SIMPLE LEFT SYMMETRIC ALGEBRAS OVER A REDUCTIVE LIE ALGEBRA

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Abstract. In [Ba], [Bu] and [M], we studied the structures of a left symmetric algebra over a real reductive Lie algebra.

In this paper, we shall give some examples of simple left symmetric algebras over a reductive Lie algebra.

I. Preliminaries.

[A] Let $\mathfrak{g}$ be a Lie algebra over K of dimension n and E^n be an affine space over K of dimension n, where K denotes the field R of all real numbers or the field C of all complex numbers.

Let $\rho = (\varphi, \pi)$ be an affine representation of $\mathfrak{g}$ in E, where $\varphi(a)$ (resp. $\pi(a)$) denotes the linear (resp. translation) part of $\rho(a)$ $(a \in \mathfrak{g})$. ρ is called *admissible affine representation of* $\mathfrak{g}$ *in* E if π is a linear isomorphism of $\mathfrak{g}$ onto E. For a given linear representation φ of $\mathfrak{g}$ in E, if there exists a point P of E such that $\pi(x) = \varphi(x)P$ $(x \in \mathfrak{g})$ is a linear isomorphism of $\mathfrak{g}$ onto E, φ is called an *admissible affine representation of* $\mathfrak{g}$ *in* E *at the point* P.

Let A be a left symmetric algebra over $\mathfrak{g}$. Denote by $L(a)$ (resp. $R(a)$) the left (resp. right) multiplication of A by an element a. Then the mapping $\tilde{L}$ of $\mathfrak{g}$ into the Lie algebra $\mathrm{aff}(A)$ of all infinitesimal affine transformations on A defined by

$$\tilde{L}(a) = (L(a), a)$$

is an admissible affine representation of $\mathfrak{g}$ in A, which is called *the left affine representation of a left symmetric algebra* A *over* $\mathfrak{g}$.

Let $\rho = (\varphi, \pi)$ be an admissible affine representation of $\mathfrak{g}$ in E. Define a binomial product in $\mathfrak{g}$ by the formula

$$ab = \pi^{-1}(\varphi(a)\pi(b)) \quad (a, b \in \mathfrak{g}).$$

Then the algebra $A = (\mathfrak{g}, \rho)$ with the above multiplication is a left symmetric algebra over $\mathfrak{g}$ ([S], [M]).

[B] For an element $a = (a_{ij}, a_i)$ of $\mathrm{aff}(E)$, denote by $\overline{a}$ a vector field on an affine space $E(x_1, x_2, \ldots, x_n)$ with a system $(x_1, x_2, \ldots, x_n)$ of affine coordinates defined by

$$\overline{a} = -\sum(a_{ij}x_j + a_i)\frac{\partial}{\partial x_i}.$$

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For an affine representation $\rho = (\varphi, \pi)$ of $\mathfrak{g}$ in E, denote by $F_\rho(x)$ (resp. $F_\varphi(x)$) a polynomial on E defined by

$$F_\rho(x)\omega_0 = \overline{\rho(a_1)} \wedge \overline{\rho(a_2)} \wedge \cdots \wedge \overline{\rho(a_n)} \quad \left(\text{resp. } F_\varphi(x)\omega_0 = \overline{\varphi(a_1)} \wedge \overline{\varphi(a_2)} \wedge \cdots \wedge \overline{\varphi(a_n)}\right),$$

where $\{a_i\}$ is a base of $\mathfrak{g}$ and ω_0 denotes the tensor field defined by

$$\omega_0 = \left(\frac{\partial}{\partial x_1}\right) \wedge \left(\frac{\partial}{\partial x_2}\right) \wedge \cdots \wedge \left(\frac{\partial}{\partial x_n}\right).$$

The polynomial $F_\rho(x)$ (resp. $F_\varphi(x)$) is uniquely determined by $(\mathfrak{g}, \rho)$ (resp. $(\mathfrak{g}, \varphi)$), up to a constant multiple. Denote this polynomial by $F_\rho = |\rho(\mathfrak{g})|$ (resp. $F_\varphi = |\varphi(\mathfrak{g})|$) and call it *the polynomial for* $(\mathfrak{g}, \rho)$ (resp. $(\mathfrak{g}, \varphi)$).

For an affine representation $\rho = (\varphi, \pi)$ of $\mathfrak{g}$ in E and an infinitesimal character χ of $\mathfrak{g}$, a polynomial $F(x)$ on E is called a *relative invariant of* $(\mathfrak{g}, \rho)$ (resp. $(\mathfrak{g}, \varphi)$) *corresponding to* χ if the following equality holds:

$$L_{\overline{\rho(a)}}F = \chi(a)F \quad \left(\text{resp. } L_{\overline{\varphi(a)}}F = \chi(a)F\right),$$

where $L_{\overline{X}}$ denotes the Lie differentiation with respect to a vector field $\overline{X}$.

We can prove the following ([M]).

Lemma 1. *Let $\rho = (\varphi, \pi)$ be an affine representation of $\mathfrak{g}$ in E, and F_ρ (resp. F_φ) the polynomial for $(\mathfrak{g}, \rho)$ (resp. $(\mathfrak{g}, \varphi)$). Then F_ρ (resp. F_φ) is a relative invariant of $(\mathfrak{g}, \rho)$ (resp. $(\mathfrak{g}, \varphi)$) corresponding to an infinitesimal character χ defined by*

$$\chi(a) = \operatorname{Tr} \operatorname{ad} a - \operatorname{Tr} \varphi(a) \quad (a \in \mathfrak{g}).$$

For a left symmetric algebra A over $\mathfrak{g}$, we have

$$L(a) - R(a) = \operatorname{ad} a \quad (a \in \mathfrak{g}).$$

Thus we have the following.

Corollary. *Let $F = \left|\tilde{L}(\mathfrak{g})\right|$ be the polynomial for a left symmetric algebra A over $\mathfrak{g}$. Then it is a relative invariant of $(\mathfrak{g}, \tilde{L})$ corresponding to $\chi(a) = -\operatorname{Tr} R(a)$ $(a \in \mathfrak{g})$.*

Lemma 2. *Let F and G be relative invariants of an affine representation $(\mathfrak{g}, \rho)$ in E corresponding to the same infinitesimal character χ. If $(\mathfrak{g}, \rho)$ is admissible, then G coincides with F up to a constant multiple.*

In fact, we have $L_{\overline{\rho(a)}}(G/F) = 0$ $(a \in \mathfrak{g})$. Thus, if $(\mathfrak{g}, \rho)$ is admissible, then G/F is a constant. □

[C] Let A be a left symmetric algebra over a Lie algebra $\mathfrak{g}$, and h a symmetric bilinear form on A. h is called *of Hessian type* ([S]) if, for $x, y, z \in A$, the following equality holds:

$$h(xy, z) + h(y, xz) = h(yx, z) + h(x, yz).$$

Put

$$h(x, y) = \operatorname{Tr} R(xy) \quad (x, y \in A).$$

h is a symmetric bilinear form on A of Hessian type. It is called *the canonical 2-form on A*.
A is called *non degenerate* if the canonical 2-form is non degenerate.

Lemma 3. *Let A be a left symmetric algebra over a Lie algebra $\mathfrak{g}$ satisfying the following conditions:*

(1) *A has an identity e,*

(2) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \{e\}$.

Then non trivial symmetric bilinear forms h_1 and h_2 on A of Hessian type are conformal.

In fact, for $x, y \in A$, there exist $z \in [\mathfrak{g}, \mathfrak{g}]$ and $\alpha \in K$ such that $xy = z + \alpha e$. Moreover, for any symmetric bilinear form h of Hessian type, the following equalities hold:

$$h([x, y], e) = 0 \quad \text{and} \quad h(x, y) = h(e, xy) \quad (x, y \in A).$$

Therefore, we have

$$h_i(x, y) = h_i(e, xy) = \alpha h_i(e, e) \quad (i = 1, 2).$$

□

Lemma 4. *Let B be an ideal of a left symmetric algebra A with a symmetric bilinear form h of Hessian type. Denote by $B^\perp$ the orthogonal complement of B in A with respect to h. Then $B^\perp$ is a subalgebra of A.*

In fact, for $x, y \in B^\perp$ and $b \in B$,

$$h(b, xy) = h(bx, y) + h(x, by) - h(xb, y) = 0.$$

□

Let A be a left symmetric algebra over $\mathfrak{g}$ corresponding to an admissible affine representation $\rho = (\varphi, \pi)$ in E, and F_φ the polynomial for $(\mathfrak{g}, \varphi)$, where there exists a point P of E such that $\pi(a) = \varphi(a)P$ $(a \in \mathfrak{g})$.

Denote by g a tensor field of type $(0, 2)$ on a domain $\Omega = \{x \in E\ ;\ F_\varphi(x) \neq 0\}$ defined by

$$g_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \left(\log |F_\varphi|\right).$$

Denote by h a symmetric bilinear form on $A = (\mathfrak{g}, \rho)$ defined by

$$h(a, b) = g\left(\overline{\rho(a)}, \overline{\rho(b)}\right)\Big|_{x=0} \quad (a, b \in A).$$

h is called a *symmetric bilinear form defined by* F_φ.

We obtain the following ([M]).

Lemma 5. *A symmetric bilinear form h on A defined by the polynomial $F = \left|\tilde{L}(\mathfrak{g})\right|$ for a left symmetric algebra A coincides with the canonical 2-form on A.*

For admissible affine representations $(\mathfrak{g}, \rho)$ and $(\mathfrak{g}, \rho')$ in E, $(\mathfrak{g}, \rho')$ is called *F-equivalent to* $(\mathfrak{g}, \rho)$, if the polynomial $F_{\varphi'}$ for $(\mathfrak{g}, \varphi')$ coincides with the polynomial F_φ for $(\mathfrak{g}, \varphi)$, up to a constant multiple.

By the definition of a symmetric bilinear form defined by F_φ, we obtain the following.

Lemma 6. *For two admissible affine representations $(\mathfrak{g}, \rho)$ and $(\mathfrak{g}, \rho')$ in E, if they are F-equivalent, then the rank of the symmetric bilinear form on $A' = (\mathfrak{g}, \rho')$ defined by $F_{\varphi'}$ coincides with that of the symmetric bilinear form on $A = (\mathfrak{g}, \rho)$ defined by F_φ.*

[D] Let G be a connected Lie group of dimension n over K, $\mathfrak{g}$ its Lie algebra, and E an affine space over K of dimension n.

Denote by Φ a linear representation of G in E, and χ a character of G. A polynomial $F(x)$ on E is called a *relative invariant for (G, Φ) corresponding to χ* if

$$F(\Phi(g)x) = \chi(g)F(x) \quad (x \in E,\ g \in G).$$

Denote by φ (resp. the same letter χ) the induced linear representation (resp. the induced infinitesimal character) of $\mathfrak{g}$. Then F is a relative invariant of $(\mathfrak{g}, \varphi)$ corresponding to χ.

Let Ψ be a mapping of a domain $\Omega = \{x \in E\ ;\ F(x) \neq 0\}$ into an affine space $E^*(y_1, y_2, \ldots, y_n)$ of dimension n defined by

$$y_i = \left(\frac{1}{F(x)}\right)\left(\frac{\partial F(x)}{\partial x_i}\right) \quad (1 \leq i \leq n).$$

Then it can be easily proved that

$$\Psi(\Phi(g)x) = \Phi^*(g)\Psi(x) \quad (x \in \Omega,\ g \in G),$$

where Φ^* denotes the contragradient representation of G in E^*.

Lemma 7. *Let $(\mathfrak{g}, \varphi)$ be an admissible affine representation of $\mathfrak{g}$ in E at a point P, $(\mathfrak{g}, \varphi^*)$ the induced contragradient representation of $\mathfrak{g}$ in E^*, and $A = (\mathfrak{g}, \rho)$ a left symmetric algebra over $\mathfrak{g}$ corresponding to $(\mathfrak{g}, \varphi)$ at P. Then the following conditions are mutually equivalent.*

(1) *$(\mathfrak{g}, \varphi^*)$ is admissible at $Q = \Psi(P)$,*

(2) *the Hessian of the mapping Ψ does not vanish at P,*

(3) *A is non degenerate.*

Proof. Denote by $H(x)$ the Hessian matrix of the mapping Ψ. The mapping Ψ is a diffeomorphism in a neighbourhood of P if and only if $(\mathfrak{g}, \varphi^*)$ is admissible at the point $Q = \Psi(P)$. Moreover, since we have $H(x)_{ij} = g_{ij}$ $(1 \leq i, j \leq n)$, by Lemmas 5 and 6, we obtain the equivalence of (2) and (3). □

[E]
Lemma 8. *Let A be a left symmetric algebra over a Lie algebra $\mathfrak{g}$. Let B be a minimal commutative ideal of A.*

Assume that the Lie algebra $\mathfrak{b}$ of B is contained in the center $\mathfrak{C}$ of $\mathfrak{g}$. Then

(1) *B is simple, or*

(2) *B is nilpotent.*

Proof. Assume that the semi simple part S of an associative algebra B is non trivial. Then S is decomposed into a direct sum $\bigoplus_{i=1}^{r} S_i$ of simple algebras S_i $(1 \leq 1 \leq r)$.

First we shall prove that $r = 1$. In fact, denote by e_i $(1 \leq i \leq r)$ the identity element of S_i. Put $B_i = Be_i$ $(1 \leq i \leq r)$. Then, for $x \in A$ and $b \in B$, since $\mathfrak{b}$ is contained in the center, we have

$$x(be_i) = x(e_i b) = (xe_i)b = (e_i x)b = e_i(xb) \in B_i.$$

Thus B_i is an ideal of A. This implies that $r = 1$ and S is simple.

Next denote by e the identity element of S. Put

$$N_0 = \{n \in N \; ; \; ne = 0\}.$$

Similarly as above, we can easily prove that N_0 is an ideal of A. Thus we obtain that $N_0 = \{0\}$ and e is the identity of B. Now, for $n \in N$ and $x \in A$, we have

$$xn = x(en) = (xe)n \in N,$$

that is, N is an ideal of A. Thus, again by the minimality of B, we have $N = \{0\}$ and $B = S$.

□

[F] In this section, let $\mathfrak{g} = \mathfrak{S} \oplus \mathfrak{C}$ be a reductive Lie algebra over K of dimension n, where $\mathfrak{S}$ (resp. $\mathfrak{C}$) denotes the semi simple ideal ($\neq \{0\}$) (resp. the center) of $\mathfrak{g}$.

Let $\rho = (\varphi, \pi)$ be an admissible affine representation of $\mathfrak{g}$ in E^n, and $A = (\mathfrak{g}, \rho)$ a left symmetric algebra over $\mathfrak{g}$ corresponding to ρ.

Assume that

$$\deg\left(\varphi \,\middle|\, \mathfrak{S}\right) = \dim \mathfrak{g}. \qquad (*)$$

Lemma 9. *Under the assumption $(*)$, let B be a non commutative minimal ideal of A, then there exists a subalgebra $\overline{B}$ of A such that*

(1) *$A = B \oplus \overline{B}$, semi direct sum with $B\overline{B} = 0$,*

(2) *B (resp. $\overline{B}$) has a right identity.*

For the proof, see [M].

Lemma 10. *Under the assumption $(*)$, let B be a non degenerate minimal commutative ideal of A, then there exists an ideal $\overline{B}$ of A such that $A = B \oplus \overline{B}$ (direct sum).*

Proof. Let $\mathfrak{b}$ be the Lie algebra of B. Then, since $\mathfrak{b}$ is contained in the center and B is non degenerate, B is simple, by Lemma 8. Denote by $B^{\perp}$ the orthogonal complement of B with respect to the canonical 2-form h of A. Then, since B is non degenerate, $B^{\perp}$ is a subalgebra of A satisfying $A = B \oplus B^{\perp}$.
Moreover, by the assumption $(*)$, A has a right identity. Thus, by the Lemma below, we have $BB^{\perp} = 0$. This implies that $B^{\perp}$ is an ideal of A. □

Lemma. *Let A be a left symmetric algebra, B an ideal of A, and $\overline{B}$ a subalgebra of A satisfying $A = B \oplus \overline{B}$ (semi direct sum).*

If the following conditions (1) and (2) are satisfied, then $B\overline{B} = 0$.

(1) *$B \perp \overline{B}$ with respect to the canonical 2-form h of A,*

(2) *A (resp. B) has a right identity e (resp. e_1).*

Proof. For $b \in B$, we have $b(e - e_1) = 0$. Thus, by (1), $e_2 = e - e_1$ is an element of $\overline{B}$. Moreover, for $c \in \overline{B}$, we have $ce_2 = c$ and $ce_1 = 0$. This implies that, for $b \in B$ and $c \in \overline{B}$, we have

$$bc = (bc)e_1 = b(ce_1) + (cb)e_1 - c(be_1) = 0.$$

□

[G] Let $E = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ be an affine space over K of dimension n^2, where $V_i = K^n(x_{i1}, x_{i2}, \ldots, x_{in})$ denotes an affine space over K with a system $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ of affine coordinates.

Denote by $F(x)$ the polynomial defined by

$$F(x) = \det(x_1, x_2, \ldots, x_n).$$

Lemma 11. *Let $\overline{X}$ be an infinitesimal linear transformation on E defined by $X = E_n \otimes c$, $(c = (c_{ij}) \in \mathrm{gl}(n, K))$. Then we have*

$$L_{\overline{X}} F = -(\mathrm{Tr}\, X)\, F.$$

In fact, it can be easily proved that

$$L_{\overline{X}} F = \begin{cases} 0, & c = e_{ij} \ \ (i \neq j); \\ -F, & c = e_{ii}, \end{cases}$$

where e_{ij} denotes the matrix unit in $\mathrm{gl}(n, K)$. □

Let $E' = W_1 \oplus W_2 \oplus \cdots \oplus W_{n+1}$ be an affine space over K of dimension $n(n+1)$, where $W_i = K^n(x_{i1}, x_{i2}, \ldots, x_{in})$ denotes an affine space over K of dimension n with a system $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ of affine coordinates.

Denote by $F(i)$ $(1 \leq i \leq n+1)$ the polynomial on E' defined by

$$F(i) = \det(x_1, x_2, \ldots, \hat{x_i}, \ldots, x_{n+1}).$$

Similarly as above, we can easily prove the following.

Lemma 12. *Let $\overline{X}$ be an infinitesimal linear transformation on E' defined by $X = E_n \otimes e_{ij}$, where e_{ij} denotes the matrix unit in* $\mathrm{gl}(n+1, K)$. *Then we have*

$$L_{\overline{X}}F(k) = \begin{cases} -F(k), & i = j \neq k, \\ (-1)^{i-j}F(i), & j = k \neq i, \\ 0, & \textit{otherwise.} \end{cases}$$

II. Let $\mathfrak{g} = \mathfrak{S} \oplus \mathfrak{C}$ be a reductive Lie algebra over K, where $\mathfrak{S}$ (resp. $\mathfrak{C}$) denotes the semi simple ideal ($\neq \{0\}$) (resp. the center) of $\mathfrak{g}$.

In the following sections, we shall give some examples of simple left symmetric algebras $A = (\mathfrak{g}, \varphi)$ over $\mathfrak{g}$.

[A] Let $E = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ be an affine space over K of dimension n^2, where $V_i = K^n(x_{i1}, x_{i2}, \ldots, x_{in})$ denotes an affine space over K with a system $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ of affine coordinates.

Put $\mathfrak{S} = \mathrm{sl}(n, K)$ and $\mathfrak{C} = \{e\}$.

Denote by φ a linear representation of $\mathfrak{g}$ in E defined by

$$\varphi|\mathfrak{S} = \mathrm{id} \otimes E_n,$$
$$\varphi(e) = E_n \otimes a \quad (a \in \mathrm{gl}(n, K)).$$

Denote by F a polynomial on E defined by

$$F = \det(x_1, x_2, \ldots, x_n),$$

and by P a point in E defined by

$$P = (e_1, e_2, \ldots, e_n),$$

where $\{e_i\}$ denotes the canonical base of K^n.

Theorem 1.

(1) $(\mathfrak{g}, \varphi)$ *is admissible at some point in E if and only if* $\mathrm{Tr}\, a \neq 0$.

(2) *If $(\mathfrak{g}, \varphi)$ is admissible, then F is the polynomial for $(\mathfrak{g}, \varphi)$ and $(\mathfrak{g}, \varphi)$ is admissible at P.*

(3) *Let $A = (\mathfrak{g}, \varphi)$ be a left symmetric algebra corresponding to an admissible affine representation $(\mathfrak{g}, \varphi)$ at P. Then A is simple and non degenerate.*

(4) *A has a right identity.*

Proof. (1) It is clear that a Lie subalgebra $\{\overline{s \otimes E_n},\ s \in \mathrm{sl}(n, K)\}$ of the Lie algebra of all infinitesimal linear transformations on E spans the tangent space at P of a hypersurface through P defined by $\{x \in E\ ;\ F(x) = F(P)\}$. Moreover $\overline{E_n \otimes a}$ is transversal to the hypersurface if and only if $\mathrm{Tr}\, a \neq 0$. Thus we obtain (1).

(2) It is clear that F is a relative invariant corresponding to the infinitesimal character $-(\mathrm{Tr}\, \varphi)$. Therefore, if $(\mathfrak{g}, \varphi)$ is admissible, F is the polynomial for $(\mathfrak{g}, \varphi)$, by Lemma 2.

(3) Simplicity is followed from the fact that $\dim \mathfrak{C} = 1$. Moreover, since F coincides with the polynomial for a non degenerate associative algebra $\mathrm{gl}(n, K)$, $A = (\mathfrak{g}, \varphi)$ is non degenerate, by Lemma 7.

(4) is followed from the fact that $\deg \varphi|\mathfrak{S} = \dim \mathfrak{g}$. □

[B] Let $E = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be an affine space over K of dimension $\sum_{i=1}^{r} n_i^2$, where

$$V_i = \bigoplus_{j=1}^{n_i} V_{ij} \quad (1 \leq i \leq r),$$
$$V_{ij} = K^{n_i}(x(i)_{j1}, x(i)_{j2}, \ldots, x(i)_{jn_i}) \ (1 \leq i \leq r,\, 1 \leq j \leq n_i)$$

denotes an affine space over K with a system $x(i)_j = (x(i)_{j1}, x(i)_{j2}, \ldots, x(i)_{jn_i})$ of affine coordinates.

Let $\mathfrak{g} = \mathfrak{S} \oplus \mathfrak{C}$ be a reductive Lie algebra over K, where

$$\mathfrak{S} = \bigoplus_{i=1}^{r} \mathfrak{S}_i, \quad \mathfrak{S}_i = \mathrm{sl}(n_i, K) \quad (1 \leq i \leq r,\, n_i \geq 2),$$

$\mathfrak{C} = \{e_1, e_2, \ldots, e_r\}$ denotes the center of $\mathfrak{g}$ spanned by $\{e_1, e_2, \ldots, e_r\}$ over K.

Denote by P a point of E defined by $P = \sum_{i=1}^{r} P_i$, where $P_i = (e(i)_1, e(i)_2, \ldots, e(i)_{n_i}) \in V_i$ $(1 \leq i \leq r)$, $\{e(i)_1, e(i)_2, \ldots, e(i)_{n_i}\}$ denotes the canonical base of K^{n_i}.

Denote by φ a linear representation of $\mathfrak{g}$ in E defined as follows:

$$\varphi|V_i = \varphi_i \quad (1 \leq i \leq r),$$
$$\varphi_i(s_j) = \delta_{ij}\left(s_j \otimes E_{n_j}\right) \quad (s_j \in \mathfrak{S}_j),$$
$$\varphi_i(e_j) = E_{n_i} \otimes a(j,i) \quad (1 \leq i, j \leq r),$$

where $a(j,i) = (a(j,i)_k)_{k=1,2,\ldots,n_i}$ denotes an element of $D(n_i, K)$ of all diagonal matrices in $\mathrm{gl}(n_i, K)$.

Denote by $F(i)$ $(1 \leq i \leq r)$ a polynomial on V_i defined by

$$F(i) = \det(x(i)_1, x(i)_2, \ldots, x(i)_{n_i}),$$

and put $F = \prod_{i=1}^{r} F(i)$.

Theorem 2.

(1) *$(\mathfrak{g}, \varphi)$ is admissible at some point in E if and only if* $\det(\mathrm{Tr}\, a(j,i))_{i,j=1,2,\ldots,r} \neq 0$.

(2) *If $(\mathfrak{g}, \varphi)$ is admissible at some point in E, then F is the polynomial for $(\mathfrak{g}, \varphi)$ and $(\mathfrak{g}, \varphi)$ is admissible at P.*

(3) *Let $A = (\mathfrak{g}, \varphi)$ be a left symmetric algebra over $\mathfrak{g}$ corresponding to an admissible affine representation $(\mathfrak{g}, \varphi)$ at P. Then A is non degenerate.*

(4) *If B is a proper ideal of A, then it is expressed as a sum $\bigoplus_{j=1}^{s} A_{i_j}$, where $\{i_1, i_2, \ldots, i_s\}$ is a subset of $\{1, 2, \ldots, r\}$ and $A_i = \pi^{-1}(V_i)$ $(1 \leq i \leq r)$.*

Proof. (1) it is clear that a Lie subalgebra $\left\{\overline{\varphi(s)}\,;\ s \in \mathfrak{S}\right\}$ of the Lie algebra of all infinitesimal linear transformations on E spans the tangent space at P of a submanifold of E defined by

$$\{x \in E\ ;\ F(i)(x) = F(i)(P),\ \ 1 \leq i \leq r\}.$$

Moreover, we have

$$L_{\overline{\varphi(e_j)}}F(i) = -(\operatorname{Tr} a(j,i))F(i) \quad (1 \leq i,j \leq r).$$

This implies (1).

(2) is followed from Lemma 2.

(3) By (1), a relative invariant F corresponding to an infinitesimal character $-(\operatorname{Tr}\varphi)$ coincides with the polynomial for $(\mathfrak{g},\varphi)$. Moreover, since the symmetric bilinear form on V_i defined by $F(i)$ $(1 \leq i \leq r)$ is non degenerate, A is non degenerate.

(4) Denote by $A_i = \pi^{-1}(V_i)$ the inverse image of V_i of the linear isomorphism π defined by $\pi(x) = \varphi(x)P$ $(x \in \mathfrak{g})$. Then A_i is a left ideal of A. Denote by $\mathfrak{g}_i$ $(1 \leq i \leq r)$ the Lie algebra of A_i. Then, since $\mathfrak{g}_i \supset \mathfrak{S}_i$ and the codimension of $\mathfrak{S}_i$ in $\mathfrak{g}_i = 1$, A_i is a simple subalgebra of A.

Let B be a proper ideal of A, and $\mathfrak{b}$ the Lie algebra of B. Because of $\deg\varphi|\mathfrak{S} = \dim\mathfrak{g}$, we have $\mathfrak{b} \not\supset \mathfrak{S}$. Moreover, because of simplicity of A_i, if $\mathfrak{b} \not\supset \mathfrak{S}_i$, then we have $\mathfrak{b} \cap \mathfrak{S}_i = \{0\}$. Thus, after a suitable choice of indeces if necessary, we may assume that there exists an integer s $(1 \leq s \leq r)$ such that

$$\mathfrak{b} \supset \mathfrak{S}_i,\ \ 1 \leq i \leq s \text{ and } \mathfrak{b} \cap \mathfrak{S}_i = \{0\}\,,\ \ s+1 \leq j \leq r.$$

Assume that $\mathfrak{b} \supset \mathfrak{S}_i$. Then, since

$$\pi(\mathfrak{S}_i) \subsetneqq \pi(\mathfrak{S}_i\mathfrak{S}_i) = \varphi(\mathfrak{S}_i)\pi(\mathfrak{S}_i) \subset \pi(\mathfrak{b}),$$

we have $\pi(\mathfrak{b}) \supset V_i$. This completes the proof of (4). □

Example. In the above theorem, put $\{\varphi(e_i)\}_{1\leq i\leq r}$ as follows:

Non vanishing terms of $\{a(i,j)_k\}$ are

$$\begin{cases} a(1,1)_1 = 1, & a(1,r)_2 = 2r, \\ a(i,i)_1 = 2i-1, & a(i,i-1)_2 = 2i-2 \ \ (2 \leq i \leq r). \end{cases}$$

Then $(\mathfrak{g},\varphi)$ is admissible and the left symmetric algebra $A = (\mathfrak{g},\varphi)$ corresponding to an admissible affine representation $(\mathfrak{g},\varphi)$ at P is simple.

[C] Let V be an affine space over K of dimension m. A commutative algebra Δ over K is called *a commutative algebra ober* V, if Δ is a commutative subalgebra of $\mathrm{gl}(V)$ consisting of upper triangular linear transformations of V with respect to some fixed base of V such that

(1) $\dim\Delta = m$,

(2) the semi simple part of the algebra Δ is spanned by the identity transformation of V.

Let $a = (a_{ij})$ and b_i $(2 \leq i \leq m)$ be matrices in $\mathrm{gl}(m,K)$ expressed as

$$a_{ij} = \begin{cases} 1, & j = i+1, \\ 0, & \text{otherwise, and} \end{cases}$$
$$b_i = e_{1i} \quad (2 \leq i \leq m),$$

where e_{ij} denotes the matrix unit in $\mathrm{gl}(m,K)$. Then a commutative algebra over K spanned by $\{\mathrm{id}, a, a^2, \ldots, a^{m-1}\}$ (resp. $\{\mathrm{id}, b_2, \ldots, b_m\}$) is a commutative algebra over K^m. We call it *a commutative algebra over* K^m *of 1st kind* (resp. *2nd kind*).

Let $(m_1, m_2, \dots, m_r)$ be a r-tuple of positive integers m_i such that $m_1+m_2+\dots+m_r = m$. A commutative algebra Δ over V is called of type $(m_1, m_2, \dots, m_r)$ if there exists a direct sum decomposition $V = \bigoplus_{i=1}^{r} V_i$ and a set of commutative algebras $\Delta(i)$ over V_i such that

(1) $\dim V_i = m_i \quad (1 \leq i \leq r)$,

(2) $\Delta = \bigoplus_{i=1}^{r} \Delta(i)$.

A commutative algebra $\Delta = \bigoplus_{i=1}^{r} \Delta(i)$ over V of type $(m_1, m_2, \dots, m_r)$ is called *of 1st kind* (resp. *2nd kind*) if $\Delta(i)$ is of 1st kind for any i (resp. $r = 1$ and Δ is of 2nd kind).

Let $\mathfrak{g} = \mathfrak{S} \oplus \mathfrak{C}$ be a reductive Lie algebra of dimension $n(n+1)$, where $\mathfrak{S} = \mathrm{sl}(n, K)$ and $\mathfrak{C}$ denotes the center of $\mathfrak{g}$ of dimension $n+1$.

Denote by φ a linear representation of $\mathfrak{g}$ into an affine space E over K of dimension $n(n+1)$ defined as follows:

(1) $\varphi|\mathfrak{S} = \mathrm{id} \otimes E_{n+1}$,

(2) there exists a r-tuple $(m_1, m_2, \dots, m_r)$ of positive integers m_i such that $m_1 + m_2 + \dots + m_r = n+1$ such that

$$\varphi|\mathfrak{C} = E_n \otimes \varphi_0,$$

where $\varphi_0(\mathfrak{C})$ is a commutative algebra over K^{n+1} of type $(m_1, m_2, \dots, m_r)$. We call $(\mathfrak{g}, \varphi)$ *of type* $(m_1, m_2, \dots, m_r)$.

Let $E = \bigoplus_{i=1}^{n+1} V_i$ be a direct sum decomposition of E, where $V_i = K^n(x_{i1}, x_{i2}, \dots, x_{in})$ $(1 \leq i \leq n+1)$ denotes an affine space over K with a system $x_i = (x_{i1}, x_{i2}, \dots, x_{in})$ of affine coordinates.

Denote by $F(i)$ a polynomial defined by

$$F(i) = \det(x_1, x_2, \dots, \hat{x_i}, \dots, x_{n+1}).$$

By Lemma 12, it is easily showed that if $(\mathfrak{g}, \varphi)$ is of type $(m_1, m_2, \dots, m_r)$, then

$$F = \prod_{j=1}^{r} F(i_j)^{m_j}$$

is a relative invariant of $(\mathfrak{g}, \varphi)$ corresponding to the infinitesimal character $\chi = -(\mathrm{Tr}\,\varphi)$, where

$$\begin{cases} i_1 = 1, \\ i_2 = m_1 + 1, \\ \quad \cdots \\ i_r = m_1 + m_2 + \dots + m_{r-1} + 1. \end{cases}$$

Denote by P a point of E defined by

$$P = (e_1, e_2, \dots, e_n, e_1 + e_2 + \dots + e_n),$$

where $\{e_i\}$ denotes the canonical base of K^n. Then we have $F(P) \neq 0$. Thus, by Lemma 2, we obtain the following.

Proposition 1. *Let $(\mathfrak{g}, \varphi)$ be a linear representation in E of type $(m_1, m_2, \dots, m_r)$. If $(\mathfrak{g}, \varphi)$ is admissible at some point, then*

(1) *F is the polynomial for $(\mathfrak{g}, \varphi)$,*

(2) *$(\mathfrak{g}, \varphi)$ is admissible at P.*

Let $(\mathfrak{g}, \varphi)$ be a linear representation of $\mathfrak{g}$ in E of type $(m_1, m_2, \dots, m_r)$. We may assume that $m_1 \geq m_2 \geq \cdots \geq m_r$.

First we shall prove the following.

Theorem 3. *Let $(\mathfrak{g}, \varphi)$ be a linear representation of $\mathfrak{g}$ in E of type $(m_1, m_2, \dots, m_r)$. If it is of 1st kind or of 2nd kind, then it is admissible.*

Proof. Let $(\mathfrak{g}, \varphi)$ be a linear representation of $\mathfrak{g}$ in E of type $(m_1, m_2, \dots, m_r)$. By the definition of $(\mathfrak{g}, \varphi)$, there exists an element e of $\mathfrak{C}$ such that $\varphi(e) = E_{n(n+1)}$. Therefore, by the definition of $(\mathfrak{g}, \varphi)$, the action of $\{\varphi(s) \ \ (s \in \mathfrak{S}) \text{ and } \varphi(e)\}$ on E is equivalent to that of $\{x \otimes E_{n+1} \ ; \ x \in \mathrm{gl}(n, K)\}$ on E.

Denote by Q a point of E defined by

$$^{t}Q = (e_1 + e_2 + \cdots + e_n, e_1, e_2, \dots, e_n).$$

Then the action of $\{x \otimes E_{n+1} \ ; \ x \in \mathrm{gl}(n, K)\}$ at a point Q is expressed as follows:

$$(e_{11} \otimes E_{n+1}, e_{21} \otimes E_{n+1}, \dots, e_{nn} \otimes E_{n+1})\, Q = \begin{bmatrix} E_n & E_n & \cdots & E_n \\ E_n & & & 0 \\ & E_n & & \\ & & \ddots & \\ 0 & & & E_n \end{bmatrix}.$$

Denote by $\mathfrak{C}_0 = \{a_1, a_2, \dots, a_n\}$ a subalgebra of $\mathfrak{C}$ such that $\mathfrak{C} = \{e\} \oplus \mathfrak{C}_0$.

Put

$$(\varphi(a_1), \dots, \varphi(a_n))\, Q = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_{n+1} \end{bmatrix},$$

where $D_i \ \ (1 \leq i \leq n+1)$ denotes a matrix in $\mathrm{gl}(n, K)$, and

$$\overline{D} = D_1 - \sum_{i=2}^{n+1} D_i.$$

Then the subspace $\varphi(\mathfrak{g})Q$ is spanned by column vectors of the following matrix:

$$\begin{bmatrix} E_n & E_n & \cdots & E_n & D_1 \\ E_n & & & 0 & D_2 \\ & E_n & & & D_3 \\ & & \ddots & & \\ 0 & & & E_n & D_{n+1} \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} 0 & \cdots & & 0 & \overline{D} \\ E_n & & & 0 & D_2 \\ & E_n & & & D_3 \\ & & \ddots & & \\ 0 & & & E_n & D_{n+1} \end{bmatrix}$$

After a suitable choice of a base $\{a_i\}$ of $\mathfrak{C}_0$, we can easily prove the following:

(1) Assume that $(\mathfrak{g}, \varphi)$ is of type $(m_1, m_2, \ldots, m_r)$ and of 1st kind.

(i) If $r = n+1$, then $m_1 = m_2 = \cdots = m_r = 1$ and $\overline{D} = -E_n$.

(ii) If $r \leq n$, then $m_1 \geq 2$ and $\overline{D}$ is expressed as follows:

$$\overline{D} = \begin{bmatrix} \overline{D_1} & & & 0 \\ & \overline{D_2} & & \\ & & \ddots & \\ 0 & & & \overline{D_r} \end{bmatrix}$$

where

$$\overline{D_1} = \left.\begin{bmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ \cdots & & \ddots & & \\ & & & 1 & \\ -1 & -1 & \cdots & -1 & 1 \end{bmatrix}\right\} m_1 - 1, \text{ and}$$

$$\overline{D_i} = \left.\begin{bmatrix} -1 & & & & 0 \\ -1 & -1 & & & \\ \cdots & & \ddots & & \\ & & & -1 & \\ -1 & -1 & \cdots & -1 & -1 \end{bmatrix}\right\} m_i \quad (2 \leq i \leq r).$$

(2) If $(\mathfrak{g}, \varphi)$ is of 2nd kind, then

$$\overline{D} = E_n.$$

Hence $(\mathfrak{g}, \varphi)$ is admissible at a point Q. □

Next let $A = (\mathfrak{g}, \varphi)$ be an algebra over $\mathfrak{g}$ corresponding an admissible affine representation $(\mathfrak{g}, \varphi)$ at P of type $(m_1, m_2, \ldots, m_r)$.

By the definition , there exists an element $e \in \mathfrak{C}$ such that $\varphi(e) = E_{n(n+1)}$. Denote by $\overline{B}$ a linear subspace of A spanned by $\{s \in \mathfrak{S} \text{ and } e\}$. Then $\overline{B}$ is a subalgebra of A which is isomorphic to an associative algebra $\mathrm{gl}(n, K)$.

In fact, for $s, s' \in \mathfrak{S}$, there exist $t \in \mathfrak{S}$ and $\alpha \in K$ such that $ss' = t + \alpha E_n$ in $\mathrm{gl}(n, K)$. Therefore, by the definition of φ, we have $ss' = t + \alpha e$ in A. Thus $\overline{B}$ is a subalgebra of A which is isomorphic to $\mathrm{gl}(n, K)$.

Let h (resp. h_1) be the canonical 2-form on A (resp. $\overline{B}$). Then, by Lemma 3, $h|\overline{B}$ and h_1 are conformal. Moreover, since $\overline{B}$ is isomorphic to $\mathrm{gl}(n, K)$, h_1 is non degenerate. Thus, denote by $A^\perp$ the orthogonal complement of A with respect to h, we have $\overline{B} \cap A^\perp = \{0\}$.

By Lemma 4, we obtain the following.

Proposition 2. *Let $A = (\mathfrak{g}, \varphi)$ be a left symmetric algebra as above. Then $A^\perp$ is a subalgebra of A of dimension $\leq n$ satisfying $\overline{B} \cap A^\perp = \{0\}$.*

Let B be a proper ideal of A, and $\mathfrak{b}$ its Lie algebra. By the definition of φ, we have $\mathfrak{b} \not\supset \mathfrak{S}$. Moreover, since $B \cap \overline{B} = \{0\}$ and $\pi(\mathfrak{b})$ is $\varphi(\mathfrak{S})$-invariant, we have

$$\dim B = n \text{ and } \mathfrak{b} \subset \mathfrak{C}.$$

Therefore, by Lemmas 8 and 10, B is nilpotent. Thus we have $r = 1$ and $\varphi(\mathfrak{C})$ is a commutative algebra on K^{n+1} of 2nd kind.

Conversely, assume that $(\mathfrak{g}, \varphi)$ satisfies the conditions described as above. Then $(\mathfrak{g}, \varphi)$ is admissible at a point Q, by Theorem 3. Therefore, by Lemma 2, it is admissible at a point P. Denote by $A = (\mathfrak{g}, \varphi)$ a left symmetric algebra corresponding to an admissible affine representation $(\mathfrak{g}, \varphi)$ at P. Then $\pi^{-1}(V_1)$ is a left ideal of A contained in $\mathfrak{C}$. Thus it is an ideal of A of dimension n (which is contained in the radical $R(A)$ of A). This proves the following.

Theorem 4. *Let $A = (\mathfrak{g}, \varphi)$ be a left symmetric algebra over $\mathfrak{g}$ corresponding to an admissible affine representation $(\mathfrak{g}, \varphi)$ at P of type $(m_1, m_2, \ldots, m_r)$. If A has a proper ideal B, then*

(1) $\dim B = n$,

(2) $r = 1$,

(3) $\varphi_0(\mathfrak{C})$ *is of 2nd kind.*

Conversely, if $(\mathfrak{g}, \varphi)$ is a linear representation satisfying the above conditions (2) *and* (3)*, then it is admissible at P and the corresponding algebra $A = (\mathfrak{g}, \varphi)$ has a commutative nilpotent ideal of dimension n.*

Remark. Let $A = (\mathfrak{g}, \varphi)$ be a left symmetric algebra over $\mathfrak{g}$ corresponding to an admissible affine representation at P of type $(m_1, m_2, \ldots, m_r)$. Then the radical $R(A)$ of A is non trivial if and only if $r = 1$.

Let $(\mathfrak{g}, \varphi)$ be an admissible affine representation in E of type $(m_1, m_2, \ldots, m_r)$ with $m_1 \geq m_2 \geq \cdots \geq m_r$. Since $F = \prod_{j=1}^{r} F(i_j)^{m_j}$ is a relative invariant of $(\mathfrak{g}, \varphi)$ corresponding to the infinitesimal character $\chi = -(\operatorname{Tr} \varphi)$, it is the polynomial for $(\mathfrak{g}, \varphi)$, by Lemma 2.

Denote by $(\mathfrak{g}, \varphi^*)$ the contragradient representation of $(\mathfrak{g}, \varphi)$. Then $(\mathfrak{g}, \varphi^*)$ is a linear representation of $\mathfrak{g}$ in E^*, where

$$\begin{aligned} E^* &= V_1^* \oplus V_2^* \oplus \cdots \oplus V_{n+1}^*, \\ V_i^* &= K^n(y_{i1}, y_{i2}, \ldots, y_{in}) \end{aligned}$$

denotes an affine space over K with a system $y_i = (y_{i1}, y_{i2}, \ldots, y_{in})$ of affine coordinates. Put

$$F^* = F^*(m_1)^{m_1} F^*(m_1 + m_2)^{m_2} \cdots F^*(m_1 + \cdots + m_r)^{m_r}$$

where $F^*(i)$ denotes the polynomial on E^* defined by

$$F^*(i) = \det(y_1, y_2, \dots, \hat{y_i}, \dots, y_{n+1}).$$

Then it is clear that F^* is a relative invariant of $(\mathfrak{g}, \varphi^*)$ corresponding to the infinitesimal character $\chi^* = -(\mathrm{Tr}\,\varphi^*)$.

Denote by Ω the domain in E defined by $\Omega = \{x \in E\ ;\ F(x) \neq 0\}$, and by Ψ a mapping of Ω into E^* defined by

$$y_{ij} = \left(\frac{1}{F(x)}\right) \frac{\partial}{\partial x_{ij}} (F(x)) \quad (x \in \Omega),$$

for $1 \leq i \leq n+1$, $1 \leq j \leq n$. Put

$$(i; j, k) = \left(\left(\frac{1}{F(i)}\right)\left(\frac{\partial}{\partial x_{jk}} F(i)\right)\right)(P),$$

for $1 \leq i, j \leq n+1$, $1 \leq k \leq n$.

By a direct computation, we obtain the following.

Lemma 13. *Non vanishing terms of $(i; j, k)$ are as follows:*

(1) $(i; j, i) = -1,\ i \neq j,\ 1 \leq i, j \leq n,$

(2) $(i; n+1, i) = 1,\ 1 \leq i \leq n,$

(3) $(i; j, j) = 1,\ i \neq j,\ 1 \leq i, j \leq n,$

(4) $(n+1, j, j) = 1,\ 1 \leq j \leq n.$

Denote by $\Psi(P)(j,k)$ the (j,k)-component of the point $\Psi(P)$ in E^*.

Lemma 14.

(1) *In the case that $r = n+1$ and $m_i = 1$ $(1 \leq i \leq n+1)$,*

$$\Psi(P)(j,k) = \begin{cases} -1, & j \neq k,\ 1 \leq j \leq n, \\ n, & j = k,\ 1 \leq j \leq n, \\ 1, & j = n+1,\ 1 \leq k \leq n. \end{cases}$$

(2) *In the case that $m_1 \geq 2$.*

$$\Psi(P)(j, m_1) = \begin{cases} n+1, & j = m_1, \\ 0, & \textit{otherwise}. \end{cases}$$

In fact, in the first case, we have

$$\Psi(P)(j,k) = \sum_{i=1}^{n+1} (i; j, k).$$

In the second case, we have

$$\Psi(P)(j, m_1) = \sum_{s=1}^{r} m_s (i_s; j, m_1).$$

Using them and the above Lemma, we obtain the desired results. □

In the case that $m_1 \geq 2$. By the above Lemma 14, we have $F^*(m_1)(\Psi(P)) = 0$, that is, $F^*(\Psi(P)) = 0$. Therefore, by Lemma 8, the Hessian of the mapping Ψ vanishes at P. Thus $A = (\mathfrak{g}, \varphi)$ is degenerate, by Lemmas 6 and 7.

Next consider the case that $r = n+1$. In this case, $(\mathfrak{g}, \varphi)$ is admissible at P, by Theorem 3. Moreover, by the above Lemma 14, we have

$$F^*(i)(\Psi(P)) = (-1)^{n-i}(n+1)^{n-1}, \;\; 1 \leq i \leq n,$$
$$F^*(n+1)(\Psi(P)) = (n+1)^{n-1}.$$

Hence we have $F^*(\Psi(P)) \neq 0$. Since F^* is a relative invariant of $(\mathfrak{g}, \varphi^*)$ corresponding to $\chi^* = -(\operatorname{Tr}\varphi^*)$ and $F^*(\Psi(P)) \neq 0$, the Hessian of Ψ does not vanish at P, by Lemma 8. This implies that $A = (\mathfrak{g}, \varphi)$ is non degenerate, by Lemma 6.

Thus we obtain the following.

Theorem 5. *Let $(\mathfrak{g}, \varphi)$ be a linear representation of type $(m_1, m_2, \dots, m_r)$.*

(1) *If $r = n + 1$, then it is admissible at a point P, and the corresponding algebra $A = (\mathfrak{g}, \varphi)$ is non degenerate.*

(2) *If $1 \leq r \leq n$ and $(\mathfrak{g}, \varphi)$ is admissible at some point, then the corresponding algebra $A = (\mathfrak{g}, \varphi)$ is degenerate.*

References

[Ba] O. Baues, Left symmetric algebras for gl(n), Transactions A.M.S. 351(1999), 2979–2996.

[Bu] D. Burde, Left invariant affine structures on reductive Lie groups, J. Algebra 181(1996), 884–902.

[M] A. Mizuhara, Left symmetric algebras over a real reductive Lie algebra, to appear in Math. Japonica.

[S] H. Shima, Homogeneous Hessian manifolds, Ann. Inst. Fourier, Grenoble 30(1980), 91–128.

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