THE SQUARE PARTIAL SUMS OF THE FOURIER TRANSFORM OF RADIAL FUNCTIONS IN THREE DIMENSIONS

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Received May 14, 2001; revised August 2, 2001

Abstract. Pinsky, Stanton and Trapa [3] showed, for example, that the spherical partial sum of the Fourier series or the Fourier transform of the characteristic function of the n-dimensional ball $|x| \le 1$ diverges at $x = 0$ when $n \ge 3$. We point out that, in three dimensions, the square partial sum of it converges at $x = 0$. Let φ be a function of bounded variation with compact support. We show that, for the radial function $f(x) = \varphi(|x|)$, $x \in \mathbb{R}^3$, the square partial sum of the Fourier transform converges to $\varphi(+0)$ at $x = 0$.

1. Introduction

In one dimension, the behavior of Fourier series at a point depends only on the behavior of the function in a neighborhood of that point. In particular, if the function is zero on an interval, then the Fourier series converges to zero on that interval. However, in two or more dimensions, this localization property does not generally hold.

In two or more dimensions, there are many ways to add up the terms of the Fourier series; spherical partial sum, rectangular partial sum, square partial sum, etc. Pinsky, Stanton and Trapa [3] and Kuratsubo [1] and [2] investigated the convergence of the spherical partial sums of radial functions. Pinsky, Stanton and Trapa [3] showed, for example, that the spherical partial sum of the Fourier series or the Fourier transform of the characteristic function of the n-dimensional ball $|x| \le 1$ diverges at $x = 0$ when $n \ge 3$.

In this paper, we study the convergence of the square partial sums of radial functions in three dimensions. Let φ be a function of bounded variation with compact support. We show that, for the radial function $f(x) = \varphi(|x|)$, $x \in \mathbb{R}^3$, the square partial sum of the Fourier transform converges to $\varphi(+0)$ at $x = 0$. In the case that f is the characteristic function of the 3-dimensional ball $|x| \le 1$, this convergence is equivalent to the following:

$$\lim_{\lambda \to +\infty} \int_{|z|<\lambda} \frac{\sin z_1 \sin z_2 \sin z_3}{z_1 z_2 z_3} \, dz_1 dz_2 dz_3 = \pi^3,$$
$$\text{where} \quad |z| = \sqrt{{z_1}^2 + {z_2}^2 + {z_3}^2}.$$

Our result is for the Fourier transform. It is unknown whether the square partial sums of the Fourier series converges, though the square partial sum of the Fourier series of the characteristic function of the 3-dimensional ball seems to converge by a numerical data. Our method is not valid for the case $x \ne 0$.

Goffman and Liu [5] showed that the square partial sums of the Fourier series of a function f has the localization property for $f \in W^1_{n-1}(\mathbb{T}^n)$. However, the characteristic function of the 3-dimensional ball $|x| \le 1$ is not in $W^1_2(\mathbb{T}^3)$.

2000 *Mathematics Subject Classification.* Primary 42B05.

Key words and phrases. Fourier transform, square partial sum, radial function, bounded variation.

In the next section we state definitions and main results. We give a proof in the third section. To prove the main result we use the measure of the intersection of the cube $\{\xi \in \mathbb{R}^3 : |\xi_k| \leq \lambda, k = 1, 2, 3\}$ and the ball $\{\xi \in \mathbb{R}^3 : |\xi| \leq r\}$. For convenience we give a calculation of this measure in the fourth section.

2. Definitions and main results

$\mathbb{Z}^n$ denotes the n-dimensional integer lattice, whose points are written $m = (m_1, ..., m_n)$, where m_k are any integers. $\mathbb{T}^n$ denotes the n-dimensional torus, whose points are written $(x_1, ..., x_n)$, where $-\pi \leq x_k \leq \pi$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, whose points are written $x = (x_1, ..., x_n)$, and for $x, y \in \mathbb{R}^n$, its inner product xy denotes $\sum_{k=1}^n x_k y_k$.

The Fourier series of a function F on $\mathbb{T}^n$, its spherical partial sum $S_\lambda^{\rm sph}$ and its square partial sum $S_\lambda^{\rm sq}$ are defined by

$$\hat{F}(m) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(x) e^{-imx}\, dx, \quad m \in \mathbb{Z}^n,$$

$$S_\lambda^{\rm sph}(x) = \sum_{|m|<\lambda} \hat{F}(m) e^{imx}, \quad |m| = \sqrt{\sum_{k=1}^n {m_k}^2}, \quad x \in \mathbb{T}^n,$$

$$S_\lambda^{\rm sq}(x) = \sum_{|m_k|<\lambda : k=1,2,\cdots,n} \hat{F}(m) e^{imx}, \quad x \in \mathbb{T}^n.$$

The Fourier transform of a function f on $\mathbb{R}^n$, its spherical partial sum $f_\lambda^{\rm sph}$ and its square partial sum $f_\lambda^{\rm sq}$ are defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx, \quad \xi \in \mathbb{R}^n,$$

$$f_\lambda^{\rm sph}(x) = \int_{|\xi|<\lambda} \hat{f}(\xi) e^{i\xi x}\, d\xi, \quad |\xi| = \sqrt{\sum_{k=1}^n {\xi_k}^2}, \quad x \in \mathbb{R}^n,$$

$$f_\lambda^{\rm sq}(x) = \int_{|\xi_k|<\lambda : k=1,2,\cdots,n} \hat{f}(\xi) e^{i\xi x}\, d\xi, \quad x \in \mathbb{R}^n.$$

Pinsky, Stanton and Trapa [3] investigated the spherical partial sum of the Fourier series of the radial functions.

Theorem A ([3])**.** *If $0 < a \leq \pi$, $n \geq 3$, then the spherical partial sum of the n-dimentional Fourier series of the characteristic function of the ball $|x| \leq a$ diverges at the center $x = 0$.*

Theorem B ([3])**.** *Let $F(x) = \varphi(|x|)$ for $x \in \mathbb{T}^3$, where $0 < a \leq \pi$, $\varphi(r)$ is a smooth function on the interval $[0, a]$ and $\varphi(r) = 0$ for $r > a$. If $\varphi(a) = 0$ then the spherical partial sum of the Fourier series converges for all $x \in \mathbb{T}^3$. Conversely, if the spherical partial sum of the Fourier series converges at $x = 0$ then $\varphi(a) = 0$.*

Theorem C ([3])**.** *Let $f(x) = \varphi(|x|)$ for $x \in \mathbb{R}^3$, where $\varphi(r)$ has a continuous derivative on the interval $[0, a]$ and $\varphi(r) = 0$ for $r > a$. Then the spherical partial sum of the Fourier transform converges at $x = 0$ if and only if $\varphi(a) = 0$.*

Pinsky, Stanton and Trapa [3] also stated a example with graphs as follows:

$$F(x) = \begin{cases} 1 & \text{for } |x| \le a, \\ 0 & \text{for } |x| > a, \end{cases} \qquad x \in \mathbb{T}^3, \quad 0 < a \le \pi,$$

$$\hat{F}(0) = \frac{a^3}{6\pi^2},$$

$$\hat{F}(m) = \frac{1}{2\pi^2|m|}\left(-\frac{a\cos a|m|}{|m|} + \frac{\sin a|m|}{|m|^2}\right), \quad m \ne 0 = (0,0,0).$$

The spherical partial sum of this series for $a = \pi$, $\lambda = \sqrt{134}$ and $(x_1, x_2, x_3) = (x, 0, 0)$ has been graphed using Mathematica and is depicted in Fig. 1. The partial sum for $\lambda = \sqrt{155}$ is depicted in Fig. 2.

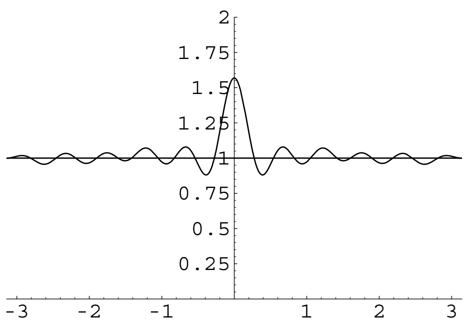

Fig. 1: $S_\lambda^{\mathrm{sph}}(x, 0, 0)$, $\lambda = \sqrt{134}$

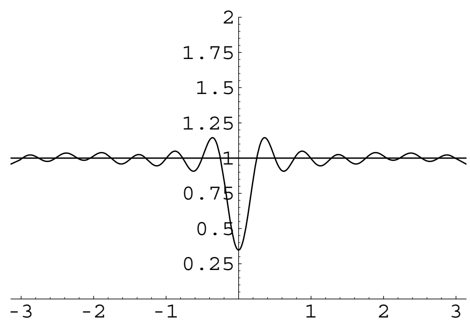

Fig. 2: $S_\lambda^{\mathrm{sph}}(x, 0, 0)$, $\lambda = \sqrt{155}$

On the other hand, the square partial sums for $\lambda = 10$ and for $\lambda = 20$ are depicted in Fig. 3 and 4, respectively. We point out that the square partial sum seems to converge at $x = 0$.

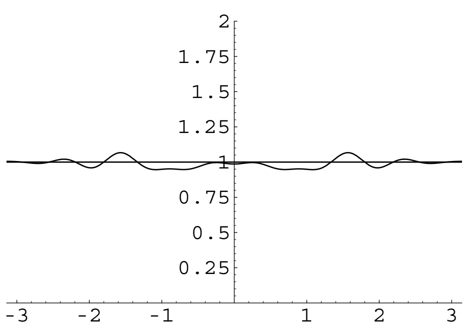

Fig. 3: $S_\lambda^{\mathrm{sq}}(x, 0, 0)$, $\lambda = 10$

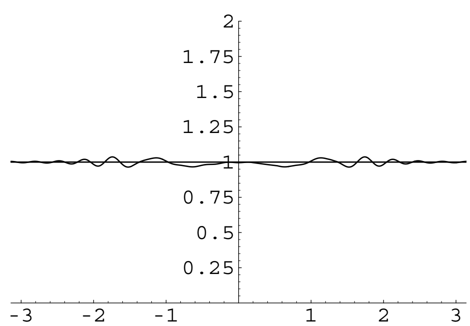

Fig. 4: $S_\lambda^{\mathrm{sq}}(x, 0, 0)$, $\lambda = 20$

Our main results are as follows:

Theorem 2.1. *Let $f(x) = \varphi(|x|)$ for $x \in \mathbb{R}^3$, where φ is a function of bounded variation with compact support. Then*

$$f_\lambda^{\mathrm{sq}}(0) \to \varphi(+0) \quad \text{as} \quad \lambda \to +\infty.$$

Since

$$\begin{aligned} f_\lambda^{\mathrm{sq}}(0) &= \int_{|\xi_k|<\lambda:k=1,2,3} \hat{f}(\xi)\,d\xi \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} f(y)\left(\int_{|\xi_k|<\lambda:k=1,2,3} e^{-i\xi y}\,d\xi\right)dy \\ &= \frac{1}{\pi^3}\int_{\mathbb{R}^3} f(y)\frac{\sin\lambda y_1 \sin\lambda y_2 \sin\lambda y_3}{y_1y_2y_3}\,dy, \end{aligned}$$

we have the following:

Corollary 2.2. *Let φ be a function of bounded variation. Then, for all $a>0$,*

$$\frac{1}{\pi^3}\int_{|y|<a} \varphi(|y|)\frac{\sin\lambda y_1 \sin\lambda y_2 \sin\lambda y_3}{y_1y_2y_3}\,dy \to \varphi(+0) \quad \textit{as} \quad \lambda\to+\infty.$$

Let $\varphi(r)=1$ and $a=1$ in Corollary 2.2. Then we have the following:

Corollary 2.3.

$$\int_{|z|<\lambda} \frac{\sin z_1 \sin z_2 \sin z_3}{z_1z_2z_3}\,dz_1\,dz_2\,dz_3 \to \pi^3 \quad \textit{as} \quad \lambda\to+\infty.$$

3. Proof

We prove the theorem in this section.

Lemma 3.1 (Kuratsubo [2])**.** *Suppose $a>0$ and φ is a function of bounded variation. Let*

$$f(x) = \begin{cases} \varphi(|x|) & \textit{for } |x|\le a, \\ 0 & \textit{for } |x|>a. \end{cases}$$

Then we have

$$\hat{f}(\xi) = \varphi(a)\hat{\chi_a}(\xi) - \int_0^a \hat{\chi_t}(\xi)\,d\varphi(t),$$

where

$$\chi_t(x) = \begin{cases} 1 & \textit{for } |x|\le t, \\ 0 & \textit{for } |x|>t, \end{cases} \quad x\in\mathbb{R}^n,$$

and $d\varphi$ is the Lebesgue-Stieltjes measure generated by φ.

In the case $n=3$, by Lemma([3], P123), we have

$$\hat{\chi_t}(\xi) = H(t,|\xi|), \quad \text{where} \quad H(t,r) = -\frac{t\cos tr}{2\pi^2r^2} + \frac{\sin tr}{2\pi^2r^3}.$$

Let $\operatorname{supp}\varphi\subset[0,a]$. Then, by Lemma 3.1, we have

$$\hat{f}(\xi) = A(|\xi|), \quad \text{where} \quad A(r) = \varphi(a)H(a,r) - \int_0^a H(t,r)\,d\varphi(t).$$

Let $V(r)$ be the measure of the intersection of the cube $\{\xi\in\mathbb{R}^3 : |\xi_k|\le\lambda, k=1,2,3\}$ and the ball $\{\xi\in\mathbb{R}^3 : |\xi|\le r\}$. Then $V'(r)=0$ for $r\ge\sqrt{3}\lambda$ and

$$\begin{aligned} f_\lambda^{\mathrm{sq}}(0) &= \int_{|\xi_k|<\lambda:k=1,2,3} \hat{f}(\xi)\,d\xi = \int_0^{\sqrt{3}\lambda} A(r)V'(r)\,dr \\ &= \varphi(a)\int_0^{\sqrt{3}\lambda} H(a,r)V'(r)\,dr - \int_0^{\sqrt{3}\lambda}\left(\int_0^a H(t,r)\,d\varphi(t)\right)V'(r)\,dr. \end{aligned}$$

Let

$$I(t,\lambda)=\int_0^{\sqrt{3}\lambda} H(t,r)V'(r)\,dr.$$

Since $V'(r)$ is continuous and $V'(r)=4\pi r^2$ for small $r>0$, $H(t,r)V'(r)$ is continuous on $[0,a]\times[0,\sqrt{3}\lambda]$. By Fubini's theorem we have

$$f_\lambda^{\mathrm{sq}}(0)=\varphi(a)I(a,\lambda)-\int_0^a I(t,\lambda)\,d\varphi(t).$$

If we show that

$$\text{(3.1)}\qquad I(t,\lambda)\text{ is bounded}\quad\text{and}\quad I(t,\lambda)\to\begin{cases}1 & \text{for } t>0,\\ 0 & \text{for } t=0,\end{cases}\quad\text{as}\quad \lambda\to+\infty,$$

then we have

$$f_\lambda^{\mathrm{sq}}(0)\to\varphi(a)-(\varphi(a)-\varphi(+0))=\varphi(+0)\quad\text{as}\quad\lambda\to+\infty.$$

It remains only to prove (3.1). By elementary calculations (see Section 4 for example), we have

$$V(r)=\begin{cases}V_1(r) & \text{for } 0<r<\lambda,\\ V_2(r) & \text{for } \lambda\le r<\sqrt{2}\lambda,\\ V_3(r)=V_{31}(r)+V_{32}(r) & \text{for } \sqrt{2}\lambda\le r<\sqrt{3}\lambda,\end{cases}$$

where

$$\begin{aligned}
V_1(r)=&\frac{4}{3}\pi r^3,\\
V_2(r)=&-\frac{8\pi r^3}{3}+6\pi r^2\lambda-2\pi\lambda^3,\\
V_{31}(r)=&-\frac{8\pi r^3}{3}+6\pi r^2\lambda-2\pi\lambda^3,\\
V_{32}(r)=&8\lambda^3\sqrt{-2+\frac{r^2}{\lambda^2}}-24r^2\lambda\arctan\sqrt{-2+\frac{r^2}{\lambda^2}}+8\lambda^3\arctan\sqrt{-2+\frac{r^2}{\lambda^2}}\\
&+8r^3\arctan\left(-1+\frac{r^2}{\lambda^2}+\frac{r}{\lambda}\sqrt{-2+\frac{r^2}{\lambda^2}}\right)\\
&-8r^3\arctan\left(-1+\frac{r^2}{\lambda^2}-\frac{r}{\lambda}\sqrt{-2+\frac{r^2}{\lambda^2}}\right),
\end{aligned}$$

and

$$\begin{aligned}
V_1'(r)=&4\pi r^2,\quad V_2'(r)=12\pi r\lambda-8\pi r^2,\quad V_{31}'(r)=12\pi r\lambda-8\pi r^2,\\
V_{32}'(r)=&-48r\lambda\arctan\sqrt{-2+\frac{r^2}{\lambda^2}}\\
&+24r^2\arctan\left(-1+\frac{r^2}{\lambda^2}+\frac{r}{\lambda}\sqrt{-2+\frac{r^2}{\lambda^2}}\right)\\
&-24r^2\arctan\left(-1+\frac{r^2}{\lambda^2}-\frac{r}{\lambda}\sqrt{-2+\frac{r^2}{\lambda^2}}\right).
\end{aligned}$$

Let

$$I(t,\lambda)=I_1(t,\lambda)+I_2(t,\lambda)+I_{31}(t,\lambda)+I_{32}(t,\lambda),$$

and

$$I_1(t,\lambda)=\int_0^{\lambda} H(t,r)V_1'(r)\,dr, \qquad I_2(t,\lambda)=\int_{\lambda}^{\sqrt{2}\lambda} H(t,r)V_2'(r)\,dr,$$
$$I_{31}(t,\lambda)=\int_{\sqrt{2}\lambda}^{\sqrt{3}\lambda} H(t,r)V_{31}'(r)\,dr, \quad I_{32}(t,\lambda)=\int_{\sqrt{2}\lambda}^{\sqrt{3}\lambda} H(t,r)V_{32}'(r)\,dr.$$

Then we have

$$(3.2)\quad I_1(t,\lambda)=\int_0^{\lambda}\left(-\frac{t\cos tr}{2\pi^2r^2}+\frac{\sin tr}{2\pi^2r^3}\right)4\pi r^2\,dr$$
$$=\int_0^{\lambda}\left(-\frac{2t\cos tr}{\pi}+\frac{2\sin tr}{\pi r}\right)dr=-\frac{2\sin t\lambda}{\pi}+\frac{2}{\pi}\int_0^{t\lambda}\frac{\sin z}{z}\,dz,$$

and

$$(3.3)\quad I_2(t,\lambda)+I_{31}(t,\lambda)=\int_{\lambda}^{\sqrt{3}\lambda}\left(-\frac{t\cos tr}{2\pi^2r^2}+\frac{\sin tr}{2\pi^2r^3}\right)\left(12\pi r\lambda-8\pi r^2\right)\,dr$$
$$=\int_{\lambda}^{\sqrt{3}\lambda}\left(-\frac{6\lambda t\cos tr}{\pi r}+\frac{6\lambda\sin tr}{\pi r^2}+\frac{4t\cos tr}{\pi}-\frac{4\sin tr}{\pi r}\right)dr$$
$$=\left[-\frac{6\lambda\sin tr}{\pi r}\right]_{\lambda}^{\sqrt{3}\lambda}+\int_{\lambda}^{\sqrt{3}\lambda}\frac{4t\cos tr}{\pi}\,dr-\frac{4}{\pi}\int_{t\lambda}^{\sqrt{3}t\lambda}\frac{\sin z}{z}\,dz$$
$$=\left(\frac{4}{\pi}-\frac{2\sqrt{3}}{\pi}\right)\sin\sqrt{3}t\lambda+\frac{2\sin t\lambda}{\pi}-\frac{4}{\pi}\int_{t\lambda}^{\sqrt{3}t\lambda}\frac{\sin z}{z}\,dz.$$

By the change of variables, $u=\sqrt{-2+r^2/\lambda^2}$, we have

$$I_{32}(t,\lambda)=\int_0^1 H\left(t,\lambda\sqrt{2+u^2}\right)V_{32}'\left(\lambda\sqrt{2+u^2}\right)\frac{\lambda u}{\sqrt{2+u^2}}\,du$$
$$=\int_0^1 E_1(u)t\cos\left(t\lambda\sqrt{2+u^2}\right)\frac{\lambda u}{\sqrt{2+u^2}}\,du+\int_0^1 E_2(u)\sin\left(t\lambda\sqrt{2+u^2}\right)du$$
$$=I_{321}(t,\lambda)+I_{322}(t,\lambda),\quad \text{say,}$$

where

$$E_1(u)=-\frac{1}{2\pi^2r^2}V_{32}'(r)$$
$$=\frac{24}{\pi^2\sqrt{2+u^2}}\arctan u$$
$$-\frac{12}{\pi^2}\left(\arctan\left(1+u^2+u\sqrt{2+u^2}\right)-\arctan\left(1+u^2-u\sqrt{2+u^2}\right)\right),$$
$$E_2(u)=\frac{1}{2\pi^2r^3}V_{32}'(r)\frac{\lambda u}{\sqrt{2+u^2}}=-E_1(u)\frac{u}{2+u^2}.$$

Since $E_1(1)=2\sqrt{3}/\pi-4/\pi$ and $E_1(0)=0$, we have

$$I_{321}(t,\lambda)=\left(\frac{2\sqrt{3}}{\pi}-\frac{4}{\pi}\right)\sin\sqrt{3}t\lambda-\int_0^1 {E_1}'(u)\sin\left(t\lambda\sqrt{2+u^2}\right)du,$$
$$\text{where}\quad {E_1}'(u)=\frac{48(2+u^2)-24(u+u^3)\arctan u}{\pi^2\sqrt{2+u^2}(2+3u^2+u^4)}.$$

Let

$$G\left(t,\lambda\right)=\int_0^1 E(u)\sin\left(t\lambda\sqrt{2+u^2}\right)du\quad\text{and}\quad E=-{E_1}'+E_2.$$

Then

$$I_{32}(t,\lambda)=\left(\frac{2\sqrt{3}}{\pi}-\frac{4}{\pi}\right)\sin\sqrt{3}t\lambda+G\left(t,\lambda\right).\tag{3.4}$$

By (3.2), (3.3) and (3.4) we have

$$I\left(t,\lambda\right)=\frac{2}{\pi}\int_0^{t\lambda}\frac{\sin z}{z}\,dz-\frac{4}{\pi}\int_{t\lambda}^{\sqrt{3}t\lambda}\frac{\sin z}{z}\,dz+G\left(t,\lambda\right).$$

We note that

$$|I(t,\lambda)|\le\frac{6}{\pi}\left|\int_0^{\pi}\frac{\sin z}{z}\,dz\right|+\sup_{0\le u\le 1}|E(u)|=C<+\infty,$$

$$\int_0^{t\lambda}\frac{\sin z}{z}\,dz\to\begin{cases}\dfrac{\pi}{2} & \text{for } t>0,\\ 0 & \text{for } t=0,\end{cases}\quad\text{as}\quad\lambda\to+\infty,$$

$$\int_{t\lambda}^{\sqrt{3}t\lambda}\frac{\sin z}{z}\,dz\to 0\quad\text{as}\quad\lambda\to+\infty.$$

For all $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|\int_0^{\delta}E(u)\sin\left(t\lambda\sqrt{2+u^2}\right)du\right|\le\delta\sup_{0\le u\le 1}|E(u)|<\frac{\varepsilon}{2}.$$

By the change of variables $v=\sqrt{2+u^2}$, and using the Riemann-Lebesgue theorem, we have

$$\left|\int_{\delta}^{1}E(u)\sin\left(t\lambda\sqrt{2+u^2}\right)du\right|=\left|\int_{\sqrt{2+\delta^2}}^{\sqrt{3}}E\left(\sqrt{v^2-2}\right)\frac{v}{\sqrt{v^2-2}}\sin t\lambda v\,dv\right|<\frac{\varepsilon}{2},$$

for large λ. Hence $G(t,\lambda)\to 0$ as $\lambda\to+\infty$. Therefore we heve (3.1). The proof is complete.

4. The measure of the intersection of the cube and the ball

For a measurable set Ω, we denote its measure by $|\Omega|$. Let

$$Q_\lambda=\{(x,y,z)\in\mathbb{R}^3:|x|\le\lambda,|y|\le\lambda,|z|\le\lambda\},$$
$$B_r=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2+z^2\le r^2\},$$

and

$$V(r)=V(\lambda,r)=|Q_\lambda\cap B_r|,\quad g(t)=V(1,t)=|Q_1\cap B_t|.$$

Then

$$V(r)=V(\lambda,r)=\lambda^3V(1,r/\lambda)=\lambda^3g(r/\lambda),\quad V'(r)=\lambda^2g'(r/\lambda).$$

In this section we calculate $g(t)$ to get $V(r)$. Let

$$g(t)=\begin{cases}g_1(t) & 0<t\le 1,\\ g_2(t) & 1<t\le\sqrt{2},\\ g_3(t) & \sqrt{2}<t<\sqrt{3},\\ g_4(t) & \sqrt{3}\le t.\end{cases}$$

We show that

$$g_1(t) = \frac{4}{3}\pi t^3, \tag{4.1}$$

$$g_2(t) = -\frac{8\pi t^3}{3} + 6\pi t^2 - 2\pi, \tag{4.2}$$

$$\begin{aligned} g_3(t) = &-\frac{8\pi t^3}{3} + 6\pi t^2 - 2\pi \\ &+ 8\sqrt{t^2-2} - \left(24t^2-8\right)\arctan\sqrt{t^2-2} \\ &+ 8t^3\arctan\left(t^2-1+t\sqrt{t^2-2}\right) - 8t^3\arctan\left(t^2-1-t\sqrt{t^2-2}\right), \end{aligned} \tag{4.3}$$

$$g_4(r) = 8. \tag{4.4}$$

If we can show this, then we have that

$$\begin{aligned} {g_1}'(t) &= 4\pi t^2, \\ {g_2}'(t) &= -8\pi t^2 + 12\pi t, \\ {g_3}'(t) &= -8\pi t^2 + 12\pi t - 48t\arctan\sqrt{t^2-2} \\ &\quad + 24t^2\arctan\left(t^2-1+t\sqrt{t^2-2}\right) - 24t^2\arctan\left(t^2-1-t\sqrt{t^2-2}\right), \\ {g_4}'(r) &= 0. \end{aligned}$$

Actually, since $(\arctan v)' = 1/\left(1+v^2\right)$, we have

$$\left(8\sqrt{t^2-2} - \left(24t^2-8\right)\arctan\sqrt{t^2-2}\right)' = -\frac{16t^3}{(t^2-1)\sqrt{t^2-2}} - 48t\arctan\sqrt{t^2-2},$$

and

$$\begin{aligned} &\left(8t^3\arctan\left(t^2-1+t\sqrt{t^2-2}\right) - 8t^3\arctan\left(t^2-1-t\sqrt{t^2-2}\right)\right)' \\ &\qquad = \frac{16t^3}{(t^2-1)\sqrt{t^2-2}} + 24t^2\arctan\left(t^2-1+t\sqrt{t^2-2}\right) \\ &\qquad\qquad - 24t^2\arctan\left(t^2-1-t\sqrt{t^2-2}\right). \end{aligned}$$

Fig. 5 and Fig. 6 are the graphs of $g(t)$ and $g'(t)$, respectively.

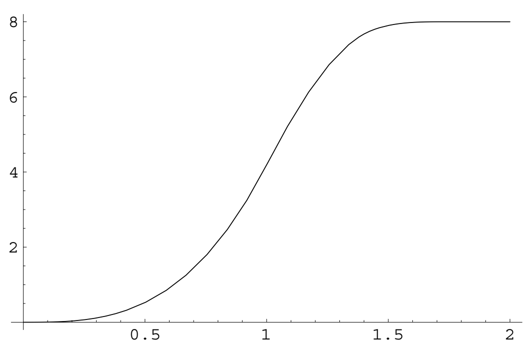

Fig. 5: $g(t)$

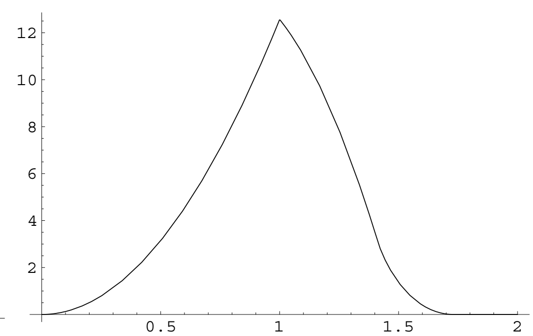

Fig. 6: $g'(t)$

(4.1) and (4.4) are clear. In the following we show (4.2) and (4.3).

For the case $t > 1$ (see Fig. 7), let

$$K_t = \{(x, y, z) \in B_t : x > 1\}.$$

For the case $t > \sqrt{2}$ (see Fig. 8 and Fig. 9), let

$$L_t = \{(x, y, z) \in B_t : x > 1, z > 1\}.$$

Then

$$g_2(t) = |B_t| - 6|K_t| \quad \text{and} \quad g_3(t) = |B_t| - 6|K_t| + 12|L_t|.$$

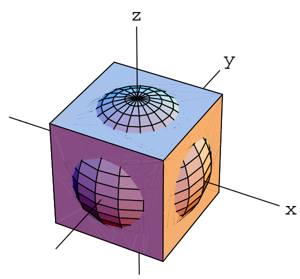

Fig. 7: Q_1 and B_t, $1 < t < \sqrt{2}$

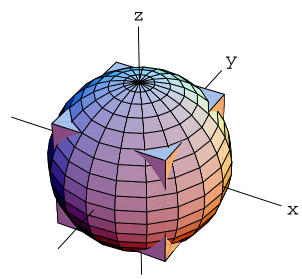

Fig. 8: Q_1 and B_t, $\sqrt{2} < t < \sqrt{3}$

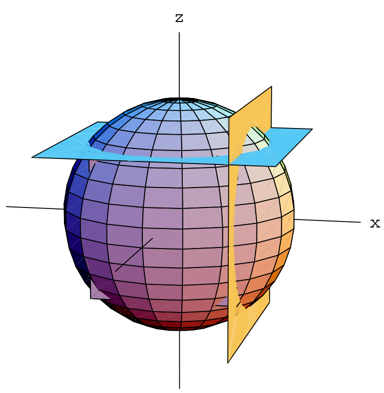

Fig. 9: L_t, $\sqrt{2} < t < \sqrt{3}$

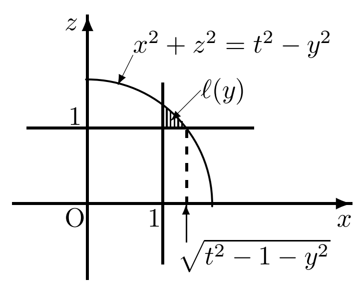

Fig. 10: $\ell(y)$, $-\sqrt{t^2 - 2} \leq y \leq \sqrt{t^2 - 2}$, $\sqrt{2} < t < \sqrt{3}$

Let

$$k(x) = |\{(y, z) \in \mathbb{R}^2 : (x, y, z) \in K_t\}|.$$

Then

$$|K_t| = \int_1^t k(x)\, dx = \int_1^t \pi \left(t^2 - x^2\right)\, dx = \pi \left(\frac{2}{3}t^3 - t^2 + \frac{1}{3}\right) \quad \text{for} \quad t > 1.$$

Hence we have (4.2).

Let (see Fig. 10)

$$\ell(y) = |\{(x,z) \in \mathbb{R}^2 : (x,y,z) \in L_t\}|.$$

Then

$$|L_t| = \int_{-\sqrt{t^2-2}}^{\sqrt{t^2-2}} \ell(y)\,dy = 2\int_0^{\sqrt{t^2-2}} \ell(y)\,dy \quad \text{for} \quad \sqrt{2} < t < \sqrt{3},$$

and

$$\ell(y) = \int_1^{\sqrt{t^2-1-y^2}} \left(\sqrt{t^2-y^2-x^2} - 1\right) dx$$

$$\text{for} \quad \sqrt{2} < t < \sqrt{3} \quad \text{and} \quad -\sqrt{t^2-2} \le y \le \sqrt{t^2-2}.$$

Let

$$R(u) = \frac{1}{2}\left(u\sqrt{1-u^2} + \arctan\left(\frac{u}{\sqrt{1-u^2}}\right)\right).$$

Then we have $R'(u) = \sqrt{1-u^2}$. Hence

$$\begin{aligned}\int_1^{\sqrt{t^2-1-y^2}} \sqrt{t^2-y^2-x^2}\,dx &= \int_{1/\sqrt{t^2-y^2}}^{\sqrt{t^2-1-y^2}/\sqrt{t^2-y^2}} \left(t^2-y^2\right)\sqrt{1-u^2}\,du \\ &= \left(t^2-y^2\right)\left(R\left(\frac{\sqrt{t^2-1-y^2}}{\sqrt{t^2-y^2}}\right) - R\left(\frac{1}{\sqrt{t^2-y^2}}\right)\right) \\ &= \frac{1}{2}\left(t^2-y^2\right)\left(\arctan\sqrt{t^2-1-y^2} - \arctan\left(\frac{1}{\sqrt{t^2-1-y^2}}\right)\right).\end{aligned}$$

Let

$$\begin{aligned}T(y) = {}&\left(t^2y - \frac{y^3}{3}\right)\left(\arctan\sqrt{t^2-1-y^2} - \arctan\left(\frac{1}{\sqrt{t^2-1-y^2}}\right)\right) \\ &- \frac{y}{3}\sqrt{t^2-1-y^2} - \left(t^2+\frac{1}{3}\right)\arctan\left(\frac{y}{\sqrt{t^2-1-y^2}}\right) \\ &+ \frac{2t^3}{3}\left(\arctan\left(\frac{t^2-1+ty}{\sqrt{t^2-1-y^2}}\right) - \arctan\left(\frac{t^2-1-ty}{\sqrt{t^2-1-y^2}}\right)\right).\end{aligned}$$

Then

$$T'(y) = \left(t^2-y^2\right)\left(\arctan\sqrt{t^2-1-y^2} - \arctan\left(\frac{1}{\sqrt{t^2-1-y^2}}\right)\right),$$

and

$$\ell(y) = \frac{1}{2}T'(y) - \sqrt{t^2-1-y^2} + 1.$$

Now we have

$$\begin{aligned}\int_0^{\sqrt{t^2-2}} T'(y)\,dy &= T\left(\sqrt{t^2-2}\right) - T(0) \\ &= -\frac{1}{3}\sqrt{t^2-2} - \left(t^2+\frac{1}{3}\right)\arctan\sqrt{t^2-2} \\ &\quad + \frac{2t^3}{3}\left(\arctan\left(t^2-1+t\sqrt{t^2-2}\right) - \arctan\left(t^2-1-t\sqrt{t^2-2}\right)\right),\end{aligned}$$

and

$$2\int_0^{\sqrt{t^2-2}} \sqrt{t^2-1-y^2}\,dy = 2\int_0^{\sqrt{t^2-2}/\sqrt{t^2-1}} \left(t^2-1\right)\sqrt{1-u^2}\,du$$
$$= \left(t^2-1\right)\left(R\left(\frac{\sqrt{t^2-2}}{\sqrt{t^2-1}}\right) - R(0)\right) = \sqrt{t^2-2} + \left(t^2-1\right)\arctan\sqrt{t^2-2}.$$

Hence

$$\begin{aligned}|L_t| =& 2\int_0^{\sqrt{t^2-2}} \ell(y)\,dy \\ =& \int_0^{\sqrt{t^2-2}} T'(y)\,dy - 2\int_0^{\sqrt{t^2-2}} \sqrt{t^2-1-y^2}\,dy + 2\int_0^{\sqrt{t^2-2}} dy \\ =& \frac{2}{3}\sqrt{t^2-2} - 2t^2\arctan\sqrt{t^2-2} + \frac{2}{3}\arctan\sqrt{t^2-2} \\ & + \frac{2t^3}{3}\arctan\left(t^2-1+t\sqrt{t^2-2}\right) - \frac{2t^3}{3}\arctan\left(t^2-1-t\sqrt{t^2-2}\right).\end{aligned}$$

Then we have (4.3).

5. Acknowledgement

The authors would like to thank Professor Shigehiko Kuratsubo for his many suggestions.

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