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ON THE YOSIDA-HEWITT DECOMPOSITION AND RÜTTIMANN DECOMPOSITION OF STATES

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ABSTRACT. We study Yosida–Hewitt decompositions of measures on orthomodular posets and lattices. We discuss the existence and uniqueness of decompositions into a convex combination of a completely additive measure and a measure which is either weakly purely finitely additive or filtering. Then we are interested in the case of c-positive orthomodular lattices which have "enough" completely additive measures. For this class of lattices we find several conditions equivalent to the requirement that the Yosida–Hewitt decomposition coincides with the Rüttimann decomposition. We present an example which shows that the uniqueness of the Yosida–Hewitt decomposition does not imply the equality of the decompositions. As an application of the construction technique used therein, we answer a problem posed by G. Rüttimann: Can any completely additive Jordan measure be expressed as a difference of completely additive positive measures? We prove that the completely additive Jordan measures on the c-positive orthomodular lattices allow for such an expression whereas for general orthomodular lattices such a decomposition is generally not available.

1 Introduction Orthomodular lattices (OMLs) or, more generally, orthomodular posets (OMPs), are common generalizations of Boolean algebras and lattices of projections in Hilbert spaces. They are often interpreted as "quantum logics" i.e., as underlying structures in the logico-algebraic approach to quantum mechanics [14, 26, 33]. In this interpretation the states of a quantum mechanical experiment correspond to "states on quantum logics", the latter being probability measures. The study of potential "physical states" and their behavior have initiated an intense investigation of measures on OMLs and OMPs (see e.g. [5, 8, 10, 12, 13, 19, 22, 28] etc.). Several results of the Boolean or operator (noncommutative) measure theory have successfully been generalized to OMLs and OMPs, several have been found difficult or impossible to generalize, and several are still open for generalization.

In this paper we are interested in the decompositions of states into convex combinations of two states, one of them being completely additive and the other being "far" from completely additive. In accordance with the choice of the properties of the second state, we obtain several types of decompositions. Thus, we obtain the Yosida–Hewitt decomposition, when the second state is assumed to be weakly purely finitely additive (wpfa), and the Rüttimann decomposition when the second state is assumed to be a filtering state [7, 8, 29]. In [8] it is noticed that, as proved in a previous paper, the Yosida–Hewitt decomposition always exists but it need *not* be unique. The Rüttimann decomposition need not exist, but when it does exist, it is unique. Moreover, when the Rüttimann decomposition exists, it has to coincide with the Yosida–Hewitt decomposition. In this paper we pursue the measure

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decomposition problem for c-positive OMPs, i.e., for those OMPs in which every nonzero element can be nonzero evaluated in some completely additive state. The class of c-positive OMPs seems important within the mathematical foundations of quantum mechanics as can be seen in [2, 11, 17, 26]. We show that in the c-positive OMPs the existence of the Rüttimann decomposition is found to be equivalent with other conditions, e.g., with the wpfa-heredity and the #-heredity of the space of completely additive states. On the other hand, the uniqueness of the Yosida–Hewitt decomposition is found to be a strictly weaker condition—this fact we show by a fairly nontrivial example of Section 6.

The results concerning states may be alternatively formulated in terms of Jordan measures. In connection with this, G. Rüttimann formulated a question of whether there is a completely additive Jordan measure which cannot be expressed as a difference of two completely additive positive measures. In Section 7 we answer this question in the positive. Nevertheless, we show that such an example does not exist among OMPs which are c-positive and wpfa-hereditary.

2 Basic definitions

Let us first introduce basic notions (see [26] and [14]).

An *orthomodular poset* (OMP) is a pentuple $(L, \leq, 0, 1, ')$, where L is a set with at least two elements, $\leq$ is a partial order on L with respect to which 0 is a least and 1 is a greatest element, $'$ is an order reversing unary operation in L called *orthocomplementation*, and where the following additional conditions are satisfied:

(i) $p'' = p$, and $p \wedge p' = 0$ for all $p \in L$,

(ii) if $p, q \in L$ with p, q *orthogonal* (i.e., $p \leq q'$), then $p \vee q$ exists in L,

(iii) if $p, q \in L$ with $p \leq q$, then $q = p \vee (p' \wedge q)$ (the *orthomodular law*).

Since there is no danger of misunderstanding, we will only use L to denote the OMP $(L, \leq, 0, 1, ')$. If, in addition, $(L, \leq)$ is a lattice, then L is called an *orthomodular lattice* (OML). As known, a distributive orthomodular lattice is a Boolean algebra, and an example of a prominent nondistributive orthomodular lattice is the lattice of projections in a Hilbert space or, more generally, in a von Neumann algebra. Further examples of (finite or infinite) OMPs can be found, e.g., in [17, 22].

Unless stated otherwise, we assume in this paper that L is an OMP. Let p be an element of L. Then the interval $L_p = \{x \in L : x \leq p\}$ inherits in a natural way the structure of L in the sense that if $\leq_p$ is the restriction of $\leq$ to L_p and $'^p$ is defined by putting $x'^p = x' \wedge p$, the relation $\leq_p$ is an order and $'^p$ is an orthocomplementation which makes $(L_p, \leq_p, 0, p, '^p)$ an OMP [26]. Thus, L_p is an OMP. Moreover, an interval in an OML is an OML.

Let us introduce some more definitions and notations that came into existence in the development of noncommutative measure theory. A subset M of L is called *orthogonal* if its elements are pairwise orthogonal. A function $\mu\colon L \to \mathbb{R}$ is called *additive* (resp., *completely additive*) if the equality $\sum_{x \in M} \mu(x) = \mu(\bigvee M)$ holds for all orthogonal subsets $M \subseteq L$ which are finite (resp., which have a supremum in L; in this case, the absolute convergence of the series is required). An additive function $\mu\colon L \to [0, \infty)$ is called a *positive measure*. A difference of two positive measures is called a *Jordan measure*. We denote by $J^+(L)$ the set of all positive measures on L and by $J(L) = J^+(L) - J^+(L)$ the set of all Jordan measures on L. The set of all positive (resp. Jordan) measures on L which are completely additive is denoted by $J_c^+(L)$ (resp. $J_c(L)$). The positive measures on L which send $1 \in L$ into $1 \in \mathbb{R}$ are called *states*. The set of all states (resp. completely additive states) on L is denoted by $\Omega(L)$ (resp., $\Omega_c(L)$).

The following notion will play a crucial role in the sequel [29].

Definition 2.1 A subset I of an OMP L is called *filtering* if for any $p \in L \setminus \{0\}$ there is $q \in I \setminus \{0\}$ such that $q \leq p$.

Note that the filtering subsets of L are exactly those subsets that cover from below all nonzero elements of L. For many examples of filtering sets we refer to [8]. Let us only recall an interesting property of filtering sets which will be used later.

Lemma 2.2 *Let F be a filtering set in L. If M is a maximal orthogonal subset of F, then $\bigvee M = 1$.*

PROOF: If $p \in L$ is an upper bound of M different from 1, then the relation $p' \neq 0$ implies that F contains a nonzero element $q \leq p'$. The system $M \cup \{q\}$ is orthogonal and is strictly larger than M. This contradicts the maximality of M. Therefore the element $1 \in L$ is the only upper bound of M. □

An element a in an OMP L is called an *atom* if a covers 0, i.e., if $L_a = \{0, a\}$. We say that L is *atomic* if the set of all atoms is filtering. From the definition of filtering sets it easily follows that a set which contains a filtering subset is itself a filtering set. Further, a filtering set contains all atoms of L. Moreover, for atomic OMPs we have the equivalence—a subset of an atomic OMP is filtering if and only if it contains all atoms.

Definition 2.3 A Jordan measure μ on L is called *filtering* if $\ker \mu := \{q \in L : \mu | L_q = 0\}$ is a filtering set.

In particular, a filtering Jordan measure vanishes at all atoms. If an OMP L contains a finite maximal Boolean subalgebra, then the zero measure is the only filtering positive measure and L does not admit any filtering state. Let us denote by $J_f(L)$ the set of all filtering Jordan measures on L, and let us extend this notation to positive measures and states: $J_f^+(L) = J_f(L) \cap J^+(L)$, $\Omega_f(L) = J_f(L) \cap \Omega(L)$.

A state μ is called *weakly filtering* if the set

$$\ker \mu \cup \bigcap_{\alpha \in \Omega_c(L)} \ker \alpha$$

is filtering. Let us denote by $\Omega_{wf}(L)$ the set of all weakly filtering states on L.

The study of the decomposition of measures into sums of two measures with given properties can be easily translated into decompositions of states. It is obvious that in this case we have to consider convex combinations instead of sums. Convex combinations of states are sufficient to describe all properties we deal with (all the notions we use are preserved by multiplying a state by a positive constant), only the zero measure cannot be taken into consideration any more. Moreover, the use of states allows us to look at the problem from the geometrical point of view, since, due to a result by Shultz [31], the state spaces of OMPs as well as the state spaces of OMLs are exactly all compact convex subsets of locally convex topological vector spaces.

We will often deal with faces of subsets in the state spaces. The reason is that some special classes of states on a OMP L constitute a face of $\Omega(L)$ and we want to consider the decomposition with respect to these faces.

Definition 2.4 Let C be a compact convex subset of a locally convex Hausdorff topological linear space. A subset F of C is a *face* of C if all $\alpha, \beta, \gamma \in C$, where γ is a convex combination of α, β with nonzero coefficients, satisfy the equivalence

$$\gamma \in F \Longleftrightarrow \alpha, \beta \in F \ .$$

Proposition 2.5 *Let L be an OMP. The set $J_f(L)$ of all filtering Jordan measures on L is a linear subspace of $J(L)$. The set $\Omega_f(L)$ of all filtering states is a (possibly empty) face of $\Omega(L)$.*

In some special cases upon representing any state by a point we can "visualize" the result obtained: A state is a convex combination of two states α and β iff it belongs to the line segment that has these two states for extreme points. We denote this set by $conv\,\{\alpha, \beta\}$.

Definition 2.6 A state $\mu \in \Omega(L) \setminus \Omega_c(L)$ is *weakly purely finitely additive* (*wpfa state*) if the condition $\mu \in conv\,\{\lambda, \nu\}$ for some $\lambda \in \Omega_c(L)$, $\nu \in \Omega(L)$ implies $\mu = \nu$. We denote by $\Omega_{wpfa}(L)$ the set of all wpfa states on L.

This definition describes the situation when a wpfa state cannot be written as a convex combination of a completely additive state and an arbitrary state with nonzero coefficients. An alternative definition of wpfa states would make use of the partial order $\ll_S$ on $\Omega(L)$ defined by

$$\lambda \ll_S \mu \iff (\lambda = \mu \text{ or } \exists \nu \in \Omega(L) \setminus \{\mu\} :\ \mu \in conv\,\{\lambda, \nu\})\ .$$

A state $\mu \in \Omega(L)$ is wpfa iff

$$\forall \lambda \in \Omega_c(L) : \lambda \not\ll_S \mu\ .$$

From the geometrical point of view, a state on L is wpfa iff it generates a face disjoint from $\Omega_c(L)$. Equivalently, the wpfa states are exactly the elements of the union of all faces of $\Omega(L)$ disjoint from $\Omega_c(L)$. For further equivalent conditions, see [8].

3 Heredity and decompositions of states Starting from a state μ defined on L, we can obtain a state μ_p on any subinterval L_p, assuming that μ does not vanish on p. It is sufficient to normalize its restriction. More explicitly, let us define, for $\mu \in \Omega(L)$ and $p \in L \setminus \ker \mu$, the state μ_p by letting $\mu_p(x) = \frac{\mu(x)}{\mu(p)}$ for all $x \in L_p$. The fact that several properties of the original state are preserved by this (normalized) restriction is generically called "heredity". For instance, the normalized restriction of a filtering state is always filtering and the normalized restriction of a completely additive state is completely additive [8, Prop. 8.3]. A similar statement need not hold for the wpfa states. When all normalized restrictions of the wpfa states are wpfa, we refer to it as the wpfa-heredity.

Definition 3.1 [29] An OMP L is called *wpfa-hereditary* if

$$\forall \mu \in \Omega_{wpfa}(L)\ \forall p \in L \setminus \ker \mu : \mu_p \in \Omega_{wpfa}(L_p)\ .$$

The latter condition can be equivalently written in the following form:

$$\forall \mu \in \Omega_{wpfa}(L)\ \forall p \in L \setminus \ker \mu\ \forall \lambda \in \Omega_c(L_p) : \lambda \not\ll_S \mu_p\ .$$

Proposition 3.2 [8, Prop. 8.8] *If $\Omega_f(L) = \Omega_{wpfa}(L)$, then L is wpfa-hereditary.*

The *Yosida–Hewitt decomposition* of a state is a decomposition to a convex combination of a completely additive state with a wpfa state. The Yosida–Hewitt decomposition always exists but it need not be unique (see [6, 7, 8, 29] for more details).

G. Rüttimann introduced filtering states in [29] with the aim to study the uniqueness of the Yosida–Hewitt decomposition of measures. Filtering states are particular wpfa states, so that the decomposition of a state into a convex combination of a completely additive state with a filtering state is a special kind of the Yosida–Hewitt decomposition. We call it the *Rüttimann decomposition* [8]. Alike the Yosida–Hewitt decomposition, the Rüttimann decomposition need not exist in general, but when it exists, it is unique [8, Cor. 7.4].

Another notion of heredity can be introduced as follows. It should be noted that it has been defined more generally for a face of $\Omega_c(L)$ in [8], here we need it only for the whole of $\Omega_c(L)$.

Definition 3.3 Let L be an OMP. We say that $\Omega_c(L)$ is *#-hereditary* if

$$\forall \mu \in \Omega_{wpfa}(L)\ \forall \nu \in \Omega_c(L)\ \forall p \in L \setminus (\ker \mu \cup \ker \nu)\ \forall \lambda \in \Omega(L_p) : \lambda \not\ll_S \mu_p \text{ or } \lambda \not\ll_S \nu_p\ .$$

In the paper [8], various kinds of decomposition of states defined on OMPs have been studied. One of the main results in this direction is the following proposition.

Proposition 3.4 *Let L be an OMP. Then the following conditions are equivalent:*

1. $\Omega_c(L)$ *is #-hereditary.*
2. *each state on L is a convex combination of a completely additive state and a weakly filtering state,*
3. $\Omega_{wf}(L) = \Omega_{wpfa}(L)$,

Any of these conditions implies the uniqueness of the Yosida–Hewitt decomposition.

The proof follows from the results of [8] which are indicated in Fig. 1. Its principal part is a lemma proved by Rüttimann in [29] which we enclose below in a more general version taken from [8].

$$\begin{tikzcd}
\Omega_f(L) = \Omega_{wpfa}(L) \arrow[r, leftrightarrow] & \Omega(L) = conv\,(\Omega_f(L) \cup \Omega_c(L)) \arrow[d] & \\
& \text{the wpfa-heredity of } L \arrow[d] & \\
\Omega_{wf}(L) = \Omega_{wpfa}(L) \arrow[r, leftrightarrow] & \Omega(L) = conv\,(\Omega_{wf}(L) \cup \Omega_c(L)) \arrow[r, leftrightarrow] \arrow[d] & \#\text{-heredity of } \Omega_c(L) \\
& \text{the uniqueness of the Yosida–Hewitt decomposition} &
\end{tikzcd}$$

Figure 1: Implications between conditions related to completely additive states

Lemma 3.5 [29] *Let L be an OMP. If $p \in L$, $\mu \in J^+(L)$ and $\nu \in J_c^+(L)$ such that $\mu(p) < \nu(p)$, then there exists $q \in L_p \setminus \{0\}$ such that $\mu(x) < \nu(x)$ for all $x \in L_q \setminus \{0\}$.*

4 The class of c-positive OMPs In this section we extend Prop. 3.4 in the case when the OMP in question has "enough" completely additive states. The reason is that in this case various decompositions will be found to coincide.

Definition 4.1 [29] An OMP L is called *c-positive* if

$$\forall p \in L \setminus \{0\} \; \exists \mu \in \Omega_c(L) : \mu(p) > 0.$$

If L is a c-positive OMP, then $\bigcap_{\mu \in \Omega_c(L)} \ker \mu = \{0\}$. Thus, the weakly filtering states coincide with filtering states and the decomposition dealt with in Prop. 3.4 coincides with the Rüttimann decomposition. Many conditions studied in [8, Sections 8 and 9] become equivalent in the class of c-positive OMPs. Among other results, we obtain the reverse implication to that established in Prop. 3.2:

Proposition 4.2 [8] *If L is an OMP that is c-positive and wpfa-hereditary, then $\Omega_f(L) = \Omega_{wpfa}(L)$.*

Let us show by example that the latter proposition would not remain valid without c-positivity.

Example 4.3 Let B be the Borel σ-algebra on the real line. We take the σ-ideal Δ of all meager sets in B. The quotient Boolean algebra $A = B/\Delta$ does not admit any completely additive state, so $\Omega_c(A) = \emptyset$, $\Omega_{wpfa}(A) = \Omega(A)$, and the same holds for any nontrivial interval in A. Thus, A is wpfa-hereditary. On the other hand, there is a state μ on A which is strictly positive, i.e., $\mu(x) > 0$ for all $x \in A \setminus \{0\}$ (see [24]). As μ is not filtering, $\Omega_{wpfa}(A) \neq \Omega_f(A)$.

To present generalizations of [7, 29], and to prepare the stage for the next section, let us formulate some equivalent conditions for positive measures, too. To denote special sets of positive measures, we use the same indices we have used for the sets of states. In particular, we write $J^+_{wpfa}(L)$ to denote the set of all wpfa positive measures on L.

Theorem 4.4 *Let L be a c-positive OMP. The following conditions are equivalent:*

1. *L is wpfa-hereditary.*
2. *$\Omega_c(L)$ is $\#$-hereditary.*
3. (The existence of the Rüttimann decomposition of states) *Each state on L is a convex combination of a completely additive state and a filtering state.*
4. (The existence of the Rüttimann decomposition of positive measures) *Each positive measure on L is a sum of a completely additive positive measure and a filtering positive measure.*
5. *Each state on L is a convex combination of a completely additive state and a weakly filtering state.*
6. *Each positive measure on L is a sum of a completely additive positive measure with a weakly filtering positive measure.*
7. *$\Omega_{wpfa}(L) = \Omega_f(L)$.*
8. *$J^+_{wpfa}(L) = J^+_f(L)$.*

The proof follows from the equality $\Omega_f(L) = \Omega_{wf}(L)$ (valid for c-positive OMPs) and from the relations implied by Fig. 1.

The Hilbert lattices form an important class of c-positive OMLs (to show it, one uses the celebrated Gleason Theorem [12]). The following theorem due to Aarnes proves the conditions of Th. 4.4 for Hilbert lattices.

Theorem 4.5 [1] *Let L be the lattice of projections in a real or complex Hilbert space H. Each state on L can be uniquely expressed as a convex combination of a completely additive state with a filtering state (i.e., with a state which vanishes at all finite-dimensional projections of H).*

The latter theorem can be generalized to Jordan measures, resp. σ-additive Jordan measures, see [10, Ths. 3.2.28, 3.2.29]. As an alternative approach, a similar result is obtained in [4] using Th. 2.14 and 3.8 from Takesaki's monograph [32]. A further generalization to the OMP of splitting subspaces of an inner product space is done in [9] and [10, Th. 4.3.4], see also [27]. Since the splitting subspace OMP does not have any completely additive state if the inner product space is not Hilbert, regular Jordan measures have to be used instead of completely additive states. We refer to [15] and [27] for related results on completeness of inner product spaces and its relations to posets of subspaces.

The conditions of Th. 4.4 imply the uniqueness of the Yosida–Hewitt decomposition. Also, in this case of c-positive OMPs, the reverse implication does not hold as we show in Section 6. First we need to introduce some techniques.

5 Constructions with orthomodular lattices In this sections we summarize construction techniques that will be used in the sequel. Let us refer to [3, 17, 22, 23, 26] for details.

Proposition 5.1 *Let $\mathcal{F}$ be a family of OMLs. Let us consider the cartesian product $L = \prod_{K\in\mathcal{F}} K$ and let us endow it with the ordering $\leq_L$ and orthocomplementation ${}'^L$ defined pointwise, i.e., for all $a, b \in L$, $a = (a_K)_{K\in\mathcal{F}}$, $b = (b_K)_{K\in\mathcal{F}}$, let us define*

$$a \leq_L b \Longleftrightarrow \forall K \in \mathcal{F} : a_K \leq_K b_K \ ,$$

$$a = b'^L \Longleftrightarrow \forall K \in \mathcal{F} : a_K = b_K{}'^K \ .$$

Let us define 0_L, resp. 1_L, to be the element of $\prod_{K\in\mathcal{F}} K$ which has the K-th coordinate equal to 0_K, resp. 1_K, for all $K \in \mathcal{F}$. Then $(L, \leq_L, 0_L, 1_L, {}'^L)$ is an OML called the product *of the family $\mathcal{F}$.*

If $\mathcal{F}$ is finite, then a function $\mu\colon L \to [0,1]$ is a state iff it is a convex combination of functions of the form $\mu_Q\colon (a_K)_{K\in\mathcal{F}} \mapsto \nu_Q(a_Q)$, where $Q \in \mathcal{F}$, $\nu_Q \in \Omega(Q)$.

Definition 5.2 [23] Let $\mathcal{F}$ be a collection of orthomodular lattices such that for each $K, L \in \mathcal{F}$ the intersection $K \cap L$ is a subalgebra of both K and L and, moreover, the operations of K and of L coincide on $K \cap L$. (In particular, all OMLs in $\mathcal{F}$ have the same least element, 0, and the same greatest element, 1.) Put $P = \bigcup_{K\in\mathcal{F}} K$ and define the binary relation $\leq_P$ and the unary operation ${}'^P$ as follows:

$$a \leq_P b \Longleftrightarrow \exists K \in \mathcal{F} : (a, b \in K,\ a \leq_K b) \ ,$$

$$a = b'^P \Longleftrightarrow \exists K \in \mathcal{F} : (a, b \in K,\ a = b'^K) \ .$$

Then $(P, \leq_P, 0, 1, {}'^P)$ is called the *pasting* of the collection $\mathcal{F}$.

Sufficient conditions for a pasting to be an orthomodular lattice are given in [23]. Here we shall apply only two very special cases.

Proposition 5.3 *Let $\mathcal{F}$ be a family of OMLs. Let us take the family $\mathcal{G}$ consisting of copies of all OMLs in $\mathcal{F}$ arranged in such a way that they are disjoint except they have the same least element, 0, and the same greatest element, 1. Thus, for each $K, M \in \mathcal{G}$, $K \neq M$, we have $K \cap M = \{0, 1\}$. Then the pasting, P, of $\mathcal{G}$ is an OML called the* horizontal sum (0-1-pasting) *of the family $\mathcal{F}$.*

A function $\mu \colon P \to [0,1]$ is a state iff its restriction to each element of $\mathcal{G}$ is a state.

The following technical tool was introduced in [23, Th. 6.1] and dealt in detail in [18]. Here we add to it a characterization of Jordan measures.

Proposition 5.4 *Let K, L be OMLs. Suppose that some element $a \in K$ is an atom in K. Write $b = a'^K$ and put $M = K_b \times L$. For all $c \in K_b$, let us identify $c \in K$ with $(c, 0_L) \in M$ and $c \vee_K a \in K$ with $(c, 1_L) \in M$. Then the pasting P of $\{K, M\}$ is an OML. We say that P originated by the* substitution *of the atom a in K with the OML L.*

Let $\mu \in J(K)$ and $\nu \in J(L)$ such that $\nu(1_L) = \mu(a)$. Then there is a $\lambda \in J(P)$ such that $\lambda | K = \mu$ and $\lambda((c, d)) = \mu(c) + \nu(d)$ for all $c \in K_b$, $d \in L$. Moreover, each Jordan measure on P can be expressed in this form.

After performing the above substitution, the interval P_a becomes isomorphic to L, and the ordering inherited from K is preserved and extended canonically to P.

Definition 5.5 Let K be an OML. By a *block* in K we mean a maximal Boolean subalgebra of K. (Each OML is the union of its blocks, see [3, 17, 26].) Let a, b be elements of K. Let us define their *distance*, $\mathbf{d}_K(a, b)$, in K as the minimal n for which there exists a sequence $(B_1, \ldots, B_n)$ of blocks in K such that $a \in B_1$, $b \in B_n$ and $B_i \cap B_{i+1} \supsetneqq \{0, 1\}$ for $i = 1, \ldots, n-1$. Let us define $\mathbf{d}_K(a, b) = \infty$ if no such sequence exists, and we put $\mathbf{d}_K(a, a) = 0$ for all $a \in K$.

Proposition 5.6 *Let K be an OML. Let M be a set of atoms of K. Let $\mathbf{d}_K(a, b) \geq 5$ for each $a, b \in M$. Let us define an equivalence relation, $\approx$, on K such that $a \approx b$ iff*

- *$a = b$ or*
- *$a, b \in M$ or*
- *$a'^K, b'^K \in M$.*

For each $a \in K$, let us write $[a] = \{b \in K : b \approx a\}$. Let us take the set $L = \{[a] : a \in K\}$, and endow it with the unary operation $'^L$ and a relation $\leq_L$ defined by

$$[a]'^L = [a'^K] \, ,$$

$$[a] \leq_L [b] \iff \exists a_1 \in [a] \; \exists b_1 \in [b] : a_1 \leq_K b_1 \, .$$

Then $(L, \leq_L, 0, 1, '^L)$ is an OML. Moreover, the atoms (resp., the blocks) of L are images of the atoms (resp., the isomorphic blocks) of K under the quotient mapping. In this case, we say that L originated by identification of atoms *of M in K.*

States on L correspond to states on K which attain equal values on M.

Remark 5.7 Prop. 5.6 can be applied subsequently to more sets of atoms M_1, M_2, etc. The only problem is that the distance of the atoms of M_2 in K/M_1 may be smaller than in K. Thus, it is necessary to choose these sets in such a way that the assumption on minimal distance is not violated in the procedure.

6 Uniqueness of the Yosida–Hewitt decomposition This section brings an example of a c-positive OML with a unique Yosida–Hewitt decomposition but with the condition $\Omega_f(L) = \Omega_{wpfa}(L)$ being violated. Some steps of the construction we formulate as lemmas. These lemmas will be used in Section 7, too. Let us denote by $\mathcal{A}(L)$ the set of all atoms of L. A state μ is called *strictly positive* iff $\ker \mu = \{0\}$.

Lemma 6.1 ([21], see also [22, 34]) *There is a finite OML, J, which admits exactly one state, μ, and which satisfies the following condition:*

$$\forall a \in \mathcal{A}(J) : \mu(a) = 1/3 \, .$$

Moreover, J contains atoms a, b with $\mathbf{d}_J(a,b) \geq 2$.

PROOF: Let us take the OML J with 44 atoms, $a_0, \ldots, a_{43}$ and 44 blocks corresponding to the following maximal orthogonal sets of atoms:

$$\{a_{2i}, a_{2i+1}, a_{2i+2}\}, \quad \{a_{2i-7}, a_{2i}, a_{2i+13}\}, \quad i = 0, \ldots, 21,$$

where the indices are evaluated modulo 44. Obviously, there is a state on J attaining $1/3$ at each atom. A computer proof [16] or the arguments in [21] show that this is the only state on J. The distance of the atoms a_0, a_4 is 2. □

Lemma 6.2 *There is a finite OML, K, such that K admits exactly one state, μ, and such that μ is strictly positive. Moreover, there are atoms $b, c \in \mathcal{A}(K)$ satisfying $\mu(b) = 1/3$, $\mu(c) = 1/9$, and $\mathbf{d}_K(b,c) \geq 2$.*

PROOF: Let us take two disjoint copies, J_1 and J_2, of the OML J constructed in Lemma 6.1 and atoms $a, b \in \mathcal{A}(J_1)$ with $\mathbf{d}_{J_1}(a,b) \geq 2$. In the OML J_1, let us substitute a with J_2 (Prop. 5.4), which yields an OML, K. The only state on J_1 attains $1/3$ at a, so, after the substitution of a, each state on K attains $1/9$ at all atoms of the interval K_a. This determines the unique state on K. We may take for c any atom of K_a. □

Lemma 6.3 *There is a finite OML, H, and atoms $b, c \in \mathcal{A}(H)$ with the following properties:*

$$\mathbf{d}_H(b,c) = 4 \, ,$$

$$\forall \mu \in \Omega(H) : \mu(b) = \mu(c) \leq 1/3 \, ,$$

$$\forall r \in [0, 1/3] \; \exists! \mu \in \Omega(H) : \mu(b) = r \, .$$

Moreover, for $r \in (0, 1/3)$ the state μ in the latter formula is strictly positive.

PROOF: Let us take the OML J with atoms $b_J, c_J \in \mathcal{A}(J)$ from Lemma 6.1. Let us form the product $H_0 = 2 \times J$ (Prop. 5.1) and let us choose its atoms $b_0 = (0, b_J)$, $c_0 = (0, c_J)$. Then H_0 satisfies all the conditions except for the first one, $\mathbf{d}_{H_0}(b_0, c_0) = 2$. Let us take a disjoint copy, H_1, of H_0, with atoms $b_1, c_1 \in \mathcal{A}(H_1)$ corresponding to $b_0, c_0 \in \mathcal{A}(H_0)$. In the horizontal sum of H_0, H_1 (Prop. 5.3), let us identify c_0 with b_1 (Prop. 5.6). We obtain an OML, H. Choosing $b = b_0$, $c = c_1$, we see that $\mathbf{d}_H(b,c) = 4$ and the conditions of the lemma are satisfied. □

Lemma 6.4 *There is a finite OML, M, and atoms $b, c \in \mathcal{A}(M)$ with the following properties:*

$$\mathbf{d}_M(a,b) = 4 ,$$

$$\forall \mu \in \Omega(M) : \mu(b) = 3\mu(c) \leq 1/3 ,$$

$$\forall r \in [0, 1/3]\ \exists! \mu \in \Omega(M) : \mu(b) = r .$$

Moreover, for $r \in (0, 1/3)$ the state μ in the latter condition is strictly positive.

PROOF: The construction is almost the same as in the proof of Lemma 6.3. The only difference is that for H_1 we take $2 \times K$ instead of $2 \times J$, where K is the OML from Lemma 6.2. Thus, we obtain $\mu(c) = \mu(b)/3$. □

We are now ready to construct the desired example. Let us use the symbol $\mathbb{Z}$ for the set of all integers and, for $n \in \mathbb{Z}$, let us set $\mathbb{N}_n = \mathbb{Z} \cap [n, \infty)$.

Example 6.5 There is a c-positive OML, L, with a unique Yosida–Hewitt decomposition and with $\Omega_f(L) \neq \Omega_{wpfa}(L)$.

For $n \in \mathbb{N}_2$, let us take copies M_n of the OML M from Lemma 6.4, with atoms $b_n, c_n \in \mathcal{A}(M_n)$ corresponding to $b, c \in \mathcal{A}(M)$. For $n \in \{0, 1\}$, let us take copies M_n of the OML H from Lemma 6.3, with atoms $b_n, c_n \in \mathcal{A}(M_n)$ corresponding to $b, c \in \mathcal{A}(H)$. In the horizontal sum (Prop. 5.3) of $\{M_n\}_{n \in \mathbb{N}_0}$, let us identify c_{n-1} with b_n for all $n \in \mathbb{N}_1$ (Prop. 5.6). We obtain an OML P and atoms $b_n \in \mathcal{A}(P)$, $n \in \mathbb{N}_0$, with the following properties:

$$\mathbf{d}_P(b_k, b_n) \geq 4 \text{ whenever } k \neq n ,$$

$$\forall \mu \in \Omega(P) : \mu(b_0) = \mu(b_1) = \mu(b_2) = 3\mu(b_3) = \ldots = 3^{n-2}\mu(b_n) = \ldots ,$$

$$\forall r \in [0, 1/3]\ \exists! \mu \in \Omega(P) : \mu(b_0) = r .$$

Moreover, for $r \in (0, 1/3)$ the state μ from the latter condition is strictly positive. (These properties are direct consequences of Lemmas 6.3, 6.4.)

Let us take a Boolean algebra B with atoms q_n, $n \in \mathbb{N}_0$, such that B contains exactly all suprema of finitely many atoms and their complements. In the horizontal sum (Prop. 5.3) of P and B, let us identify (Prop. 5.6) b_n with q_n for all $n \in \mathbb{N}_0$. The result of this construction is the desired OML L. As all atoms of L are atoms of P, each state on L is uniquely determined by the value $\mu(b_0) \in [0, 1/3]$. The block of L corresponding to B is the only infinite block and, also, the only block of L which is not a block of P. This gives rise to the restriction condition

$$\sum_{n \in \mathbb{N}_0} \mu(b_n) \leq 1 .$$

Moreover, μ is completely additive iff the equality holds. All summands in the latter formula can be expressed in terms of $\mu(b_0)$. We then obtain

$$3\mu(b_0) + \mu(b_0) \cdot \sum_{k=1}^{\infty} 3^{-k} \leq 1$$

which is equivalent to

$$\mu(b_0) \leq \frac{2}{7} .$$

Hence $\mu(b_0)$ can be an arbitrary number from the interval $[0, 2/7]$. The state space of L is a line segment with extreme points ν, λ satisfying $\nu(b_0) = 0$, $\lambda(b_0) = 2/7$. The state λ is the only completely additive state on L. Thus $\Omega_c(L) = \{\lambda\}$ but $\Omega_{wpfa}(L) = \{\nu\}$. The Yosida–Hewitt decomposition is unique. As L contains finite blocks, it does not admit any filtering state. Moreover, λ is strictly positive and $\Omega_f(L) = \emptyset \neq \Omega_{wpfa}(L)$. Thus the conditions of Th. 4.4 are not satisfied. $\square$

7 Jordan decomposition of completely additive measures In this section we give a positive answer to a question posed by G. Rüttimann (presented, e.g., at the conference Quantum Strucutures '94, Prague, 1994):

> Is there a completely additive Jordan measure on an OML which does not allow for a Jordan decomposition into completely additive positive measures?

The latter question appeared to be related to the type of decomposition studied in the previous sections, although Jordan decomposition is not of the same type. The above results enabled to obtain a negative answer for wpfa-hereditary OMPs which are c-positive. The following theorem was proved by Rüttimann in [29, Ths. 6.4, 6.5] for OMLs and orthocomplete OMPs. Besides a generalization to OMPs, we give here quite a new and considerably simplified proof based on Th. 4.4.

Proposition 7.1 *If L is a c-positive wpfa-hereditary OMP, then*

$$J_c(L) = J_c^+(L) - J_c^+(L) \ .$$

PROOF: Let $\mu \in J_c(L)$. As $J_c(L) \subseteq J(L) = J^+(L) - J^+(L)$, there are positive measures $\lambda, \nu \in J^+(L)$ such that $\mu = \lambda - \nu$. According to our assumption and Th. 4.4, each positive measure on L has a unique Rüttimann decomposition. In particular, $\lambda = \alpha + \beta$, $\nu = \gamma + \delta$ for uniquely determined positive measures $\alpha, \gamma \in J_c^+(L)$ and $\beta, \delta \in J_f^+(L)$. From $\mu = (\alpha + \beta) - (\gamma + \delta)$, we obtain the equality

$$\mu - \alpha + \gamma = \beta - \delta \ .$$

The left-hand side is completely additive, the right-hand side is known to be filtering (Prop. 2.5). According to [8, Prop. 7.3], the only completely additive filtering Jordan measure is the zero measure, so $\mu - \alpha + \gamma = 0$. We obtain $\mu = \alpha - \gamma \in J_c^+(L) - J_c^+(L)$ which is the desired Jordan decomposition into a difference of positive completely additive measures. $\square$

The above theorem used two additional assumptions—wpfa heredity and c-positivity. It remains an open problem whether one of these conditions can be dropped.

In the rest of this section, we shall present a positive answer to the original Rüttimann's question, i.e., we shall exhibit a completely additive Jordan measure on an OML which does *not* allow for a Jordan decomposition into completely additive positive measures. We shall use Lemma 6.3 and the following lemma:

Lemma 7.2 *There is a finite OML, Y, with an atom y such that each Jordan measure $\mu \in J(Y)$ is uniquely determined by $\mu(1)$ and satisfies $\mu(y) = 0$.*

PROOF: According to [21] (an alternative example is given in [34]), there is a finite OML X which does not admit a nonzero measure with values in any group. In particular, $\Omega(X) = \emptyset$, $J(X) = \{0\}$. Following the method of [25], we can easily show that it is sufficient to take for Y the product $2 \times X$ and for y any atom of the factor corresponding to X. $\square$

Example 7.3 There is an OML L and a completely additive Jordan measure η on L such that η cannot be expressed as a difference of two completely additive positive measures.

Let us take three disjoint copies, Y_1, Y_2, Y_3, of the OML H of Lemma 6.3. Let us denote by $b_n, c_n \in \mathcal{A}(Y_n)$, $n = 1, 2, 3$, the atoms corresponding to the atoms $b, c \in \mathcal{A}(H)$. Further, let us take countably many disjoint copies, Y_n, $n \in \mathbb{N}_4$, of the OML Y of Lemma 7.2. Let us denote by $b_n \in \mathcal{A}(Y_n)$, $n \in \mathbb{N}_4$, the atoms corresponding to $b \in \mathcal{A}(Y)$.

Let us take a Boolean algebra B with atoms q_n, $n \in \mathbb{N}_1$, such that B contains exactly all suprema of finitely many atoms and their complements. In the horizontal sum (Prop. 5.3) of B and $\{Y_n\}_{n\in\mathbb{N}_1}$ we identify (Prop. 5.6) b_n with q_n for all $n \in \mathbb{N}_1$. The result is the desired OML L.

To carry on the argument, let us derive conditions equivalent to the complete additivity of a Jordan measure $\mu \in J(L)$. As B is the only infinite block of L, μ is completely additive iff its restriction to B is completely additive, i.e., iff

$$\text{(E1)} \qquad \sum_{n\in\mathbb{N}_1} \mu(b_n) = \mu(1)\ .$$

For each $n \in \mathbb{N}_4$, the restriction $\mu|Y_n$ is a Jordan measure on Y_n. Hence $\mu(b_n) = 0$ and (E1) may be simplified to the form

$$\text{(E2)} \qquad \mu(b_1) + \mu(b_2) + \mu(b_3) = \mu(1)\ .$$

If, moreover, μ is positive, then the restrictions $\mu|H_i$, $i = 1, 2, 3$, are multiples of states on H_i. According to Lemma 6.3,

$$\text{(E3)} \qquad \mu(b_i) \le \frac{\mu(1)}{3}, \qquad i = 1, 2, 3\ .$$

Thus, the complete additivity of a positive measure μ is equivalent to the condition

$$\text{(E4)} \qquad \mu(b_1) = \mu(b_2) = \mu(b_3) = \frac{\mu(1)}{3}\ .$$

As a consequence, any linear combination of completely additive positive measures on L satisfies (E4). In particular, all Jordan measures from $J_c^+(L) - J_c^+(L)$ satisfy (E4).

According to Lemma 6.3, for each $r_1, r_2, r_3 \in [0, 1/3]$, there is exactly one state $\mu \in \Omega(L)$ such that $\mu(b_1) = r_1$, $\mu(b_2) = r_2$, $\mu(b_3) = r_3$. We uniquely determine two states, $\lambda, \nu \in \Omega(L)$, by the conditions

$$\lambda(b_1) = 1/3, \quad \lambda(b_2) = 0, \quad \lambda(b_3) = 0\ ,$$
$$\nu(b_1) = 0, \quad \nu(b_2) = 1/3, \quad \nu(b_3) = 0\ .$$

They are not completely additive because

$$\lambda(b_1) + \lambda(b_2) + \lambda(b_3) = \nu(b_1) + \nu(b_2) + \nu(b_3) = 1/3 \neq 1 = \lambda(1) = \nu(1)\ .$$

Their difference, $\eta = \lambda - \nu$, satisfies

$$\eta(b_1) = 1/3, \quad \eta(b_2) = -1/3, \quad \eta(b_3) = 0, \quad \eta(1) = 0.$$

According to (E2), η is completely additive, $\eta \in J_c(L)$. Since it does not satisfy (E4), it cannot be expressed as a difference of two positive measures. Hence $\eta \notin J_c^+(L) - J_c^+(L)$, and η is a completely additive Jordan measure which does not allow for a Jordan decomposition into a difference of completely additive positive measures. □

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