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ESTIMATES FOR MODULI OF COEFFICIENTS OF POSITIVE TRIGONOMETRIC POLYNOMIALS

Dedicated to Professor Tsuyoshi Ando on his seventieth birthday

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Abstract. Suppose that a trigonometric polynomial

$$\tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} \alpha_k e^{ik\theta}, \quad \theta \in [0, 2\pi),$$

is positive, $\alpha_{N-1} \neq 0, N \geq 2$. Then a classical matter due to Fejér asserts that the estimate

$$|\alpha_1| \leq \alpha_0 \cos \frac{\pi}{N+1}$$

for the modulus $|\alpha_1|$ of α_1 holds and that the equality occurs only for the polynomial $\alpha_0 \tau_N(e^{i(\theta-\varphi)})$, where

$$\tau_N(e^{i\theta}) = \frac{2}{N+1} \left| \sum_{k=0}^{N-1} \left(\sin \frac{(k+1)\pi}{N+1} \right) e^{ik\theta} \right|^2, \quad \theta \in [0, 2\pi),$$

and $\varphi \in [0, 2\pi)$. In this paper, we will show that the corresponding estimate

$$|\alpha_n| \leq \alpha_0 \cos \frac{\pi}{\lceil N/n \rceil + 1}$$

for the modulus $|\alpha_n|$ of α_n is true, $1 \leq n \leq N-1$, $\lceil N/n \rceil$ the minimum integer not smaller than N/n, and that the equality for $n = n_0$ occurs only for the polynomial τ of the form

$$\tau(e^{i\theta}) = \sigma(e^{i\theta}) \tau_{\lceil N/n_0 \rceil}(e^{in_0(\theta-\varphi)}), \quad \theta \in [0, 2\pi),$$

where σ is a positive trigonometric polynomial and $\varphi \in [0, 2\pi)$.

1. Introduction.

Let S_N, where $N \geq 2$, be the $N \times N$ shift matrix, i.e.,

$$S_N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

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Then it is known by Davidson and Holbrook [1], Corollary 2, that for n with $1 \leq n \leq N-1$, $\lceil N/n \rceil$ denoting the minimum integer not smaller than N/n (which in fact is $[(N-1)/n]+1$), the numerical radius

$$w((S_N)^n) = \sup_{||\zeta||=1} |\langle (S_N)^n \zeta, \zeta \rangle|$$

of the power $(S_N)^n$ of S_N coincides with $\cos \dfrac{\pi}{\lceil N/n \rceil + 1}$. But, in the case when $n = 1$, Haagerup and de la Harpe [3], Proposition 1 (and T. Yoshino [5], Lemmas 6 and 7, p.134, also) proves that, given a unit vector $\zeta \in C^N$, the equality

$$\langle S_N \zeta, \zeta \rangle = \cos \frac{\pi}{N+1}$$

holds if and only if

$$\zeta = e^{i\varphi} \zeta_1 \quad \text{for some} \quad \varphi \in [0, 2\pi),$$

where ζ_1 is the vector in C^N of which mth coordinate is

$$\left(\frac{2}{N+1} \right)^{1/2} \sin \frac{m\pi}{N+1}, \quad 1 \leq m \leq N.$$

Haagerup and de la Harpe observed further that this serves to lead us to the classical matter due to Fejér ([2]; [4], 8.4) which asserts that if a trigonometric polynomial

$$\tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} \alpha_k e^{ik\theta}, \quad \theta \in [0, 2\pi),$$

is positive, namely

$$\tau(e^{i\theta}) \geq 0 \text{ for any } \theta \in [0, 2\pi),$$

and not identically zero (or equivalently $\alpha_0 > 0$) with $\alpha_{N-1} \neq 0$, then one has the estimate

$$|\alpha_1| \leq \alpha_0 \cos \frac{\pi}{N+1}$$

for the modulus of α_1, and the equality occurs only for the polynomial $\alpha_0 \tau_N(e^{i(\theta - \varphi)})$, where

$$\tau_N(e^{i\theta}) = \frac{2}{N+1} \left| \sum_{k=0}^{N-1} \left(\sin \frac{(k+1)\pi}{N+1} \right) e^{ik\theta} \right|^2, \quad \theta \in [0, 2\pi).$$

It is easy for us to give the corresponding estimates for the moduli $|\alpha_n|$ of the nth coefficients α_n of τ, $-N+1 \leq n \leq N-1$ (but for the case $n = 0$ we give an appropriate understanding). In fact, By the Fejér-Riesz theorem (See [2], [4]), there exists a polynomial

$$\sigma(e^{i\theta}) = \sum_{k=0}^{N-1} \gamma_k e^{ik\theta}$$

such that

$$\tau(e^{i\theta}) = |\sigma(e^{i\theta})|^2 = \sum_{k,l=0}^{N-1} \gamma_k \bar{\gamma}_l e^{i(k-l)\theta}$$

(So it is immediate that $\alpha_{-n} = \bar{\alpha}_n, \quad -N+1 \leq n \leq N-1$). Let ζ be the vector in C^N of which kth coordinate are $\gamma_{k-1}, \ 1 \leq k \leq N$. Then we have

$$\alpha_0 = ||\zeta||^2 \quad \text{and} \quad \alpha_n = \langle (S_N)^n \zeta, \zeta \rangle, \quad 1 \leq n \leq N-1.$$

Therefore, by [1], Corollary 2, it actually follows that

$$|\alpha_n| = ||\zeta||^2 \left| \left\langle (S_N)^n \frac{\zeta}{||\zeta||}, \frac{\zeta}{||\zeta||} \right\rangle \right| \leq \alpha_0 \cos \frac{\pi}{\lceil N/n \rceil + 1}.$$

We will devote ourselves in the following two sections to determining the polynomial τ for which the equality

$$|\alpha_n| = \alpha_0 \cos \frac{\pi}{\lceil N/n \rceil + 1}, \quad 1 \leq n \leq N-1,$$

occurs. In the last section, an application will be given to positive "operator-valued" trigonometric polynomials.

2. Unit vectors which attain the numerical radius of $(S_N)^n$.

For the sake of convenience, we identify, through the canonical manner, the space C^N with a subspace of the space $C^{\lceil N/n \rceil} \otimes C^n$, and accordingly the power $(S_N)^n$ of S_N with the operator $P_n(S_{\lceil N/n \rceil} \otimes I_n)\Big|_{C^N}$ which restricts the operator $P_n(S_{\lceil N/n \rceil} \otimes I_n)$ on C^N, I_n the $n \times n$ unit matrix, P_n the orthogonal projection from $C^{\lceil N/n \rceil} \otimes C^n$ onto C^N.

Let $\xi_k \in C^{\lceil N/n \rceil}$ be the unit vector of which mth coordinate is

$$\left(\sum_{\nu=1}^{\lceil N/n \rceil} \sin^2 \frac{k\nu\pi}{\lceil N/n \rceil + 1} \right)^{-1/2} \sin \frac{km\pi}{\lceil N/n \rceil + 1}, \quad 1 \leq m \leq \lceil N/n \rceil,$$

and $\iota_l \in C^n$ the unit vector of which lth coordinate is 1 and others 0. Then the vectors $\xi_k \otimes \iota_l, \ 1 \leq k \leq \lceil N/n \rceil, \ 1 \leq l \leq n$, make an orthonormal basis for $C^{\lceil N/n \rceil} \otimes C^n$.

Lemma 1 *Let $1 \leq n \leq N-1$, and let $\zeta \in C^N$ be a unit vector. Then*

$$\langle (S_N)^n \zeta, \zeta \rangle = \cos \frac{\pi}{\lceil N/n \rceil + 1}$$

occurs if and only if $\zeta \in C^N$ is of the form

$$\zeta = P_n(\xi_1 \otimes \eta),$$

where $\eta = \sum_{l=1}^{r} \beta_l \iota_l$ with $\sum_{l=1}^{r} |\beta_l|^2 = 1$, $r = N - (\lceil N/n \rceil - 1)\, n$.

Proof. First assume that n divides N, that $\zeta \in C^N$ is a unit vector and that

$$\langle (S_N)^n \zeta, \zeta \rangle = \cos \frac{\pi}{N/n + 1}.$$

Put

$$\zeta = \sum_{1\le k\le N/n,\ 1\le l\le n} \beta_{k,l}\xi_k \otimes \iota_l, \quad \text{with} \quad \sum_{1\le k\le N/n,\ 1\le l\le n} |\beta_{k,l}|^2 = 1.$$

Then, since

$$\mathrm{Re}(S_{N/n})\xi_k = \left(\cos\frac{k\pi}{N/n+1}\right)\xi_k,\ 1 \le k \le N/n,$$

we have

$$\begin{aligned}
\cos\frac{\pi}{N/n+1} &= \left\langle (S_{N/n}\otimes I_n)\zeta, \zeta \right\rangle \\
&= \left\langle \mathrm{Re}(S_{N/n}\otimes I_n)\sum_{k,l}\beta_{k,l}\xi_k\otimes\iota_l, \sum_{k',l'}\beta_{k',l'}\xi_{k'}\otimes\iota_{l'} \right\rangle \\
&= \sum_{k,k',l,l'} \beta_{k,l}\bar{\beta}_{k',l'}\left\langle \mathrm{Re}(S_{N/n})\xi_k, \xi_{k'}\right\rangle \langle \iota_l, \iota_{l'}\rangle \\
&= \sum_{k,k',l,l'} \beta_{k,l}\bar{\beta}_{k',l'}\left\langle \left(\cos\frac{k\pi}{N/n+1}\right)\xi_k, \xi_{k'}\right\rangle \langle \iota_l, \iota_{l'}\rangle \\
&= \sum_{k,l} |\beta_{k,l}|^2 \cos\frac{k\pi}{N/n+1}.
\end{aligned}$$

This shows that $\beta_{k,l} = 0$ for $k \ge 2$. So, putting $\beta_l = \beta_{1,l}$, we have

$$\eta = \sum_{l=1}^{n} \beta_l \iota_l \quad \text{and} \quad \sum_{l=1}^{n} |\beta_l|^2 = 1.$$

Next assume that n does not divide N, and that a unt vector $\zeta \in C^N$ satisfies

$$\langle (S_N)^n\zeta, \zeta\rangle = \cos\frac{\pi}{\lceil N/n\rceil + 1}.$$

Then we have $\left\langle (S_{\lceil N/n\rceil}\otimes I_n)\zeta, \zeta\right\rangle = \cos\frac{\pi}{\lceil N/n\rceil+1}$. It follows that ζ is of the form

$$\zeta = \xi_1 \otimes \sum_{l=1}^{n} \beta_l \iota_l$$

with $\sum_{l=1}^{n} |\beta_l|^2 = 1$. But one has $\beta_l = 0$ if $l > r$, since ζ is in C^N. **QED**

3. Positive polynomial for which the modulus of α_n attains the bound.

Now we will show the aimed theorem in this paper:

Theorem 2 *Suppose that a trigonometric polynomial*

$$\tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} \alpha_k e^{ik\theta}, \quad \theta \in [0, 2\pi),$$

is positive and such that $\alpha_{N-1} \neq 0,\ N \geq 2$. *If* $1 \leq n_0 \leq N-1$, *and the equality*

$$|\alpha_{n_0}| = \alpha_0 \cos \frac{\pi}{\lceil N/n_0 \rceil + 1}$$

holds, then τ *is of the form*

$$\tau(e^{i\theta}) = \sigma(e^{i\theta})\tau_{\lceil N/n_0 \rceil}(e^{in_0(\theta-\varphi)}), \quad \theta \in [0, 2\pi),$$

where σ *is a positive trigonometric polynomial of degree* $r_0 - 1$, $r_0 = N - (\lceil N/n_0 \rceil - 1)n_0$, $\tau_{\lceil N/n_0 \rceil}$ *the trigonometric polynomial already introduced and* $\varphi \in [0, 2\pi)$. *Moreover, for any* $n \neq n_0,\ 1 \leq n \leq N-1$, *one has*

$$|\alpha_n| < \alpha_0 \cos \frac{\pi}{\lceil N/n \rceil + 1}.$$

Conversely, for the polynomial $\sigma(e^{i\theta})\tau_{\lceil N/n_0 \rceil}(e^{in_0(\theta-\varphi)})$, *the modulus* $|\alpha_{n_0}|$ *of* α_{n_0} *is equal to* $\alpha_0 \cos \dfrac{\pi}{\lceil N/n_0 \rceil + 1}$.

Proof. By the Fejér-Riesz theorem one has a polynomial

$$\sigma(e^{i\theta}) = \sum_{k=0}^{N-1} \gamma_k e^{ik\theta}$$

such that

$$\tau(e^{i\theta}) = |\sigma(e^{i\theta})|^2 = \sum_{k,l=0}^{N-1} \gamma_k \bar{\gamma}_l e^{i(k-l)\theta}.$$

Assume that the equality

$$|\alpha_{n_0}| = \alpha_0 \cos \frac{\pi}{\lceil N/n_0 \rceil + 1}$$

holds for $n_0, 1 \leq n_0 \leq N-1$.

First we let $\alpha_0 = 1$ and $\alpha_{n_0} \geq 0$. The vector ζ of which kth coordinate is γ_{k-1} $(1 \leq k \leq N)$ achieves the numerical radius $w((S_N)^{n_0})$ of the matrix $(S_N)^{n_0}$, so, by Lemma 1, ζ is of the form

$$\zeta = P_{n_0}\left(\xi_1 \otimes \sum_{l=1}^{r_0} \beta_l \iota_l\right), \quad \sum_{l=1}^{r_0} |\beta_l|^2 = 1,$$

where P_{n_0} is the orthogonal projection from $C^{\lceil N/n_0 \rceil} \otimes C^{n_0}$ onto C^N, ξ_1 the unit vector in $C^{\lceil N/n_0 \rceil}$ of which kth coordinate is

$$\left(\frac{2}{\lceil N/n_0 \rceil + 1}\right)^{1/2} \sin \frac{k\pi}{\lceil N/n_0 \rceil + 1}, \quad 1 \leq k \leq \lceil N/n_0 \rceil,$$

ι_l the unit vector in C^{n_0} of which lth coordinate is 1 and others 0, $1 \leq l \leq r_0$. Then we have

$$\gamma_k = \beta_l \left(\frac{2}{\lceil N/n_0 \rceil + 1}\right)^{1/2} \sin \frac{(j+1)\pi}{\lceil N/n_0 \rceil + 1}$$

if $k = l + n_0 j - 1,\ 1 \leq l \leq r_0,\ 0 \leq j \leq \lceil N/n_0 \rceil - 1,$ and $\gamma_k = 0$ otherwise. Therefore, we have

$$\begin{aligned} \tau(e^{i\theta}) &= \left| \sum_{k=0}^{N-1} \gamma_k e^{ik\theta} \right|^2 \\ &= \frac{2}{\lceil N/n_0 \rceil + 1} \left| \sum_{j=0}^{\lceil N/n_0 \rceil - 1} \sum_{l=1}^{r_0} \beta_l \sin \frac{(j+1)\pi}{\lceil N/n_0 \rceil + 1} e^{i(l-1+n_0 j)\theta} \right|^2 \\ &= \left| \sum_{l=1}^{r_0} \beta_l e^{i(l-1)\theta} \right|^2 \left(\frac{2}{\lceil N/n_0 \rceil + 1} \left| \sum_{j=1}^{\lceil N/n_0 \rceil - 1} \sin \frac{(j+1)\pi}{\lceil N/n_0 \rceil + 1} e^{in_0 j\theta} \right|^2 \right), \end{aligned}$$

and $\beta_{r_0} \neq 0$. Therefore, putting

$$\sigma(e^{i\theta}) = \left| \sum_{l=1}^{r_0} \beta_l e^{i(l-1)\theta} \right|^2, \quad \theta \in [0, 2\pi),$$

which in fact is positive, we have

$$\tau(e^{i\theta}) = \sigma(e^{i\theta}) \tau_{\lceil N/n_0 \rceil}(e^{in_0\theta}), \quad \theta \in [0, 2\pi).$$

Assume that

$$|\alpha_{n_1}| = \cos \frac{\pi}{\lceil N/n_1 \rceil + 1}$$

holds for $n_1,\ 1 \leq n_1 \leq N-1$. Then ζ is of the form

$$\zeta = P'_{n_1}(\xi'_1 \otimes \sum_{l=1}^{r_1} \beta'_l \iota'_l),$$

where P'_{n_1} the projection from $C^{\lceil N/n_1 \rceil} \otimes C^{n_1}$ onto C^N, ξ'_1 the vector in $C^{\lceil N/n_1 \rceil}$ of which kth coordinate is

$$e^{i\psi_k} \left(\frac{2}{\lceil N/n_1 \rceil + 1} \right)^{1/2} \sin \frac{k\pi}{\lceil N/n_1 \rceil + 1}, \quad \psi_k \in [0, 2\pi), \quad 1 \leq k \leq \lceil N/n_1 \rceil,$$

ι'_l the vector in C^{n_1} of which lth coordinate is 1 and others 0 and $r_1 = N - (\lceil N/n_1 \rceil - 1)\, n_1$. Therefore, we have

$$P'_{n_1}(\xi'_1 \otimes \sum_{l=1}^{r_1} \beta'_l \iota'_l) = P_{n_0}(\xi_1 \otimes \sum_{l=1}^{r_0} \beta_l \iota_l).$$

But it occurs only when $r_1 = r_0$ and $n_1 = n_0$.

Now we turn to the general case. We apply the foregoing argument to the positive trigonometric polynomial

$$\tilde{\tau}(e^{i\theta}) = \tau(e^{i(\theta-\varphi)})/\alpha_0, \quad \theta \in [0, 2\pi),$$

$\varphi = \mathrm{Arg}\alpha_{n_0}/n_0$. Then we have the desired conclusion.

Conversely, let

$$\tau(e^{i\theta}) = \sigma(e^{i\theta}) \tau_{\lceil N/n_0 \rceil}(e^{in_0(\theta-\varphi)}), \quad \theta \in [0, 2\pi),$$

where σ is a positive trigonometric polynomial, then we can easily have the equality

$$|\alpha_{n_0}| = \alpha_0 \cos \frac{\pi}{\lceil N/n_0 \rceil + 1}.$$

QED

4. An application to operator-valued trigonometric polynomials.

Theorem 2 yields the estimates for numerical radii of operators which are coefficients of positive operator-valued trigonometric polynomials:

Corollary 3 *Let A_k be bounded operators on a Hilbert space H, $-N+1 \leq k \leq N-1$, $N \geq 2$. Suppose that*

$$\tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} A_k e^{ik\theta} \geq O$$

for any $\theta \in [0, 2\pi)$. Then, $A_0 \geq O$ and one has

$$w(A_n) \leq ||A_0|| \cos \frac{\pi}{\lceil N/n \rceil + 1}, \quad 1 \leq n \leq N-1.$$

Proof. Let $\zeta \in H$ and $||\zeta|| = 1$. Then

$$\tau_\zeta(e^{i\theta}) = \sum_{k=-N+1}^{N-1} \langle A_k \zeta, \zeta \rangle e^{ik\theta}, \quad \theta \in [0, 2\pi),$$

is a positive trigonometric polynomial. So it follows that $A_0 \geq O$. If $\langle A_0 \zeta, \zeta \rangle > 0$, then we know that the inequality

$$| \langle A_n \zeta, \zeta \rangle | \leq \langle A_0 \zeta, \zeta \rangle \cos \frac{\pi}{\lceil N/n \rceil + 1}$$

holds. If $\langle A_0 \zeta, \zeta \rangle = 0$, then we have $\langle A_n \zeta, \zeta \rangle = 0$, $1 \leq n \leq N-1$, and so, we know that the above inequality turns out to be trivial. Hence we have

$$w(A_n) \leq ||A_0|| \cos \frac{\pi}{\lceil N/n \rceil + 1}.$$

QED

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