $\widehat{B}(\xi_0)$ is a Banach subalgebra of $\mathcal{L}_A(E)$. We denote by $C^*(\widehat{B}(\xi_0))$ the C^*-subalgebra of $\mathcal{L}_A(E)$ generated by $\widehat{B}(\xi_0)$.

In this section, we consider examples arising from a finite groupoid, an r-discrete groupoid and a compact groupoid. Other examples are considered in Section 7. Let G be a second countable locally compact Hausdorff groupoid. We keep the notations in Example 3.6 except for $C_0(G^{(0)})$. Here we denote by A the C^*-algebra $C_0(G^{(0)})$. Let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be the multiplicative unitary operator constructed in Example 3.6. Then we have $(W\xi)(x, y) = \xi(y, xy)$ for $\xi \in C_c(G^2(ss))$ and $(x, y) \in G^{(2)}$. Note that we have $(W^*\xi)(x, y) = \xi(yx^{-1}, x)$ for $\xi \in C_c(G^{(2)})$ and $(x, y) \in G^2(ss)$. We denote by $C_r^*(G)$ the reduced groupoid C^*-algebras. (As for the definition of the reduced groupoid C^*-algebra, see [19], [22].)

Example 4.2. Let G be a finite groupoid and let $\{\lambda_u\}$ be a right Haar system such that λ_u is a counting measure on G_u. Then we have $A = C(G^{(0)})$ and $E = C(G)$. The A-valued inner product of E is given by $< \xi, \eta > (u) = \sum_{x \in G_u} \overline{\xi(x)}\eta(x)$. We have $E \otimes_\psi E = C(G^2(ss))$ and the A-valued inner product of $E \otimes_\psi E$ is given by

$$< \xi, \eta > (u) = \sum_{s(x)=s(y)=u} \overline{\xi(x, y)}\eta(x, y).$$

We have $E \otimes_\phi E = C(G^{(2)})$ and the A-valued inner product of $E \otimes_\phi E$ is given by

$$< \xi, \eta > (u) = \sum_{\substack{s(x)=r(y) \\ s(y)=u}} \overline{\xi(x, y)}\eta(x, y).$$

We set $M = \max\{|G_u|; u \in G^{(0)}\}$, where $|G_u|$ is the number of elements of G_u. Define an element ξ_0 of E by $\xi_0(x) = M^{-1/2}$ for all $x \in G$. Then ξ_0 has the properties (E1) and (E2).

We have $\pi_{\xi_0}(\xi)\zeta = M^{-1/2}\xi * \zeta$, where $\xi * \zeta$ is the convolution product defined by

$$(\xi * \zeta)(x) = \sum_{y \in G_{s(x)}} \xi(xy^{-1})\zeta(y).$$

Therefore we have $B(\xi_0) = C_r^*(G)$. Since we have $\widehat{\pi}_{\xi_0}(\xi) = \theta_{\xi,\xi_0}$, we have $C^*(\widehat{B}(\xi_0)) = \mathcal{K}_A(E)$. Define an element η_0 of E by $\eta_0 = \chi_{G^{(0)}}$, where $\chi_{G^{(0)}}$ is the characteristic function of $G^{(0)}$. Then η_0 has the properties (E1) and (E2). Since we have $\pi_{\eta_0}(\xi) = \theta_{\xi,\eta_0}$, we have $C^*(B(\eta_0)) = \mathcal{K}_A(E)$. We have $\widehat{\pi}_{\eta_0}(\xi) = m(\xi)$, where $m(\xi)$ is the multiplication operator on E defined by $(m(\xi)\zeta)(x) = \xi(x)\zeta(x)$. Therefore we have $\widehat{B}(\eta_0) = C(G)$.

Example 4.3. Let G be an r-discrete groupoid [24, I.2.6]. Note that $G^{(0)}$ is open and closed in G and that G_u is discrete for every $u \in G^{(0)}$. Let $\{\lambda_u\}$ be a right Haar system such that λ_u is the counting measure on G_u. Since we have $\|\xi\|_\infty \le \|\xi\|_E$ for $\xi \in C_c(G)$, E is a subspace of $C_0(G)$. Fix an element f of A such that $\|f\|_\infty = 1$. Define an element η_0 of E by $\eta_0 = f\chi_{G^{(0)}}$. Then η_0 has the properties (E1) and (E2). We have $\pi_{\eta_0}(\xi) = \theta_{\xi,\eta_0}$. If the support of f is $G^{(0)}$, then we have $C^*(B(\eta_0)) = \mathcal{K}_A(E)$. We have $\widehat{\pi}_{\eta_0}(\xi) = m(\phi(\overline{f})\xi)$, where $m(\eta)$ is the multiplication operator on E. If f is real-valued, then we have $\widehat{\pi}_{\eta_0}(\xi)^* = \widehat{\pi}_{\eta_0}(\overline{\xi})$. Therefore, if f is real-valued and the support of f is $G^{(0)}$, then we have $\widehat{B}(\eta_0) = C_0(G)$.

Example 4.4. Let G be a compact groupoid and let $\{\lambda_u\}$ be a right Haar system such that $\lambda_u(G) = 1$ for all $u \in G^{(0)}$. Define an element ξ_0 of E by $\xi_0(x) = 1$ for all $x \in G$. Then ξ_0 has the properties (E1) and (E2). Note that $C(G)$ is a dense subspace of E. For $\xi, \zeta \in C(G)$, we have $\pi_{\xi_0}(\xi)\zeta = \xi * \zeta$, where $\xi * \zeta$ is the convolution product defined by

$$(\xi * \zeta)(x) = \int \xi(xy^{-1})\zeta(y)\, d\lambda_{s(x)}(y).$$

Therefore we have $B(\xi_0) = C_r^*(G)$. Since we have $\widehat{\pi}_{\xi_0}(\xi) = \theta_{\xi,\xi_0}$, we have $C^*(\widehat{B}(\xi_0)) = \mathcal{K}_A(E)$.

5. Operators associated with inclusions of C^*-algebras

In this section, we study a multiplicative unitary operator associated with a general inclusion of C^*-algebras. Let A_1 be a C^*-algebra and let A_0 be a C^*-subalgebra of A_1. In this section, we do not need to assume that A_1 and A_0 are unital. Let E_1 be a Hilbert A_0-module and let ϕ_1 be a $*$-homomorphism of A_1 to $\mathcal{L}_{A_0}(E_1)$. We denote by ϕ_0 the restriction of ϕ_1 to A_0. Set $E_2 = E_1 \otimes_{\phi_0} E_1$ and define a $*$-homomorphism ϕ_2 of A_1 to $\mathcal{L}_{A_0}(E_2)$ by $\phi_2 = \phi_1 \otimes \iota$. In general, we set $E_n = E_{n-1} \otimes_{\phi_0} E_1$. We denote by A the C^*-algebra $\mathcal{L}_{A_0}(E_1, \phi_1)$ and by E the normed space $\mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2))$. Then E is a right A-module with the right A-action defined by $(xa)(\xi) = x(a\xi)$ for $x \in E$, $a \in A$ and $\xi \in E_1$. Define an A-valued inner product of E by $< x, y >= x^*y$ for $x, y \in E$. Then E is a Hilbert A-module. Define $*$-homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $(\phi(a)x)(\xi) = (a \otimes I)x(\xi)$ and $(\psi(a)x)(\xi) = (I \otimes a)x(\xi)$ respectively for $a \in A$, $x \in E$ and $\xi \in E_1$. We denote by i the inclusion map of A into $\mathcal{L}_{A_0}(E_1)$.

Proposition 5.1. *There exists an A_0-linear bounded map U of $E \otimes_i E_1$ to E_2 such that $U(x \otimes \xi) = x(\xi)$ for $x \in E$ and $\xi \in E_1$. Moreover the following equalities hold:*

$$\begin{aligned} < U\alpha, U\beta > &=< \alpha, \beta > && \textit{for } \alpha, \beta \in E \otimes_i E_1, \\ U(\phi(a) \otimes I) &= (a \otimes I)U && \textit{for } a \in A, \\ U(\psi(a) \otimes I) &= (I \otimes a)U && \textit{for } a \in A, \\ U(I \otimes \phi_1(a)) &= \phi_2(a)U && \textit{for } a \in A_1. \end{aligned}$$

The proof is straightforward and we omit it. Note that U may not be adjointable. We can define the following A_0-linear bounded operators;

$$\begin{aligned} I \otimes_{\phi\otimes\iota} U &: E \otimes_\phi E \otimes_i E_1 \longrightarrow E \otimes_{i\otimes\iota} E_2, \\ U \otimes_{\phi_0} I &: E \otimes_{i\otimes\iota} E_2 \longrightarrow E_3, \\ I \otimes_{\psi\otimes\iota} U &: E \otimes_\psi E \otimes_i E_1 \longrightarrow E \otimes_{\iota\otimes i} E_2, \\ I \otimes_{\iota\otimes\phi_0} U &: E_1 \otimes_{\iota\otimes\phi_0} (E \otimes_i E_1) \longrightarrow E_3. \end{aligned}$$

There exists an isomorphism S of $E \otimes_{\iota\otimes i} E_2$ onto $E_1 \otimes_{\iota\otimes\phi_0} (E \otimes_i E_1)$ as Hilbert A_0-modules such that $S(x \otimes (\xi \otimes \eta)) = \xi \otimes (x \otimes \eta)$ for $x \in E$ and $\xi,\, \eta \in E_1$. Define an A_0-linear bounded operator V of $E \otimes_\phi E \otimes_i E_1$ to E_3 by

$$V = (U \otimes_{\phi_0} I)(I \otimes_{\phi\otimes\iota} U),$$

and define an A_0-linear bounded operator $\widetilde{V}$ of $E \otimes_\psi E \otimes_i E_1$ to E_3 by

$$\widetilde{V} = (I \otimes_{\iota\otimes\phi_0} U) S (I \otimes_{\psi\otimes\iota} U).$$

We summarize the properties of V and $\widetilde{V}$ in the following proposition. The proof is easy and we omit it.

Proposition 5.2. *The operators V and $\widetilde{V}$ satisfies the following equalities;*

$$\begin{aligned} &< V\alpha, V\beta > = < \alpha, \beta > && \textit{for } \alpha,\, \beta \in E \otimes_\phi E \otimes_i E_1, \\ &< \widetilde{V}\alpha, \widetilde{V}\beta > = < \alpha, \beta > && \textit{for } \alpha,\, \beta \in E \otimes_\psi E \otimes_i E_1, \\ &V(x \otimes y \otimes \xi) = (x \otimes_{\phi_0} I_{E_1}) y(\xi) && \textit{for } x,\, y \in E \textit{ and } \xi \in E_1, \\ &\widetilde{V}(x \otimes y \otimes \xi) = (I_{E_1} \otimes_{\phi_0} x) y(\xi) && \textit{for } x,\, y \in E \textit{ and } \xi \in E_1. \end{aligned}$$

In the rest of this section, we will prove the following theorem.

Theorem 5.3. *Let U, V and $\widetilde{V}$ be as above. Suppose that U is unitary and suppose that there exists an element W of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^* \widetilde{V} = W \otimes I_{E_1}$. Then W is a multiplicative unitary operator.*

Since U is unitary by the assumption, V and $\widetilde{V}$ are also unitary operators. By straightforward calculations, we have, for every $a \in A$,

$$\begin{aligned} V(\phi(a) \otimes I_E \otimes I_{E_1}) &= (a \otimes I_{E_1} \otimes I_{E_1}) V, \\ V(I_E \otimes \psi(a) \otimes I_{E_1}) &= (I_{E_1} \otimes I_{E_1} \otimes a) V, \\ V(\psi(a) \otimes I_E \otimes I_{E_1}) &= (I_{E_1} \otimes a \otimes I_{E_1}) V, \\ \widetilde{V}(I_E \otimes \phi(a) \otimes I_{E_1}) &= (a \otimes I_{E_1} \otimes I_{E_1}) \widetilde{V}, \\ \widetilde{V}(\psi(a) \otimes I_E \otimes I_{E_1}) &= (I_{E_1} \otimes I_{E_1} \otimes a) \widetilde{V}, \\ \widetilde{V}(\phi(a) \otimes I_E \otimes I_{E_1}) &= (I_{E_1} \otimes a \otimes I_{E_1}) \widetilde{V}. \end{aligned}$$

Therefore W satisfies the equations (3.1), (3.2) and (3.3). For $n \geq 2$, we set

$$E^{\otimes_\phi n} = E \otimes_\phi \cdots \otimes_\phi E \quad (n \text{ times })$$

and we define $E^{\otimes_\psi n}$ similarly. It follows from Proposition 5.1 that we have $U(\phi \otimes \iota)(a) = (i \otimes \iota)(a) U$ for $a \in A$. Therefore we can define the following operators;

$$\begin{aligned} I_E \otimes I_E \otimes U &\in \mathcal{L}_{A_0}(E^{\otimes_\phi 3} \otimes_i E_1, E^{\otimes_\phi 2} \otimes_{i\otimes\iota} E_2), \\ I_E \otimes U \otimes I_{E_1} &\in \mathcal{L}_{A_0}(E^{\otimes_\phi 2} \otimes_{i\otimes\iota} E_2, E \otimes_{i\otimes\iota\otimes\iota} E_3), \\ U \otimes I_{E_1} \otimes I_{E_1} &\in \mathcal{L}_{A_0}(E \otimes_{i\otimes\iota\otimes\iota} E_3, E_4). \end{aligned}$$

We define an element U_3 in $\mathcal{L}_{A_0}(E^{\otimes_\phi 3} \otimes_i E_1, E_4)$ by

$$U_3 = (U \otimes I_{E_1} \otimes I_{E_1})(I_E \otimes U \otimes I_{E_1})(I_E \otimes I_E \otimes U).$$

Since U is unitary by the assumption, U_3 is also a unitary operator. To prove Theorem 5.3, it is enough to prove the following proposition.

Proposition 5.4. *Set*

$$W_1 = (W \otimes_\phi I)(I \otimes_{\phi\otimes\iota} W)(W \otimes_\psi I),$$
$$W_2 = (I \otimes_{\iota\otimes\phi} W)\Sigma_{12}(I \otimes_{\psi\otimes\iota} W).$$

Then the following equation holds;

$$\begin{aligned} &U_3(W_1 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\ &= U_3(W_2 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\ &= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi. \end{aligned}$$

for x, y, $z \in E$ and $\xi \in E_1$.

In the rest of this section, we will prove Proposition 5.4. Let

$$S_\psi : E \otimes_\psi E \otimes_{\iota\otimes i} E_2 \longrightarrow E_1 \otimes_{\iota\otimes\iota\otimes\phi_0} (E \otimes_\psi E \otimes_i E_1)$$

be an isomorphism defined by $S_\psi(x \otimes \xi \otimes \eta) = \xi \otimes (x \otimes \eta)$ for $x \in E \otimes_\psi E$ and ξ, $\eta \in E_1$, and let

$$S_\phi : E \otimes_\phi E \otimes_{\iota\otimes i} E_2 \longrightarrow E_1 \otimes_{\iota\otimes\iota\otimes\phi_0} (E \otimes_\phi E \otimes_i E_1)$$

be an isomorphism defined by $S_\phi(x \otimes \xi \otimes \eta) = \xi \otimes (x \otimes \eta)$ for $x \in E \otimes_\phi E$ and ξ, $\eta \in E_1$. Set $U^{(13)} = (I \otimes_{\iota\otimes\phi_0} U)S$.

Lemma 5.5. *We have the following equalities for x, y, $z \in E$ and $\xi \in E_1$;*

$$(5.12) \quad U_3((W \otimes_\phi I) \otimes_i I_{E_1}) = (\widetilde{V} \otimes_{\phi_0} I_{E_1})(I_{E\otimes_\psi E} \otimes_{\phi\otimes\iota} U),$$

$$\begin{aligned}(5.13) \quad &((I \otimes_{\phi\otimes\iota} W) \otimes_i I_{E_1})((W \otimes_\psi I) \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\ &= (I_E \otimes_{\iota\otimes i\otimes\iota} V^*)(I_E \otimes_{\phi\otimes\iota} U^{(13)})S_\phi^*(I_{E_1} \otimes_{\phi_0\otimes\iota\otimes\iota} V^*)(I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi,\end{aligned}$$

$$\begin{aligned}(5.14) \quad &(I_E \otimes_{\iota\otimes i\otimes\iota} V^*)(I_E \otimes_{\phi\otimes\iota} U^{(13)})S_\phi^*(I_{E_1} \otimes_{\phi_0\otimes\iota\otimes\iota} V^*) \\ &= (I_{E\otimes_\psi E} \otimes_{i\otimes\iota} U^*)(\widetilde{V}^* \otimes_{\phi_0} I_{E_1}).\end{aligned}$$

Proof. Since we have $U_3 = (V \otimes_{\phi_0} I_{E_1})(I_E \otimes I_E \otimes U)$, we have the equation (5.12). The equation (5.13) follows from the following equations;

$$\begin{aligned} &((I \otimes_{\phi\otimes\iota} W) \otimes_i I_{E_1}) \\ &= (I_E \otimes_{\iota\otimes i\otimes\iota} V^*)(I_E \otimes_{\phi\otimes\iota} U^{(13)})(I_E \otimes_{\phi\otimes\iota} (I \otimes_{\psi\otimes\iota} U)), \\ &(I_E \otimes_{\phi\otimes\iota} (I \otimes_{\psi\otimes\iota} U))((W \otimes_\psi I) \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\ &= (W \otimes_{\iota\otimes i} I_{E_2})(x \otimes y \otimes (z\xi)), \\ &W \otimes_{\iota\otimes i} I_{E_2} = S_\phi^*(I_{E_1} \otimes_{\iota\otimes\iota\otimes\phi_0} V^*\widetilde{V})S_\psi, \\ &(I_{E_1} \otimes_{\iota\otimes\iota\otimes\phi_0} \widetilde{V})S_\psi(x \otimes y \otimes (z\xi)) = (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi. \end{aligned}$$

The equation (5.14) follows from the following equations;

$$I_E \otimes_{\iota\otimes i\otimes\iota} V^* = (I_{E\otimes_\psi E} \otimes_{i\otimes\iota} U^*)(I_E \otimes_{\iota\otimes i\otimes\iota} (U^* \otimes_{\phi_0} I_{E_1})),$$
$$I_E \otimes_{\phi\otimes\iota} U^{(13)} = (I_E \otimes_{\iota\otimes\phi\otimes\iota} (I_{E_1} \otimes_{\iota\otimes\phi_0} U))(I_E \otimes_{\phi\otimes\iota} S),$$
$$I_{E_1} \otimes_{\phi_0\otimes\iota\otimes\iota} V^*$$
$$= (I_{E_1} \otimes_{\iota\otimes\phi_0\otimes\iota} (I_E \otimes_{i\otimes\iota} U^*))(I_{E_1} \otimes_{\phi_0\otimes\iota\otimes\iota} (U^* \otimes_{\phi_0} I_{E_1})),$$
$$(I_E \otimes_{\iota\otimes\phi\otimes\iota} (I_{E_1} \otimes_{\iota\otimes\phi_0} U))(I_E \otimes_{\phi\otimes\iota} S)S_\phi^*(I_{E_1} \otimes_{\iota\otimes\phi_0\otimes\iota} (I_E \otimes_{i\otimes\iota} U^*))$$
$$= S^* \otimes_{\phi_0} I_{E_1}$$
$$(I_{E_1} \otimes_{\iota\otimes\phi_0\otimes\iota} (U \otimes_{\phi_0} I_{E_1}))(S \otimes_{\phi_0} I_{E_1})(I_E \otimes_{\psi\otimes\iota\otimes\iota} (U \otimes_{\phi_0} I_{E_1}))$$
$$= \widetilde{V} \otimes_{\phi_0} I_{E_1}.$$

Note that we have

$$(I_E \otimes_{\psi\otimes\iota\otimes\iota} (U \otimes_{\phi_0} I_{E_1}))^* = I_E \otimes_{\iota\otimes i\otimes\iota} (U^* \otimes_{\phi_0} I_{E_1}),$$
$$(I_{E_1} \otimes_{\iota\otimes\phi_0\otimes\iota} (U \otimes_{\phi_0} I_{E_1}))^* = I_{E_1} \otimes_{\phi_0\otimes\iota\otimes\iota} (U^* \otimes_{\phi_0} I_{E_1}).$$

□

Lemma 5.6. *We have the following equalities for $x \in E$ and $\xi_i \in E_1$ $(i = 1, 2, 3)$;*

(5.15) $$W_2 \otimes_i I_{E_1} = (I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\psi\otimes\iota\otimes\iota} V^*\widetilde{V}),$$

(5.16) $$U_3(I_E \otimes_{i\otimes\iota\otimes\iota} V^*) = U \otimes I_{E_1} \otimes I_{E_1},$$

(5.17) $$(I_E \otimes_{\iota\otimes\phi\otimes\iota} \widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} V^*)(x \otimes (\xi_1 \otimes \xi_2 \otimes \xi_3))$$
$$= U^*(\xi_1 \otimes \xi_2) \otimes x\xi_3.$$

Proof. The equation (5.15) follows from the assumption $V^*\widetilde{V} = W \otimes I_{E_1}$. The equation (5.16) follows from the equation

$$I_E \otimes_{i\otimes\iota\otimes\iota} V^* = (I_E \otimes I_E \otimes U^*)(I_E \otimes U^* \otimes I_{E_1}).$$

We will show the equation (5.17). Define a unitary operator

$$S' : E \otimes_{\iota\otimes\iota\otimes i} (E \otimes_{i\otimes\iota} E_2) \longrightarrow E \otimes_{\iota\otimes i\otimes\iota} (E \otimes_{\iota\otimes i} E_2)$$

by

$$S' = (I_E \otimes_{\iota\otimes\phi\otimes\iota} (I \otimes_{\psi\otimes\iota} U))(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} (I_E \otimes_{i\otimes\iota} U^*)).$$

Then we have $S'(x \otimes (y \otimes \xi_1 \otimes \xi_2)) = y \otimes (x \otimes (\xi_1 \otimes \xi_2))$ for $x, y \in E$ and $\xi_1, \xi_2 \in E_1$. Therefore we have

$$(I_E \otimes_{\iota\otimes\phi\otimes\iota} \widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} (I_E \otimes_{i\otimes\iota} U^*))(x \otimes (y \otimes \xi_1 \otimes \xi_2))$$
$$= (I_E \otimes_{\iota\otimes i\otimes\iota} U^{(13)})S'(x \otimes (y \otimes \xi_1 \otimes \xi_2))$$
$$= (y \otimes \xi_1) \otimes x\xi_2.$$

Then the equation (5.17) follows from the following equation

$$I_E \otimes_{\iota\otimes\iota\otimes i} V^* = (I_E \otimes_{\iota\otimes\iota\otimes i} (I_E \otimes_{i\otimes\iota} U^*))(I_E \otimes_{\iota\otimes\iota\otimes i} (U^* \otimes_{\phi_0} I_{E_1})).$$

□

Proof of Proposition 5.4. Let x, y, z be elements of E and let ξ be an element of E_1. It follows from Lemma 5.5 that we have

$$\begin{aligned}
&U_3(W_1 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\
&= U_3((W \otimes_\phi I) \otimes_i I_{E_1})((I \otimes_{\phi\otimes\iota} W) \otimes_i I_{E_1})((W \otimes_\psi I) \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\
&= U_3((W \otimes_\phi I) \otimes_i I_{E_1})(I_{E\otimes_\psi E} \otimes_{i\otimes\iota} U^*)(\widetilde{V}^* \otimes_{\phi_0} I_{E_1})(I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi \\
&= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi.
\end{aligned}$$

It follows from (5.16) and (5.17) that we have

$$\begin{aligned}
&U_3(I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} V^*)(x \otimes (\xi_1 \otimes \xi_2 \otimes \xi_3)) \\
&= (U \otimes I_{E_1} \otimes I_{E_1})(U^*(\xi_1 \otimes \xi_2) \otimes x\xi_3) \\
&= (I_{E_1} \otimes I_{E_1} \otimes x)(\xi_1 \otimes \xi_2 \otimes \xi_3).
\end{aligned}$$

for $\xi_i \in E_1$ $(i = 1, 2, 3)$. Then by (5.15) we have

$$\begin{aligned}
&U_3(W_2 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\
&= U_3(I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\psi\otimes\iota\otimes\iota} V^*\widetilde{V})(x \otimes y \otimes z \otimes \xi) \\
&= U_3(I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} V^*)(x \otimes \{(I_{E_1} \otimes y)z\xi\}) \\
&= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi.
\end{aligned}$$

□

6. Inclusions of index finite-type

In this section, we study a multiplicative unitary operator associated with an iclusion of C^*-algebras when the inclusion is of index-finite type in the sense of Watatani [35]. Let A_1 be a C^*-algebra with the identity 1, let A_0 be a C^*-subalgebra of A_1 which contains 1 and let $P_1 : A_1 \longrightarrow A_0$ be a faithful positive conditional expectation. We assume that P_1 is of index-finite type, that is, there exists a family $u_i \in A_1$ $(i = 1, \cdots, n)$ such that

$$\sum_{i=1}^{n} u_i P_1(u_i^* a) = \sum_{i=1}^{n} P_1(a u_i) u_i^* = a$$

for every $a \in A_1$ [35, 1.2.2, 2.1.6]. Then the index of P_1 is given by $\text{Index } P_1 = \sum_i u_i u_i^*$ which is an element of the center of A_1. We denote by E_1 a right A_0-module A_1 whose right A_0-action is the product in A_1. Define an A_0-valued inner product of E_1 by $< a, b >= P_1(a^*b)$ for a, $b \in E_1$. It follows from [35, 2.1.5] that there exists a positive number λ such that

$$\lambda \|a\|_{A_1} \leq \|a\|_{E_1} \leq \|a\|_{A_1}$$

for every $a \in E_1 = A_1$, where $\| \cdot \|_{A_1}$ and $\| \cdot \|_{E_1}$ denote the norms of A_1 and E_1 respectively. Therefore E_1 is complete and is a Hilbert A_0-module. Define a unital injective $*$-homomorphism $\phi_1 : A_1 \longrightarrow \mathcal{L}_{A_0}(E_1)$ by $\phi_1(a)b = ab$ for $a \in A_1$ and $b \in E_1$, where ab is the product in A_1. Then we can construct A, E, ϕ and ψ as in Section 5. Moreover we can construct the operators U, V and $\widetilde{V}$.

We denote by A_2 the C^*-algebra $\mathcal{K}_{A_0}(E_1)$ (cf. [35, 2.1.2, 2.1.3]). Note that we have $\mathcal{K}_{A_0}(E_1) = \mathcal{L}_{A_0}(E_1)$. In fact, we have $I = \sum_{i=1}^{n} \theta_{u_i,u_i}$ in $\mathcal{L}_{A_0}(E_1)$. We identify $\phi(A_1)$ with A_1 and we have inclusions $A_0 \subset A_1 \subset A_2$, which is the basic construction ([35, 2.2.10], see also [11, Chapter 2]). Let $P_2 : A_2 \longrightarrow A_1$ be the dual conditional expectation of P_1, that is, $P_2(\theta_{a,b}) = (\text{ Index } P_1)^{-1} ab^*$ for a, $b \in A_1$ [35, 2.3.3]. Note that P_2 and $P_1 \circ P_2$ are of index-finite type [35, 1.7.1, 2.3.4]. We denote by F_2 a right A_0-module A_2 whose right A_0-action is the product in A_2. Define an A_0-valued inner product of F_2 by $< \xi, \eta >= P_1 \circ P_2(\xi^*\eta)$ for

$\xi,\ \eta \in F_2 = A_2$. Then F_2 is a Hilbert A_0-module. Define a unital injective $*$-homomorphism $\widetilde{\phi}_2 : A_1 \longrightarrow \mathcal{L}_{A_0}(F_2)$ by $\widetilde{\phi}_2(a)\xi = a\xi$ for $a \in A_1$ and $\xi \in F_2$, where $a\xi$ is the product in A_2. Define a linear map $\Phi : E_2 \longrightarrow F_2$ by

$$\Phi(a \otimes b) = \theta_{a,b^*}\phi_1((\text{ Index } P_1)^{1/2})$$

for $a,\ b \in E_1$. Then Φ is an isomorphism between the Hilbert A_0-modules. Moreover we have $\Phi(\phi_2(a_1)\xi) = \widetilde{\phi}_2(a_1)\Phi(\xi)$ for $a_1 \in A_1$ and $\xi \in E_2$.

We denote by $A_0' \cap A_2$ the C^*-algebra $\{a \in A_2;\ ab = ba \text{ for every } b \in A_0\}$ and denote by $\overline{\text{lin}}\ A_1(A_0' \cap A_2)$ the closed linear subspace of A_2 generated by elements of the form ab with $a \in A_1$ and $b \in A_0' \cap A_2$. For $a \in A_1$, we denote by $C(a)$ the norm closure of the convex hull of the set consisting of elements of the form uau^* with unitary elements u of A_0. We consider the following two conditions:

(P1) $\quad A_2 = \overline{\text{lin}}\ A_1(A_0' \cap A_2)$.

(P2) $\quad A_0' \cap C(a) \neq \emptyset \quad$ for every $a \in A_1$.

The condition (P1) corresponds to the condition that $A_0 \subset A_1$ is of depth 2. For inclusions of factors, this fact is well-known (cf. [17], [29]). As for inclusions of C^*-algebras, the author is not sure that (P1) coincides precisely with the condition that $A_0 \subset A_1$ is of depth 2. Therefore we avoid the term "of depth 2".

In the following theorem, we show that the conditions (P1) and (P2) imply the assumptions of Theorem 5.3. Thus we have a multiplicative unitary operator when these conditions are satisfied.

Theorem 6.1. (1) *The operator U is unitary if and only if the condition* (P1) *is satisfied.*

(2) *Suppose that U is unitary and that the condition* (P2) *is satisfied. Then there exists an elment W of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes I_{E_1}$.*

We mention some consequences of Theorem 6.1 before proving it.

Corollary 6.2. *Suppose that the conditions* (P1) *and* (P2) *are satisfied. Then there exists a multiplicative unitary operator W in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes I_{E_1}$.*

Corollary 6.3. *Suppose that A_0 is finite-dimensional and that the condition* (P1) *is satisfied. Then there exists a multiplicative unitary operator W in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes I_{E_1}$.*

Proof. It is enough to show that (P2) is satisfied if A_0 is finite-dimensional. Let G be the group of unitary elements of A_0. Since A_0 is finite-dimensional, G is a compact group with respect to the norm topology. Therefore there exists a left Haar measure ν of G such that $\nu(G) = 1$. For $a \in A_1$, set

$$\widetilde{a} = \int_G uau^*\, d\nu(u).$$

By a standard argument, we know that $\widetilde{a}$ is an element of $C(a)$. We can also prove that $\widetilde{a}$ belongs to $A_0' \cap A_1$. Thus (P2) is satisfied. □

Now we will prove Theorem 6.1. The following proposition is useful in later arguments.

Proposition 6.4. (1) *There exists a bijection q_1 of $A_0' \cap A_1$ onto A such that $q_1(a)b = ba$ for $a \in A_0' \cap A_1$ and $b \in E_1$, where ba is the product in A_1.*

(2) *There exists a bijection q_2 of $A_0' \cap A_2$ onto E such that $q_2(a)b = \Phi^{-1}(ba)$ for $a \in A_0' \cap A_2$ and $b \in E_1$, where ba is the product in A_2.*

Proof. (1) Since P_1 is of index-finite type, it follows from [35, 1.11.3] that there exists an automorphism θ_1 of the algebra $A_0' \cap A_1$ such that $P_1(ab) = P_1(b\theta_1(a))$ for $a \in A_0' \cap A_1$ and $b \in A_1$. (Note that θ_1 is not $*$-preserving in general.) For $a \in A_0' \cap A_1$, the map

$q_1(a) : E_1 \longrightarrow E_1$ is adjointable. In fact we have $q_1(a)^* = q_1(\theta_1(a^*))$. Thus $q_1(a)$ is an element of $\mathcal{L}_{A_0}(E_1)$. It is clear that $q_1(a)$ commutes with ϕ_1. Therefore $q_1(a)$ is an element of A. On the other hand, let x be an element of A. Set $a = x(1)$. Then a belongs to $A_0' \cap A_1$ and we have $q_1(a) = x$.

(2) There exists an automorphism θ_2 of the algebra $A_0' \cap A_2$ such that $P_1 \circ P_2(ab) = P_1 \circ P_2(b\theta_2(a))$ for $a \in A_0' \cap A_2$ and $b \in A_2$. For $a \in A_0' \cap A_2$, the map $q_2(a) : E_1 \longrightarrow E_2$ is adjointable. In fact we have, for $\eta \in E_2$, $q_2(a)^*\eta = P_2(\Phi(\eta)\theta_2(a^*))$, where $\Phi(\eta)\theta_2(a^*)$ is the product in A_2. Thus $q_2(a)$ is an element of $\mathcal{L}_{A_0}(E_1, E_2)$. Since we have $q_2(a)\phi_1(a_1) = \phi_2(a_1)q_2(a)$ for every $a_1 \in A_1$, $q_2(a)$ is an element of E. On the other hand, let x be an element of E. Set $a = \Phi(x(1))$. Then a belongs to $A_0' \cap A_2$ and we have $q_2(a) = x$. □

Proof of Theorem 6.1 (1). Suppose that U is a unitary operator. Since U is surjective, for every $a \in A_2$ and every $\varepsilon > 0$, there exist $x_j \in E$ and $a_j \in E_1$ $(j = 1, \cdots, k)$ such that

$$\|U(\sum_{j=1}^{k} x_j \otimes a_j) - \Phi^{-1}(a)\| < \varepsilon.$$

It follows from Proposition 6.4 that there exists $b_j \in A_0' \cap A_2$ such that $q_2(b_j) = x_j$ for $j = 1, \cdots, k$. Then we have

$$\|\sum_{j=1}^{k} a_j b_j - a\|_{F_2} = \|\sum_{j=1}^{k} x_j(a_j) - \Phi^{-1}(a)\|_{E_2} < \varepsilon.$$

Note that the norm of the Hilbert C^*-module F_2 and the norm of the C^*-algebra A_2 are equivalent. Therefore the linear space generated by ab with $a \in A_1$ and $b \in A_0' \cap A_2$ is dense in A_2.

Conversely, suppose that the condition (P1) is satisfied. Since the norms of F_2 and A_2 are equivalent, for every $a \in A_2$ and every $\varepsilon > 0$, there exist $a_j \in A_1$ and $b_j \in A_0' \cap A_2$ $(j = 1, \cdots, k)$ such that

$$\|\sum_{j=1}^{k} a_j b_j - a\|_{F_2} < \varepsilon.$$

Set $x_j = q_2(b_j) \in E$. Then we have

$$\|U(\sum_{j=1}^{k} x_j \otimes a_j) - \Phi^{-1}(a)\| < \varepsilon.$$

Therefore $U(E \otimes_i E_1)$ is dense in E_2. Since U is isometry by Proposition 5.1, U is surjective. Therefore U is invertible. Since we have $< U\alpha, U\beta > = < \alpha, \beta >$ by Proposition 5.1, U is adjointable and we have $U^* = U^{-1}$. Thus U is a unitary operator. □

In the rest of this section, we will prove the statement (2) of Theorem 6.1. We suppose that U is unitary and that the condition (P2) is satisfied. Before proving the statement, we prepare several lemmas. We fix an arbitrary element α of $E \otimes_\psi E$ and set $\xi = V^*\widetilde{V}(\alpha \otimes 1)$, where 1 is the identity element of $A_1 = E_1$.

Lemma 6.5. *Let α and ξ be as above. Then, for every $j = 1, 2, \cdots$, there exist non-zero elements $\beta_{jk} \in E \otimes_\phi E$ and $b_{jk} \in A_0' \cap A_1 \subset E_1$ $(k = 1, \cdots, n_j)$ such that*

$$\left\| \sum_{k=1}^{n_j} \beta_{jk} \otimes b_{jk} - \xi \right\| \longrightarrow 0 \qquad (j \longrightarrow \infty).$$

Proof. Let u be a unitary element of A_0. For $\eta \in E \otimes_\phi E \otimes_i E_1$, set

$$\mathrm{Ad}\ u(\eta) = (I_{E\otimes_\phi E} \otimes \phi_1(u))\eta u^*.$$

Since we have, for every $a \in A_1$,

$$V^*\widetilde{V}(I_{E\otimes_\psi E} \otimes \phi_1(a)) = (I_{E\otimes_\phi E} \otimes \phi_1(a))V^*\widetilde{V}, \tag{6.18}$$

we have Ad $u(\xi) = \xi$ for every unitary element $u \in A_0$. For every $j = 1, 2, \cdots$, there exist non-zero elements $\beta_{jk} \in E \otimes_\phi E$ and $a_{jk} \in E_1$ $(k = 1, \cdots, n_j)$ such that

$$\sum_{k=1}^{n_j} \beta_{jk} \otimes a_{jk} \longrightarrow \xi.$$

We set $\xi_j = \sum_{k=1}^{n_j} \beta_{jk} \otimes a_{jk}$. Note that we have

$$\mathrm{Ad}\ u(\xi_j) = \sum_{k=1}^{n_j} \beta_{jk} \otimes (ua_{jk}u^*).$$

We fix j. It follows from the condition (P2) that there exists an element b_{j1} of $C(a_{j1}) \cap A_0'$. Then there exist unitary elements $u_l^{(1)}$ of A_0 and $t_l^{(1)} > 0$ $(l = 1, \cdots, m_1)$ with $\sum_{l=1}^{m_1} t_l^{(1)} = 1$ such that

$$\left\| b_{j1} - \sum_{l=1}^{m_1} t_l^{(1)} u_l^{(1)} a_{j1} (u_l^{(1)})^* \right\| \le (jn_j\|\beta_{j1}\|)^{-1}.$$

Set

$$a_{jk}^{(1)} = \sum_{l=1}^{m_1} t_l^{(1)} u_l^{(1)} a_{jk} (u_l^{(1)})^*,$$

$$\eta_j^{(1)} = \beta_{j1} \otimes b_{j1} + \sum_{k=2}^{n_j} \beta_{jk} \otimes a_{jk}^{(1)}.$$

We have

$$\begin{aligned}
\|\xi - \eta_j^{(1)}\| &= \|\sum_{l=1}^{m_1} t_l^{(1)} \mathrm{Ad}\ u_l^{(1)}(\xi - \xi_j + \xi_j) - \eta_j^{(1)}\| \\
&\le \sum_{l=1}^{m_1} t_l^{(1)} \|\mathrm{Ad}\ u_l^{(1)}(\xi - \xi_j)\| + \|\beta_{j1} \otimes (a_{j1}^{(1)} - b_{j1})\| \\
&\le \|\xi - \xi_j\| + 1/(jn_j).
\end{aligned}$$

By repeating similar arguments, we can construct elements $\eta_j^{(m)}$ of $E \otimes_\phi E \otimes_i E_1$ $(m = 1, \cdots, n_j)$ with the following properties;

$$\eta_j^{(m)} = \sum_{k=1}^{m} \beta_{jk} \otimes b_{jk} + \sum_{k=m+1}^{n_j} \beta_{jk} \otimes a_{jk}^{(m)},$$

$$\|\xi - \eta_j^{(m)}\| \le \|\xi - \xi_j\| + m/(jn_j),$$

where $b_{jk} \in A_0' \cap A_1$ $(k = 1, \cdots, m)$ and $a_{jk}^{(m)} \in A_1$ $(k = m + 1, \cdots, n_j)$. Then β_{jk} and b_{jk} have the desired property. □

Lemma 6.6. *Let α and ξ be as above. Then there exist elements β_j of $E \otimes_\phi E$ $(j = 1, 2, \cdots)$ such that $\|\beta_j \otimes 1 - \xi\| \longrightarrow 0$ $(j \longrightarrow \infty)$.*

Proof. Let β_{jk} and b_{jk} be elements as in Lemma 6.5. Since $b_{jk} \in A_0' \cap A_1$, it follows from Proposition 6.4 that we have $\beta_{jk} \otimes b_{jk} = \beta_{jk} q_1(b_{jk}) \otimes 1$. Set $\beta_j = \sum_{k=1}^{n_j} \beta_{jk} q_1(b_{jk})$. Then β_j has the desired property. □

Lemma 6.7. *Let α and ξ be as above. Then there exists a unique element β of $E \otimes_\phi E$ such that $\xi = \beta \otimes 1$.*

Proof. Let β_j be as in Lemma 6.6. We fix j and k and set $\gamma = \beta_j - \beta_k$. It follows from Proposition 6.4 that there exists $b \in A_0' \cap A_1$ such that $q_1(b) =< \gamma, \gamma >^{1/2}$. Then we have $\|b\|_{E_1} = \|\beta_j \otimes 1 - \beta_k \otimes 1\|$. Recall that there exists a positive number λ such that $\lambda \|a\|_{A_1} \leq \|a\|_{E_1}$ for every $a \in E_1$. We have $\|\beta_j - \beta_k\| = \|q_1(b)\| \leq \lambda^{-1}\|b\|_{E_1}$. Therefore we have $\|\beta_j - \beta_k\| \leq \lambda^{-1}\|\beta_j \otimes 1 - \beta_k \otimes 1\|$. Since $\{\beta_j \otimes 1\}$ is a Cauchy sequence, $\{\beta_j\}$ is also a Cauchy sequence. Thus $\{\beta_j\}$ converges to an element β of $E \otimes_\phi E$. Since $\{\beta_j \otimes 1\}$ converges to $\beta \otimes 1$, we have $\xi = \beta \otimes 1$. For $\beta' \in E \otimes_\phi E$, $\beta' \otimes 1 = 0$ implies that $\beta' = 0$. Therefore β is unique. □

Proof of Theorem 6.1 (2). It follows from Lemma 6.7 that there exists a linear map W of $E \otimes_\psi E$ to $E \otimes_\phi E$ such that $V^* \widetilde{V}(\alpha \otimes 1) = (W\alpha) \otimes 1$ for every $\alpha \in E \otimes_\psi E$. By (6.18), we have $V^* \widetilde{V}(\alpha \otimes a) = (W\alpha) \otimes a$ for $\alpha \in E \otimes_\psi E$ and $a \in E_1$. Since we have, for every $a \in A_1$,

$$\widetilde{V}^* V(I_{E \otimes_\phi E} \otimes \phi_1(a)) = (I_{E \otimes_\psi E} \otimes \phi_1(a))\widetilde{V}^* V,$$

we can prove results similar to Lemmas 6.5, 6.6 and 6.7 with respect to $\widetilde{V}^* V$. Therefore there exists a linear map W' of $E \otimes_\phi E$ to $E \otimes_\psi E$ such that $\widetilde{V}^* V(\alpha \otimes 1) = (W'\alpha) \otimes 1$ for every $\alpha \in E \otimes_\phi E$. Then we have $\widetilde{V}^* V(\alpha \otimes a) = (W'\alpha) \otimes a$ for $\alpha \in E \otimes_\phi E$ and $a \in E_1$. For every $\alpha \in E \otimes_\psi E$ and $\beta \in E \otimes_\phi E$ and a, $b \in E_1$, we have

$$\begin{aligned} < V^* \widetilde{V}(\alpha \otimes a), \beta \otimes b > &=< a, < W\alpha, \beta > b > \\ &=< a, < \alpha, W'\beta > b > . \end{aligned}$$

Therefore we have $< W\alpha, \beta >=< \alpha, W'\beta >$. Thus W is adjointable and we have $V^* \widetilde{V} = W \otimes I_{E_1}$. This completes the proof of Theorem 6.1. □

7. Crossed products by finite groups

In this section, we apply the above results to the inclusion associated with a crossed product of a C^*-algebra by a finite group. Let A_0 be a unital C^*-algebra, let G be a finite group and let α be an action of G on A. We denote by A_1 the crossed product $A_0 \rtimes_\alpha G$. Then we have the inclusion $A_0 \subset A_1$ and the canonical conditional expectation P_1 of A_1 onto A_0. We will show that the above inclusion satisfies the condition (P1) and the assumption of Theorem 5.3. Therefore we have a multiplicative unitary operator W associated with the inclusion. We will give an explicit formula for W. We will also give elements that satisfy the conditions (E1) and (E2) in Section 4.

For every finite set X, we denote by $\mathrm{Map}(X, A_0)$ the linear space of all maps from X to A_0. We identify A_1 with $\mathrm{Map}(G, A_0)$ with the following product and involution;

$$(ab)(g) = \sum_{h \in G} a(h)\alpha_h(b(h^{-1}g))$$
$$a^*(g) = \alpha_g(a(g^{-1})^*)$$

for a, $b \in \mathrm{Map}(G, A_0)$ and $g \in G$. For $a_0 \in A_0$, define an element $\tilde{a}_0$ of $\mathrm{Map}(G, A_0)$ by $\tilde{a}_0(e) = a_0$ and $\tilde{a}_0(g) = 0$ if $g \neq e$, where e is the unit of G. In the following, we identify a_0 with $\tilde{a}_0$ and we have the inclusion $A_0 \subset A_1$. Then the canonical conditional expectation P_1 of A_1 onto A_0 is given by $P_1(a) = a(e)$. Note that P_1 is of index finite type and

Index $P_1 = |G|$. In fact, $\{\delta_g; g \in G\}$ is a quasi-basis for P_1, where δ_g is the function on G such that $\delta_g(g) = 1$ and $\delta_g(h) = 0$ if $h \neq g$. We denote by $\mathcal{E}_1$ the Hilbert A_0-module $\mathrm{Map}(G, A_0)$ with the following right A_0-action and A_0-valued inner product;

$$(\xi a)(g) = \xi(g)\alpha_g(a),$$
$$< \xi, \eta > = \sum_{g \in G} \alpha_{g^{-1}}(\xi(g)^* \eta(g))$$

for ξ, $\eta \in \mathrm{Map}(G, A_0)$, $a \in A_0$ and $g \in G$. Then we can identify $\mathcal{E}_1$ with the Hilbert A_0-module E_1 defined in Section 6. With this identification, the $*$-homomorphism ϕ_1 of A_1 to $\mathcal{L}_{A_0}(E_1)$ is given by

$$(\phi_1(a)\xi)(g) = \sum_{h \in G} a(h)\alpha_h(\xi(h^{-1}g))$$

for $a \in A_1$, $\xi \in E_1$ and $g \in G$. We denote by $\mathcal{A}_2$ the $*$-algebra $\mathrm{Map}(G^2, A_0)$ with the following product and involution;

$$(ab)(h, g) = \sum_{k \in G} a(h, k)\alpha_{hk^{-1}}(b(k, g)),$$
$$a^*(h, g) = \alpha_{hg^{-1}}(a(g, h)^*)$$

for a, $b \in \mathrm{Map}(G^2, A_0)$. We can identify $\mathcal{A}_2$ with the C^*-algebra A_2 defined in Section 6. The identification is given by

$$a(\xi) = \sum_{g, h \in G} a(h, g)\alpha_{hg^{-1}}(\xi(g))\delta_h$$

for $a \in A_2$ and $\xi \in E_1$. Let $\mathcal{A}_0$ be the $*$-subalgebra of $\mathcal{A}_2$ consisting of elements a for which there exists an element a_0 in A_0 such that $a(h, g) = a_0\delta_e(hg^{-1})$ for every g, $h \in G$. Let $\mathcal{A}_1$ be the $*$-subalgebra of $\mathcal{A}_2$ consisting of elements a for which there exists an element a_1 in A_1 such that $a(h, g) = a_1(hg^{-1})$ for every g, $h \in G$. Then the basic construction $A_0 \subset A_1 \subset A_2$ is identified with $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2$.

Proposition 7.1. *The basic construction $A_0 \subset A_1 \subset A_2$ satisfies the condition* (P1).

Proof. For $a \in \mathcal{A}_2$, a belongs to $\mathcal{A}_0' \cap \mathcal{A}_2$ if and only if

$$a_0 a(h, g) = a(h, g)\alpha_{hg^{-1}}(a_0)$$

for every $a_0 \in A_0$ and $g, h \in G$. For $g \in G$, define $d_g \in \mathcal{A}_0' \cap \mathcal{A}_2$ by $d_g(h', g') = \delta_g(h')\delta_g(g')$. For $a_0 \in A_0$ and $g, h \in G$, define $f(a_0; h, g) \in \mathcal{A}_1$ by $f(a_0; h, g)(h', g') = a_0\delta_{hg^{-1}}(h'g'^{-1})$. Then we have

$$(f(a_0; h, g)d_g)(h', g') = a_0\delta_h(h')\delta_g(g').$$

Therefore we have, for $a \in \mathcal{A}_2$,

$$a = \sum_{g, h \in G} f(a(h, g); h, g)d_g.$$

This implies that $\mathcal{A}_2$ is the linear span of $\mathcal{A}_1(\mathcal{A}_0' \cap \mathcal{A}_2)$. □

We denote by $\mathcal{E}_2$ the Hilbert A_0-module $\mathrm{Map}(G^2, A_0)$ with the following right A_0-action and A_0-valued inner product;

$$(\xi a)(h, g) = \xi(h, g)\alpha_{hg^{-1}}(a),$$
$$< \xi, \eta > = \sum_{g, h \in G} \alpha_{gh^{-1}}(\xi(h, g)^* \eta(h, g))$$

for $\xi, \eta \in \mathrm{Map}(G^2, A_0)$, $a \in A_0$ and $g, h \in G$. Then we can identify $\mathcal{E}_2$ with the Hilbert A_0-module E_2 defined in Section 5. The identification is given by

$$(u \otimes_{\phi_0} v)(h, g) = u(h)\alpha_h(v(g^{-1}))$$

for u, $v \in E_1$. With this identification, the $*$-homomorphism ϕ_2 of A_1 to $\mathcal{L}_{A_0}(E_2)$ is given by

$$(\phi_2(a)\xi)(h, g) = \sum_{k \in G} a(k)\alpha_k(\xi(k^{-1}h, g))$$

for $a \in A_1$, $\xi \in E_2$ and g, $h \in G$. We denote by $\mathcal{E}_3$ the Hilbert A_0-module $\mathrm{Map}(G^3, A_0)$ with the following right A_0-action and A_0-valued inner product;

$$(\xi a)(k, h, g) = \xi(k, h, g)\alpha_{kh^{-1}g}(a)$$
$$< \xi, \eta > = \sum_{g,h,k \in G} \alpha_{g^{-1}hk^{-1}}(\xi(k, h, g)^* \eta(k, h, g))$$

for $\xi, \eta \in \mathrm{Map}(G^3, A_0)$, $a \in A_0$ and $g, h, k \in G$. Then we can identify $\mathcal{E}_3$ with the Hilbert A_0-module E_3 defined in Section 5. The identification is given by

$$(u_1 \otimes_{\phi_0} u_2 \otimes_{\phi_0} u_3)(k, h, g) = u_1(k)\alpha_k(u_2(h^{-1}))\alpha_{kh^{-1}}(u_3(g))$$

for $u_i \in E_1$ $(i = 1, 2, 3)$.

We denote by $\mathcal{A}$ the subset of $\mathrm{Map}(G^2, A_0)$ consisting of elements a with the following properties;

$$\alpha_g(a_0)a(h, g) = a(h, g)\alpha_h(a_0),$$
$$a(k, gh) = \alpha_g(a(g^{-1}k, h))$$

for $a_0 \in A_0$ and $g, h, k \in G$. Then $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{A}_2$. The product of $\mathcal{A}$ is given by

$$(ab)(h, g) = \sum_{k \in G} b(k, g)a(h, k)$$

for a, $b \in \mathcal{A}$. We can identify $\mathcal{A}$ with the C^*-algebra A defined in Section 5. The identification is given by

$$a(\xi) = \sum_{g,h \in G} \xi(g)a(h, g)\delta_h$$

for $a \in A$ and $\xi \in E_1$. We denote by $\mathcal{E}$ the subset of $\mathrm{Map}(G^3, A_0)$ consisting of elements x with the following properties;

$$\alpha_g(a_0)x(h, k, g) = x(h, k, g)\alpha_{hk^{-1}}(a_0),$$
$$x(k, l, hg) = \alpha_h(x(h^{-1}k, l, g))$$

for $a_0 \in A_0$ and $g, h, k, l \in G$. Then $\mathcal{E}$ is a Hilbert $\mathcal{A}$-module with the following right $\mathcal{A}$-action and $\mathcal{A}$-valued inner product;

$$(xa)(h, k, g) = \sum_{l \in G} a(l, g)x(h, k, l),$$
$$< x, y > (h, g) = \sum_{k,l \in G} \alpha_{hkl^{-1}}(x(l, k, h)^* y(l, k, g))$$

for $x, y \in \mathcal{E}$ and $a \in \mathcal{A}$. We can identify $\mathcal{E}$ with the Hilbert A-module E defined in Section 5. The identification is given by

$$x(\xi)(h, k) = \sum_{g \in G} \xi(g)x(h, k, g)$$

for $x \in E$, $\xi \in E_1$ and $g, h \in G$. Let ϕ and ψ be $*$-homomorphisms of A to $\mathcal{L}_A(E)$ defined in Section 5. With the above identification, ϕ and ψ are given as follows;

$$(\phi(a)x)(h,k,g) = \sum_{l \in G} x(l,k,g)a(h,l),$$
$$(\psi(a)x)(h,k,g) = \sum_{l \in G} x(h,l,g)\alpha_h(a(k^{-1},l^{-1}))$$

for $a \in A$, $x \in E$ and $g, h, k \in G$.

We denote by $\mathcal{F}$ the subset of $\mathrm{Map}(G^4, A_0)$ consisting of elements X with the following properties;

$$X(k,h,g,l)\alpha_{kh^{-1}g}(a_0) = \alpha_l(a_0)X(k,h,g,l),$$
$$X(k,h,g,nl) = \alpha_n(X(n^{-1}k,h,g,l))$$

for every $a_0 \in A_0$ and $g, h, k, l, n \in G$. Then $\mathcal{F}$ is a Hilbert $\mathcal{A}$-module with the following right $\mathcal{A}$-action and $\mathcal{A}$-valued inner product;

$$(Xa)(k,h,g,l) = \sum_{m \in G} a(m,l)X(k,h,g,m),$$
$$< X, Y > (h,g) = \sum_{k,l,n \in G} \alpha_{hn^{-1}kl^{-1}}(X(l,k,n,h)^* Y(l,k,n,g))$$

for $a \in \mathcal{A}$ and X, $Y \in \mathcal{F}$. The fact that $\mathcal{F}$ is complete is proved in the following proposition. We also show that $E \otimes_\phi E$ and $E \otimes_\psi E$ are isomorphic to $\mathcal{F}$. Note that we identify E with $\mathcal{E}$ and A with $\mathcal{A}$.

Proposition 7.2. (1) *There exists an isomorphism M of $E \otimes_\phi E$ onto $\mathcal{F}$ as Hilbert A-modules such that*

$$M(x \otimes_\phi y)(k,h,g,l) = \sum_{m \in G} y(m, g^{-1}, l)x(k,h,m)$$

for $x, y \in E$ and $g, h, k, l \in G$.

(2) *There exists an isomorphism $\widetilde{M}$ of $E \otimes_\psi E$ onto $\mathcal{F}$ as Hilbert A-modules such that*

$$\widetilde{M}(x \otimes_\psi y)(k,h,g,l) = \sum_{m \in G} y(k,m,l)\alpha_k(x(h^{-1}, g^{-1}, m^{-1}))$$

for $x, y \in E$ and $g, h, k, l \in G$.

Proof. (1) We denote by $E \odot_\phi E$ the linear space generated by elements of the form $x \otimes_\phi y$ with $x, y \in E$. Define a linear map M of $E \odot_\phi E$ to $\mathrm{Map}(G^4, A_0)$ by the formula in the statement (1) of the proposition. It is clear that the image of M is contained in $\mathcal{F}$ and that M is an A-module map. Let V be the unitary operator of $E \otimes_\phi E \otimes_i E_1$ onto E_3 defined in Section 5. It follows from Proposition 5.2 that we have, for $x, y \in E$ and $\xi \in E_1$, $V(x \otimes y \otimes \xi) = (x \otimes_{\phi_0} I_{E_1})y(\xi)$, where $(x \otimes_{\phi_0} I_{E_1})y$ is an element of $\mathcal{L}_{A_0}(E_1, E_3)$. Note that we identify E_3 with $\mathcal{E}_3$. Then we have

$$((x \otimes_{\phi_0} I_{E_1})y(\xi))(k,h,g) = \sum_{l \in G} \xi(l)M(x \otimes_\phi y)(k,h,g,l).$$

For $x_i, y_i \in E$ $(i = 1, \cdots, n)$, set $X = \sum_{i=1}^n x_i \otimes_\phi y_i \in E \odot_\phi E$ and $Y = \sum_{i=1}^n (x_i \otimes_{\phi_0} I_{E_1}) y_i \in \mathcal{L}_{A_0}(E_1, E_3)$. Then we have

$$\begin{aligned} < Y\xi, Y\xi >_{E_3} &= \sum_{g,h,k \in G} \alpha_{g^{-1}hk^{-1}}(|(Y\xi)(k,h,g)|^2) \\ &\geq \alpha_{g^{-1}hk^{-1}}(|\sum_{l \in G} \xi(l) M(X)(k,h,g,l)|^2) \end{aligned}$$

for every $\xi \in E_1$ and $g, h, k \in G$. Therefore we have $\|Y\delta_l\|^2 \geq \|M(X)(k,h,g,l)\|^2$. On the other hand, since V is unitary, we have

$$\|Y\xi\| = \|V(X \otimes \xi)\| = \|X \otimes \xi\| \leq \|X\|\|\xi\|$$

for every $\xi \in E_1$. Thus we have $\|Y\| \leq \|X\|$. Therefore we have

$$\|M(X)(k,h,g,l)\| \leq \|Y\| \leq \|X\|$$

for every $g, h, k, l \in G$. Then we can extend M to the A-module map of $E \otimes_\phi E$ to $\mathcal{F}$, which we denote again by M. By a straightforward calculation, we know that $< X_1, X_2 >=< M(X_1), M(X_2) >$ for every $X_1, X_2 \in E \otimes_\phi E$.

We will prove that M is surjective. Let X be an element of $\mathcal{F}$. Fix $g_0, h_0 \in G$. Define an element $X_{(g_0,h_0)}$ of $\mathcal{E}$ by

$$X_{(g_0,h_0)}(k,h,g) = \begin{cases} X(k, h_0, g_0, gg_0) & \text{if } h = h_0 \\ 0 & \text{if } h \neq h_0 \end{cases}$$

and define an element e_{g_0} of $\mathcal{E}$ by

$$e_{g_0}(k,h,g) = \begin{cases} 1 & \text{if } k = gh \text{ and } h = g_0^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $X = \sum_{g,h \in G} M(X_{(g,h)} \otimes_\phi e_g)$. Therefore M is surjective. It also implies that $\mathcal{F}$ is in fact a Hilbert A-module.

(2) We denote by $E \odot_\psi E$ the linear space generated by elements of the form $x \otimes_\psi y$ with $x, y \in E$. Define a linear map $\widetilde{M}$ of $E \odot_\psi E$ to $\mathrm{Map}(G^4, A_0)$ by the formula in the statement (2) of the proposition. It is clear that the image of $\widetilde{M}$ is contained in $\mathcal{F}$ and that $\widetilde{M}$ is an A-module map. Note that we have, for $x, y \in E$, $\xi \in E_1$ and $g, h, k \in G$,

$$\begin{aligned} (\widetilde{V}(x \otimes y \otimes \xi))(k,h,g) &= ((I_{E_1} \otimes_{\phi_0} x) y(\xi))(k,h,g) \\ &= \sum_{l \in G} \xi(l) \widetilde{M}(x \otimes_\psi y)(k,h,g,l). \end{aligned}$$

Then we can argue as in (1) and we can extend $\widetilde{M}$ to the A-module map of $E \otimes_\psi E$ to $\mathcal{F}$, which we denote again by $\widetilde{M}$. By a straightforward calculation, we know that $< X_1, X_2 >=< \widetilde{M}(X_1), \widetilde{M}(X_2) >$ for every $X_1, X_2 \in E \otimes_\psi E$.

We will prove that $\widetilde{M}$ is surjective. Let X be an element of $\mathcal{F}$. Fix g_0, $h_0 \in G$. Define an element $\widetilde{X}_{(g_0,h_0)}$ of $\mathcal{E}$ by

$$\widetilde{X}_{(g_0,h_0)}(k,h,g) = \begin{cases} X(k, h_0, g_0, g) & \text{if } h = g_0^{-1} h_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $X = \sum_{g,h \in G} \widetilde{M}(e_g \otimes_\psi \widetilde{X}_{(g,h)})$. This completes the proof of the proposition. $\square$

From the above proof, we have an explicit formula for M^{-1}, that is, we have

$$M^{-1}(X) = \sum_{g,h \in G} X_{(g,h)} \otimes_\phi e_g.$$

Now we can construct a multiplicative unitary operator and we have an explicit formula for it.

Theorem 7.3. *There exists a multiplicative unitary operator W in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes I_{E_1}$. Moreover W has the following form;*

$$W = M^{-1}\widetilde{M}.$$

Proof. Note that we have, for every $X \in E \otimes_\psi E$, $Y \in E \otimes_\phi E$, $\xi \in E_1$ and $g, h, k \in G$,

$$\widetilde{V}(X \otimes \xi)(k,h,g) = \sum_{l \in G} \xi(l)\widetilde{M}(X)(k,h,g,l),$$
$$V(Y \otimes \xi)(k,h,g) = \sum_{l \in G} \xi(l)M(Y)(k,h,g,l).$$

Note also that we have, for $X, Y \in E \otimes_\phi E$ and $\xi, \eta \in E_1$,

$$\begin{aligned} &< X \otimes \xi, Y \otimes \eta >_{E \otimes_\phi E \otimes_i E_1} \\ &= \sum_{g,h,k,l,n \in G} \alpha_{n^{-1}kl^{-1}}([\xi(h)M(X)(l,k,n,h)]^* \eta(g)M(Y)(l,k,n,g)). \end{aligned}$$

We identify E_3 with $\mathcal{E}_3$. Let ζ be an element of E_3. For $\mu = (k_0, h_0, g_0) \in G^3$, define an element w_μ of $\mathcal{F}$ by

$$w_\mu(k,h,g,l) = \begin{cases} 1 & \text{if } g = g_0,\ h = h_0 \text{ and } l = kh_0^{-1}g_0 \\ 0 & \text{otherwise} \end{cases}$$

and define an element ζ_μ of E_1 by

$$\zeta_\mu(g) = \begin{cases} \zeta(k_0,h_0,g_0) & \text{if } g = k_0h_0^{-1}g_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the adjoint V^* of V is given by

$$V^*\zeta = \sum_{\mu \in G^3} M^{-1}(w_\mu) \otimes \zeta_\mu.$$

For $X \in E \otimes_\psi E$ and $\xi \in E_1$, set $\zeta = \widetilde{V}(X \otimes \xi)$. Then, for $\mu = (k_0, h_0, g_0)$, we have

$$\zeta_\mu(k_0h_0^{-1}g_0) = \sum_{l \in G} \xi(l)\widetilde{M}(X)(k_0,h_0,g_0,l).$$

Since we have, for $X \in E \otimes_\psi E$, $Y \in E \otimes_\phi E$ and $\xi, \eta \in E_1$,

$$\begin{aligned} &< M^{-1}(w_\mu) \otimes \zeta_\mu, Y \otimes \eta > \\ &= \sum_{g \in G} \alpha_{g_0^{-1}h_0k_0^{-1}}(\zeta_\mu(k_0h_0^{-1}g_0)^* \eta(g)M(Y)(k_0,h_0,g_0,g)) \\ &= \sum_{g,l \in G} \alpha_{g_0^{-1}h_0k_0^{-1}}([\xi(l)\widetilde{M}(X)(k_0,h_0,g_0,l)]^* \eta(g)M(Y)(k_0,h_0,g_0,g)), \end{aligned}$$

we have

$$< V^*\widetilde{V}(X \otimes \xi), Y \otimes \eta > = < M^{-1}\widetilde{M}(X) \otimes \xi, Y \otimes \eta > .$$

This completes the proof of the theorem. ∎

Finally, we give elements of E which satisfy properties (E1) and (E2) of Section 4. Define an element ξ_0 of E by $\xi_0(k,h,g) = |G|^{-1/2}$ if $k = gh$ and $\xi_0(k,h,g) = 0$ if $k \neq gh$. Then we have $\|\xi_0\| = 1$ and $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$. We can show that ξ_0 satisfies the property (E1). In fact, the element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ is given by

$$(\pi_{\xi_0}(\xi)\zeta)(k,h,g) = |G|^{-1/2} \sum_{u,v\in G} \zeta(v,u^{-1},g)\xi(k,uh,v).$$

Then the adjoint $\pi_{\xi_0}(\xi)^*$ is given by

$$(\pi_{\xi_0}(\xi)^*\eta)(k,h,g) = |G|^{-1/2} \sum_{u,v\in G} \alpha_{kh^{-1}uv^{-1}}(\xi(v,h^{-1}u,k)^*\eta(v,u,g)).$$

We can show that ξ_0 satisfies the property (E2). In fact, the element $\widehat{\pi}_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ is given by

$$(\widehat{\pi}_{\xi_0}(\xi)\zeta)(k,h,g) = |G|^{-1/2} \sum_{u,v\in G} \zeta(ku,v,g)\alpha_{ku}(\xi(u^{-1},h,v^{-1})).$$

Then the adjoint $\widehat{\pi}_{\xi_0}(\xi)^*$ is given by

$$(\widehat{\pi}_{\xi_0}(\xi)^*\eta)(k,h,g) = |G|^{-1/2} \sum_{u,v\in G} \alpha_{kh^{-1}vu^{-1}}(\alpha_k(\xi(k^{-1}u,v,h^{-1})^*)\eta(u,v,g)).$$

Define an element η_0 of E by $\eta_0(k,h,g) = 1$ if $k = g$ and $h = e$ and $\eta_0(k,h,g) = 0$ otherwise. Then we have $\|\eta_0\| = 1$ and $W(\eta_0 \otimes_\psi \eta_0) = \eta_0 \otimes_\phi \eta_0$. We can show that η_0 satisfies the property (E1). In fact, the element $\pi_{\eta_0}(\xi)$ of $\mathcal{L}_A(E)$ is given by

$$(\pi_{\eta_0}(\xi)\zeta)(k,h,g) = \sum_{v\in G} \zeta(v,e,g)\xi(k,h,v).$$

Then the adjoint $\pi_{\eta_0}(\xi)^*$ is given by

$$(\pi_{\eta_0}(\xi)^*\eta)(k,h,g) = \begin{cases} \sum_{u,v\in G} \alpha_{kuv^{-1}}(\xi(v,u,k)^*\eta(v,u,g)) & \text{if } h = e \\ 0 & \text{otherwise.} \end{cases}$$

We can show that η_0 satisfies the property (E2). In fact, the element $\widehat{\pi}_{\eta_0}(\xi)$ of $\mathcal{L}_A(E)$ is given by

$$(\widehat{\pi}_{\eta_0}(\xi)\zeta)(k,h,g) = \sum_{v\in G} \zeta(k,v,g)\alpha_k(\xi(e,h,v^{-1})).$$

Then the adjoint $\widehat{\pi}_{\eta_0}(\xi)^*$ is given by

$$(\widehat{\pi}_{\eta_0}(\xi)^*\eta)(k,h,g) = \sum_{u\in G} \alpha_{kh^{-1}u}(\xi(e,u,h^{-1})^*\alpha_{k^{-1}}(\eta(k,u,g))).$$

The author does not characterize the associated algebras $B(\xi_0)$, $\widehat{B}(\xi_0)$, $B(\eta_0)$ and $\widehat{B}(\eta_0)$ yet. It will be interesting to know the structures of these algebras.

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