ON A NECESSARY CONDITION OF LOCAL INTEGRABILITY FOR COMPLEX VECTOR FIELD IN $\mathbb{R}^2$

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ABSTRACT. Let X be a non-solvable C^∞ complex vector field in $\mathbb{R}^2$ satisfying certain conditions. A necessary condition for the equation $Xu = 0$ to have a solution u such that $du \neq 0$ near the origin and one for the $Xu = 0$ to admit a solution u such that $du \not\equiv 0$ in any sufficiently small neighborhood of the origin are given. These are expressed by making use of an estimate.

1. INTRODUCTION

Let X_n be a nowhere-zero C^∞ complex vector field defined near a point P in $\mathbb{R}^n$. We shall say that X_n is locally integrable at P if there exist a neighborhood Ω of P and functions $u_i(i = 1, 2, \cdots, n-1)$ satisfying $X_n u_i = 0$ in Ω such that $du_1 \wedge du_2 \wedge \cdots \wedge du_{n-1}(\mathrm{P}) \neq 0$(see [13] and [14]).

The importance of the study of the local integrability originated from the papers of Lewy ([2] and [3]), where he found a holomorphic extension property of the solutions of some kind of homogeneous equations $X_n u = 0 (n = 3, 4)$ and pioneered a new type of the concept of holomorphic hull.

We know the following facts: X_n is locally integrable at P if X_n is real-analytic or locally solvable at P(see [14], for instance); there exists a X_2 which has the property that the $X_2 u = 0$ admits no non-trivial solutions in any neighborhood of the origin(Nirenberg [9]; see also [5]).

It is an open problem to obtain a necessary and sufficient condition for the local integrability of X_n that is non-analytic and not locally solvable.

In this paper, we investigate the case where $n = 2$.

We know that the equation $X_2 u = 0$ near P is transformed into that of the form

$$Lu \equiv \big(\partial_t + ia(t,x)\partial_x\big)u = 0$$

near the origin in $\mathbb{R}^2$, where $a(t,x)$ is a real-valued C^∞ function.

Now the following theorems are proved:

Theorem 1([12]). *Assume that $a(0,0) = 0$ and $a_t(0,0) \neq 0$. L is locally integrable at the origin if and only if there is a change of local coordinates such that L becomes a (non-vanishing C^∞ function) multiple of the Mizohata operator $\partial_{x_1} + ix_1\partial_{x_2}$.*

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Theorem 2([10]). *Assume that L satisfies $a(0,0)=0$ and $\partial_t a(0,0)\neq 0$. Then there exist C^∞ functions u^+, which is defined in $t\geq 0$, and u^-, which is defined in $t\leq 0$, such that $u^\pm(0,x)$ are real, $\partial_x u^\pm(0,x)>0$, and $Lu^\pm=0$. L is locally integrable at the origin if and only if the function ${u^+}^{-1}\circ u^-(0,x)$ is real analytic at the origin.*

Theorem 3([8]). *Assume that $a(t,x)=(t^{2d})'b(t,x)$, where d is a positive integer and $b(t,x)$ a positive C^∞ function. Then L is locally integrable at the origin if and only if there exist an element $\Big(Z_1(t,x),Z_2(t,x),T_0\Big)\in\mathfrak{S}$ and a function f which is holomorphic in $\mathfrak{J}=\{z\in\mathbb{C};z=Z_2(0,x),\ x\in(-T_0,T_0)\}$ and satisfies $Z_1(0,x)=f\Big(Z_2(0,x)\Big)$.*
(See [8] on the meaning of the symbol $\mathfrak{S}$.)

We can say that the above theorems are *qualitative* ones. On the other hand, in [6](see also [7]), we tried to obtain a *quantitative* condition to present a necessary condition(Theorem 3 in [6]) that is expressed *by making use of an estimate*, under the following assumption:
(i) $a(0,x)$ vanishes identically.
(ii) There is a neighborhood ω of the origin such that

$$t\big(a(t,x)-a(-t,x)\big)>0 \qquad \text{in} \quad \{t\neq 0\}\cap\omega$$

and

$$a(t,x)+a(-t,x)\geq 0 \qquad \text{in} \quad \omega.$$

In this article, we aim at giving a necessary condition for the local integrability and one* for the equation $Lu=0$ to have a non-trivial solution that are also expressed by making use of estimates, under the following assumption:
(a.1) $a_t(0,0)>0$.
(a.2) There is a neighborhood ω_0 of the origin such that

$$ta(t,x)>0, \quad ta_x(t,x)\geq 0 \quad \text{in} \quad \{t\neq 0\}\cap\omega_0.$$

Those estimates are shown in Theorems I and II stated in the next section.

2. Results

Our main results are stated as follows:

Theorem I. *Assume:*
(a.1) $a_t(0,0)>0$.
(a.2) *There is a neighborhood ω_0 of the origin such that*

$$ta(t,x)>0, \quad ta_x(t,x)\geq 0 \quad \textit{in} \quad \{t\neq 0\}\cap\omega_0.$$

If $Lu=0$ has a C^∞ solution u near the origin such that $u_x(0,0)\neq 0$, then there must exist positive constants C and G satisfying that, for every positive constant ϵ and for every positive constant ν larger than G

$$\iint_{\left\{(t,x);\ 0\leq t<\nu^{-\epsilon}\ ,\ |x|<\nu^{-\epsilon}\right\}} a_x(t,x)\,dtdx \leq C\nu^{-1-\epsilon}.$$

*If X_n is locally integrable, then the equation $X_n u=0$ trivially has a non-trivial solution. But we remark that the converse is not necessarily true: by virtue of Hölmander([1], Theorem 8.9.2), we see that there exists a X_2 having the property that the equation $X_2u=0$ has a non-trivial solution near the origin and X_2 is not locally integrable at the origin.

Theorem II. *Assume:*
(a.1) $a_t(0,0) > 0$.
(a.2) *There is a neighborhood* ω_0 *of the origin such that*

$$ta(t,x) > 0, \quad ta_x(t,x) \geq 0 \quad in \quad \{t \neq 0\} \cap \omega_0.$$

If $Lu = 0$ *has a* C^∞ *solution* u *in a neighborhood of the origin such that* $du \not\equiv 0$ *in any sufficiently small neighborhood of the origin, then there must exist a sequence of positive numbers* $\{x_m\}(m = 1, 2, \cdots)$ *which tends to* 0 *as* $m \to \infty$ *having the following property:*

There exist positive constants C_m *and* G_m *satisfying that, for every positive constant* ϵ *and for every positive constant* ν *larger than* G_m

$$\iint_{\left\{(t,x);\ 0 \leq t < \nu^{-\epsilon},\, |x - x_m| < \nu^{-\epsilon}\right\}} a_x(t,x)\, dt dx \leq C_m\, \nu^{-1-\epsilon}.$$

Let us give an example:

Example.

Set:

$$a_n = \frac{1}{n},$$

$$b_n = a_{n+1} + \frac{a_n - a_{n+1}}{2},$$

$$U_n = \left\{(t,x); b_n - \frac{a_n - a_{n+1}}{2^2} \leq t \leq b_n + \frac{a_n - a_{n+1}}{2^2},\ 0 \leq x \leq 1\right\},$$

and

$$V_n = \left\{(t,x); b_n - \frac{a_n - a_{n+1}}{2^3} \leq t \leq b_n + \frac{a_n - a_{n+1}}{2^3},\ 0 \leq x \leq 1\right\},$$

where $n \in \mathbb{N}$. Let $f_n(t,x)$ be the C^∞ function having the following properties:
(i) $0 \leq f_n(t,x) \leq a_{n+2} - a_{n+3}$.
(ii) $f_n(t,x)$ vanishes outside of U_n and equals $a_{n+2} - a_{n+3}$ in V_n.

We define the C_o^∞ function $\alpha(t,x)$ as follows:
(iii) $\alpha(t,x) = f_n(t,x)$ in U_n.
(iv) $\alpha(t,x) = 0$ in $\mathbb{R}^2_{t,x} \setminus \bigcup_{n=1}^\infty U_n$.

We have the following

Collorary. *The equation* $\left\{\partial_t + it\left(1 + \int_0^x \alpha(t,s)\, ds\right)\partial_x\right\}u = 0$ *admits no non-trivial* C^∞ *solutions in any neighborhood of the origin.*

3. Proof of Theorem I

Suppose $Lu = 0$ holds in a neighborhood Ω of the origin. Then we may assume that $ta(t,x) > 0$ and $ta_x(t,x) \geq 0$ in $\Omega \setminus \{t = 0\}$. Let us set $\mathrm{Re}\, u_x(0,0) = \alpha$ and $\mathrm{Im}\, u_x(0,0) = \beta$. Multiplying $u(t,x)$ by an appropriate complex number $\exp(i\theta)$, where θ is real, we can assume that $\alpha > 0$, $\beta < 0$, and $\alpha + \beta \neq 0$. Further let us set $\mathrm{Re}\, u_{xx}(0,0) = \gamma$ and $\mathrm{Im}\, u_{xx}(0,0) = \delta$.

For a positive number ν we define a complex number $\xi + i\eta$ in the following way:
Case 1 $\alpha + \beta > 0$. $\xi = \nu$, $\eta = 0$.
Case 2 $\alpha + \beta < 0$. $\xi = \nu$, $\eta = \nu$.

We define the function $w(t,x)$ by

$$w(t,x) = exp\{(\xi + i\eta)u(t,x)\}/exp\{(\xi + i\eta)u(0,0)\}.$$

It trivially follows that

$$Lw(t,x) = 0. \tag{3.1}$$

$$Lu_x(t,x) = -ia_x u_x(t,x). \tag{3.1'}$$

Hereafter let us set:

$$\alpha_1 = \operatorname{Re} w_x(0,0), \beta_1 = \operatorname{Im} w_x(0,0), \gamma_1 = \operatorname{Re} w_{xx}(0,0), \delta_1 = \operatorname{Im} w_{xx}(0,0),$$

$$c = \operatorname{Im} w_x(0,0) \operatorname{Re} w_{xx}(0,0) - \operatorname{Re} w_x(0,0) \operatorname{Im} w_{xx}(0,0),$$

and

$$d = \frac{\frac{a_x^2(0,0)}{a_t(0,0)} - \frac{\operatorname{Im} w_x(0,0) \operatorname{Re} w_{xx}(0,0) - \operatorname{Re} w_x(0,0) \operatorname{Im} w_{xx}(0,0)}{\left(\operatorname{Re} w_x(0,0)\right)^2 + \left(\operatorname{Im} w_x(0,0)\right)^2}}{\operatorname{Im} w_x(0,0)}.$$

Then we have the following:

Lemma 5. *There exists a positive constant G_0 such that, for every ν satisfying $\nu \geq G_0$, it holds that*

$$\alpha_1 > 0,\ \beta_1 < 0,\ \gamma_1 > 0,\ \delta_1 < 0,\ c > 0, \quad \textit{and} \quad d \geq \frac{3}{4}.$$

Proof. We have

$$\alpha_1 = \alpha\xi - \beta\eta, \beta_1 = \alpha\eta + \beta\xi,$$

$$\gamma_1 = \alpha_1^2 - \beta_1^2 + \gamma\xi - \delta\eta = (\alpha^2 - \beta^2)(\xi^2 - \eta^2) - 4\alpha\beta\xi\eta + \gamma\xi - \delta\eta,$$

and

$$\delta_1 = 2\alpha_1\beta_1 + \gamma\eta + \delta\xi = 2(\alpha^2 - \beta^2)\xi\eta + 2\alpha\beta(\xi^2 - \eta^2) + \gamma\eta + \delta\xi.$$

The symbols $K_l(\gamma,\delta)(l = 1, \cdots, 8)$ which will appear in the below denote certain polynomials, whose coefficients may depend on α or β, with respect to γ and δ.
Case 1. We have:

$$\alpha_1 = \alpha\nu,\ \beta_1 = \beta\nu$$

$$\gamma_1 = (\alpha - \beta)(\alpha + \beta)\nu^2 + K_1(\gamma,\delta)\nu, \delta_1 = 2\alpha\beta\nu^2 + K_2(\gamma,\delta)\nu$$

$$c = -\beta(\alpha^2 + \beta^2)\nu^3 + K_3(\gamma,\delta)\nu^2$$

$$d = \frac{a_x^2(0,0)}{\beta\nu a_t(0,0)} + 1 + \frac{K_4(\gamma,\delta)}{\nu}.$$

Case 2. We have:

$$\alpha_1 = (\alpha - \beta)\nu,\ \beta_1 = (\alpha + \beta)\nu$$

$$\gamma_1 = -4\alpha\beta\nu^2 + K_5(\gamma,\delta)\nu, \delta_1 = 2(\alpha - \beta)(\alpha + \beta)\nu^2 + K_6(\gamma,\delta)\nu$$

$$c = -2(\alpha+\beta)(\alpha^2+\beta^2)\nu^3 + K_7(\gamma,\delta)\nu^2$$
$$d = \frac{a_x^2(0,0)}{(\alpha+\beta)\nu a_t(0,0)} + 1 + \frac{K_8(\gamma,\delta)}{\nu}.$$

Thus we find that there exists a positive constant G_0 such that, in either case of the above it holds that

$$\alpha_1 > 0,\ \beta_1 < 0,\ \gamma_1 > 0,\ \delta_1 < 0,\ c > 0 \ \text{ and } \ d \geq \frac{3}{4}$$

for every ν satisfying $\nu \geq G_0$. □

Hereafter let $\nu \geq G_0$. Then by Lemma 5 we can take a neighborhood $\Omega_0(G_0)$ of the origin to suppose that the following hold: for every ν such that $\nu \geq G_0$,

$$\mathrm{Im}\, w_x(t,x) < 0, \tag{3.2}$$

and

$$\mathrm{Re}\, w_x(t,x) > 0 \tag{3.3}$$

in $\Omega_0(G_0)$.

Here we remark the following:
The function $i \log u_x(t,x)$ is defined as a single-valued function in $\Omega_0(G_0)$.

Since $\alpha_1 > 0$ and $\beta_1 < 0$, we can choose a positive constant b satisfying

$$2\alpha_1 + b\beta_1 = 0. \tag{3.4}$$

We set

$$W(t,x) = w(t,x) + 1 + ib.$$

We note that

$$W_x(t,x) = w_x(t,x) \quad \text{and} \quad W_t(t,x) = w_t(t,x).$$

Now, we have the following

Lemma 6. *There exists a neighborhood ω_1 of the origin satisfying*

$$\sup_{\omega_1} |W(t,x)| = |W(0,0)| = \frac{2\sqrt{\alpha_1^2+\beta_1^2}}{|\beta_1|}.$$

Proof. Since $W_x(t,x) = w_x(t,x)$, $\mathrm{Re}\, W(0,0) = 2$ and $\mathrm{Im}\, W(0,0) = b > 0$, we can take a sufficiently small neighborhood ω_1 ($\subseteq \Omega_0(G_0)$) of the origin such that the following hold:

$$\mathrm{Re}\, W(t,x) > 0. \tag{3.5.1}$$

$$\mathrm{Im}\, W(t,x) > 0. \tag{3.5.2}$$

$$\mathrm{Re}\, W_x(t,x) > \frac{\alpha_1}{2}. \tag{3.5.3}$$

$$\text{(3.5.4)} \qquad \operatorname{Im} W_x(t,x) < \frac{\beta_1}{2}.$$

Here we remark that the ω_1 is dependent of G_0 but can be chosen independently of ν. Now we set:

$$f(t,x) = \frac{1}{2}|W(t,x)|^2.$$

Since $W(0,0) = 2 + ib = 2(1 - \frac{\alpha_1}{\beta_1}i)$, we have only to prove the following:

(1) $f_t(t,x) = f_x(t,x) = 0$ in ω_1 if and only if $(t,x) = (0,0)$.
(2) $f_{tt}(0,0) < 0$ and $f_{tx}(0,0)^2 - f_{tt}(0,0)f_{xx}(0,0) < 0$.

Let us set $\operatorname{Re} W(t,x) = A(t,x)$ and $\operatorname{Im} W(t,x) = B(t,x)$. Since $LW(t,x) = 0$, we have

$$\text{(3.6)} \qquad A_t(t,x) = a(t,x)B_x(t,x), B_t(t,x) = -a(t,x)A_x(t,x).$$

First, (1) is proved in the following manner:

Let $f_t(t,x) = f_x(t,x) = 0$ in ω_1. From (3.6), this implies

$$a(t,x)\{A(t,x)B_x(t,x) - B(t,x)A_x(t,x)\} = 0,\ A(t,x)A_x(t,x) + B(t,x)B_x(t,x) = 0.$$

Suppose $t \neq 0$. Since $a(t,x) \neq 0$ for $t \neq 0$ by our assumption (a.2), we have

$$A(t,x)B_x(t,x) - B(t,x)A_x(t,x) = 0,$$

$$A(t,x)A_x(t,x) + B(t,x)B_x(t,x) = 0.$$

Since $A_x(t,x)^2 + B_x(t,x)^2 \neq 0$ by (3.5.3), and so $A(t,x) = B(t,x) = 0$. This is contradictory to (3.5.1). Therefore t must be 0. Then it follows that

$$A(0,x)A_x(0,x) + B(0,x)B_x(0,x) = 0.$$

Since there exist real numbers $\xi_j (j = 1,2,3,4)$ such that, $(0,\xi_j) \in \omega_1$ and

$$\begin{aligned}
A(0,x) &= A(0,0) + A_x(0,0)x + A_{xx}(0,\xi_1)x^2 \\
A_x(0,x) &= A_x(0,0) + A_{xx}(0,\xi_2)x \\
B(0,x) &= B(0,0) + B_x(0,0)x + B_{xx}(0,\xi_3)x^2 \\
B_x(0,x) &= B_x(0,0) + B_{xx}(0,\xi_4)x,
\end{aligned}$$

we have the following:

$$\begin{aligned}
&A(0,x)A_x(0,x) + B(0,x)B_x(0,x) = \\
&A(0,0)A_x(0,0) + \{A(0,0)A_{xx}(0,\xi_2) + A_x(0,0)^2\}x + \\
&A_x(0,0)\Big(A_{xx}(0,\xi_1) + A_{xx}(0,\xi_2)\Big)x^2 + \\
&A_{xx}(0,\xi_1)A_{xx}(0,\xi_2)x^3 + \\
&B(0,0)B_x(0,0) + \{B(0,0)B_{xx}(0,\xi_4) + B_{xx}(0,0)^2\}x + \\
&B_x(0,0)\Big(B_{xx}(0,\xi_3) + B_{xx}(0,\xi_4)\Big)x^2 + \\
&B_{xx}(0,\xi_3)B_{xx}(0,\xi_4)x^3 =
\end{aligned}$$

$$2\alpha_1 + \{2A_{xx}(0,\xi_2) + \alpha_1^2\}x + \alpha_1\Big(A_{xx}(0,\xi_1) + A_{xx}(0,\xi_2)\Big)x^2 + A_{xx}(0,\xi_1)A_{xx}(0,\xi_2)x^3 +$$

$$b\beta_1 + \{bB_{xx}(0,\xi_4) + \beta_1^2\}x + \beta_1\Big(B_{xx}(0,\xi_3) + B_{xx}(0,\xi_4)\Big)x^2 + B_{xx}(0,\xi_3)B_{xx}(0,\xi_4)x^3 =$$

$$x\Bigg\{2A_{xx}(0,\xi_2) + \alpha_1^2 + bB_{xx}(0,\xi_4) + \beta_1^2 + \Big\{\alpha_1\Big(A_{xx}(0,\xi_1) + A_{xx}(0,\xi_2)\Big) + \beta_1\Big(B_{xx}(0,\xi_3) + B_{xx}(0,\xi_4)\Big)\Big\}x + \Big(A_{xx}(0,\xi_1)A_{xx}(0,\xi_2) + B_{xx}(0,\xi_3)B_{xx}(0,\xi_4)\Big)x^2\Bigg\}$$

(by (3.4)). Therefore we have:

$$x\Bigg\{2A_{xx}(0,\xi_2) + \alpha_1^2 + bB_{xx}(0,\xi_4) + \beta_1^2 + \Big\{\alpha_1\Big(A_{xx}(0,\xi_1) + A_{xx}(0,\xi_2)\Big) + \beta_1\Big(B_{xx}(0,\xi_3) + B_{xx}(0,\xi_4)\Big)\Big\}x + \Big(A_{xx}(0,\xi_1)A_{xx}(0,\xi_2) + B_{xx}(0,\xi_3)B_{xx}(0,\xi_4)\Big)x^2\Bigg\} = 0. \tag{3.7}$$

Now we have:

$$2A_{xx}(0,0) + \alpha_1^2 + bB_{xx}(0,0) + \beta_1^2 < 0.$$

Proof. From (3.4), $b = -\frac{2\alpha_1}{\beta_1}$. Hence

$$2A_{xx}(0,0) + \alpha_1^2 + bB_{xx}(0,0) + \beta_1^2 =$$
$$\alpha_1^2 + \beta_1^2 + 2\left\{\frac{\beta_1 A_{xx}(0,0) - \alpha_1 B_{xx}(0,0)}{\beta_1}\right\} = \alpha_1^2 + \beta_1^2 + \frac{2c}{\beta_1} =$$
$$2(\alpha_1^2 + \beta_1^2)\left\{\frac{\frac{a_x(0,0)^2}{a_t(0,0)}}{\beta_1} + \frac{1}{2} - d\right\} < 0$$

by Lemma 5 and our assumption (a.1). □

Thus we find that (3.7) implies that $x = 0$ and hence $f_t(t,x) = f_x(t,x) = 0$ in ω_1 implies $(x,t) = (0,0)$. On the other hand it holds that $f_t(0,0) = f_x(0,0) = 0$ by (3.4).

Therefore, we have proved: $f_t(t,x) = f_x(t,x) = 0$ in $\omega_1 \Longleftrightarrow (x,t) = (0,0)$.

Next, (2) is proved in the following way:

Using (3.6), we have

$$f_{tx} = A_tA_x + \big(AB_x - BA_x\big)a_x(t,x) + \big(AB_{xx} - BA_{xx}\big)a(t,x),$$

$$f_{tt} = A_t^2 + B_t^2 + \big(AB_x - BA_x\big)a_t(t,x) + \big(AB_{tx} - BA_{tx}\big)a(t,x),$$

$$f_{xx} = A_x^2 + B_x^2 + AA_{xx} + BB_{xx}.$$

Hence, we have

$$f_{tt}(0,0) = \big(2\beta_1 - b\alpha_1\big)a_t(0,0)$$

and

$$f_{tx}(0,0)^2 - f_{tt}(0,0)f_{xx}(0,0) =$$

$$\big(2\beta_1 - b\alpha_1\big)^2 a_x(0,0)^2 - \big(2\beta_1 - b\alpha_1\big)\big(\alpha_1^2 + \beta_1^2 + 2\gamma_1 + b\delta_1\big)a_t(0,0) =$$

$$\big(2\beta_1 - b\alpha_1\big)^2 a_t(0,0)\Big\{\frac{a_x(0,0)^2}{a_t(0,0)} - \frac{\alpha_1^2 + \beta_1^2 + 2\gamma_1 + b\delta_1}{2\beta_1 - b\alpha_1}\Big\}.$$

Substituting $b = -\frac{2\alpha_1}{\beta_1}$, we have

$$2\beta_1 - b\alpha_1 = \frac{2\left(\alpha_1^2 + \beta_1^2\right)}{\beta_1}.$$

Hence we have

$$\frac{a_x(0,0)^2}{a_t(0,0)} - \frac{\alpha_1^2 + \beta_1^2 + 2\gamma_1 + b\delta_1}{2\beta_1 - b\alpha_1} =$$

$$\frac{a_x(0,0)^2}{a_t(0,0)} - \frac{\beta_1\gamma_1 - \alpha_1\delta_1}{\alpha_1^2 + \beta_1^2} - \frac{\beta_1}{2} = \beta_1\, d - \frac{\beta_1}{2} = \beta_1\Big(d - \frac{1}{2}\Big).$$

Therefore by Lemma 5, we see: $f_{tt}(0,0) < 0$, $f_{tx}(0,0)^2 - f_{tt}(0,0)f_{xx}(0,0) < 0$. $\square$

Now let us set $U(t,x) = i\log u_x(t,x)$(We have remarked that $U(t,x)$ is a single-valued function in ω_1). From (3.1'), we see:

$$LU(t,x) = a_x(t,x) \quad \text{in} \quad \omega_1. \tag{3.8}$$

Let ϵ be an arbitrary positive number. Taking ν large such that $\{(t,x); 0 \le t < \nu^{-\epsilon}, |x| < \nu^{-\epsilon}\} \subset \omega_1$, we set $r = \nu^{-\epsilon}$ and $D(r) = \{(t,x); 0 \le t < r, |x| < r\}$. Then we obtain the following

Lemma 7.

$$-\iint_{D(r)} a_x(t,x)W_x(t,x)\,dtdx =$$

$$\int_{\partial D(r)} W(t,x)U_t(t,x)\,dt + W(t,x)U_x(t,x)\,dx.$$

Proof. From (3.8), we have

$$\iint_{D(r)} -W_x(t,x)\Big\{U_t(t,x) + ia(t,x)U_x(t,x)\Big\}\,dtdx = \tag{3.9}$$

$$\iint_{D(r)} -a_x(t,x)W_x(t,x)\,dtdx.$$

Since $W_t(t,x) = -ia(t,x)W_x(t,x)$, the left-hand side of (3.9) =

$$\iint_{D(r)} -\Big\{W_x(t,x)U_t(t,x) - W_t(t,x)U_x(t,x)\Big\}\,dtdx =$$

$$\iint_{D(r)} d\Big\{W(t,x)dU(t,x)\Big\} =$$

$$\int_{\partial D(r)} W(t,x)U_t(t,x)\,dt + W(t,x)U_x(t,x)\,dx. \quad \square$$

From Lemma 7, we thus have:

$$\left| \iint_{D(r)} a_x(t,x)\{\operatorname{Re} W_x(t,x) + i \operatorname{Im} W_x(t,x)\}\,dtdx \right| \leq \tag{3.10}$$

$$\left| \int_{\partial D(r)} W(t,x)U_t(t,x)\,dt + W(t,x)U_x(t,x)\,dx \right|.$$

We shall estimate the right-hand side and the left-hand one of (3.10). Firstly the left-hand side is estimated as follows:

$$\left| \iint_{D(r)} a_x(t,x)\{\operatorname{Re} W_x(t,x) + i \operatorname{Im} W_x(t,x)\}\,dtdx \right| \geq$$

$$\frac{\sqrt{\alpha_1^2 + \beta_1^2}}{4} \iint_{D(r)} a_x(t,x)\,dtdx$$

Proof. The left-hand side above $\geq$

$$\frac{\left|\iint_{D(r)} a_x(t,x)A_x(t,x)\,dtdx\right| + \left|\iint_{D(r)} a_x(t,x)B_x(t,x)\,dtdx\right|}{\sqrt{2}}.$$

By (3.5.3) and (3.5.4),

$$\left| \iint_{D(r)} a_x(t,x)A_x(t,x)\,dtdx \right| =$$

$$\iint_{D(r)} a_x(t,x)A_x(t,x)\,dtdx \geq$$

$$\frac{\alpha_1}{2} \iint_{D(r)} a_x(t,x)\,dtdx.$$

$$\left| \iint_{D(r)} a_x(t,x)B_x(t,x)\,dtdx \right| =$$

$$\iint_{D(r)} a_x(t,x)\{-B_x(t,x)\}\,dtdx \geq$$

$$-\frac{\beta_1}{2} \iint_{D(r)} a_x(t,x)\,dtdx.$$

Hence we see

$$\frac{\left|\iint_{D(r)} a_x(t,x)A_x(t,x)\,dtdx\right| + \left|\iint_{D(r)} a_x(t,x)B_x(t,x)\,dtdx\right|}{\sqrt{2}} \geq$$

$$\frac{\alpha_1 - \beta_1}{2\sqrt{2}} \iint_{D(r)} a_x(t,x)\,dtdx >$$
$$\frac{\sqrt{\alpha_1^2 + \beta_1^2}}{4} \iint_{D(r)} a_x(t,x)\,dtdx. \quad \square$$

Next, since

$$\Big|\int_{\partial D(r)} W(t,x)U_t(t,x)\,dt + W(t,x)U_x(t,x)\,dx\Big| \leq$$
$$\int_{\partial D(r)} |W(t,x)||U_t(t,x)|\,dt + |W(t,x)||U_x(t,x)|\,dx,$$

we find that there exists a positive constant C_0 satisfying

$$\Big|\int_{\partial D(r)} W(t,x)U_t(t,x)\,dt + W(t,x)U_x(t,x)\,dx\Big| \leq$$
$$C_0|\partial D(r)| \sup_{\partial D(r)} |W(t,x)|.$$

Thus we have the following estimate:

$$\frac{\sqrt{\alpha_1^2 + \beta_1^2}}{4} \iint_{D(r)} a_x(t,x)\,dtdx \leq$$
$$C_0|\partial D(r)| \sup_{\partial D(r)} |W(t,x)|.$$

Making use of Lemma 5 and the above inequality, at last we have

$$\iint_{D(r)} a_x(t,x)\,dtdx \leq 8C_0 \frac{|\partial D(r)|}{|\beta_1|}.$$

From this we easily obtain the conclusion of Theorem I. $\square$

4. Proof of Theorem 2

Suppose that $Lu = 0$ holds in a neighborhood $\omega = (-\epsilon_1, \epsilon_1) \times (-\epsilon_1, \epsilon_1)$ of the origin where ϵ_1 is a positive constant. Needless to say, we may suppose that $ta(t,x) > 0$ and $ta_x(t,x) \geq 0$ in $\omega \setminus \{t = 0\}$.

Now we have the following:

Lemma 8. *There exists a sequence $\{x_m\}$ of positive numbers which tends to 0 as $m \longrightarrow \infty$ such that $u_x(0, x_m) \neq 0$ $(m = 1, 2, \cdots)$.*

Proof. Assume that there is a positive constant $\epsilon_0 \leq \epsilon_1$ such that $u_x(0,x)$ vanishes in $\{x; 0 \leq x \leq \epsilon_0\}$.

Setting $v = u_x(t,x)$, we have $Lv + ia_x(t,x)v = 0$. Since $v(0,x) = 0$ in $\{x; 0 \leq x \leq \epsilon_0\}$ and $ta(t,x) > 0$ for $t \neq 0$, applying the uniqueness theorem (see [4] or [11]), we find that v vanishes in $[-\epsilon_0, \epsilon_0] \times [0, \epsilon_0]$. Then, since L is elliptic for $t \neq 0$, we can apply the unique continuation theorem to conclude that v must vanish in ω. Thus we arrive at the contradiction that u is constant.

Hence we see that Lemma 8 holds. $\square$

By virtue of Lemma 8, we find that the same manner as is employed in the proof of Theorem 1 can be applied by only replacing the reasoning near the origin with the one near the point $(0, x_m)$ to get the conclusion of Theorem 2. $\square$

5. Proof of Corollary

It is clear that the assumptions (a.1) and (a.2) hold. Now assume the contrary. Then there exist positive constants x_m , C_m, and G_m satisfying that, for every positive constant ϵ and for every positive constant ν larger than G_m

$$\iint_{\{(t,x);\ 0 \leq t < \nu^{-\epsilon},\, |x - x_m| < \nu^{-\epsilon}\}} a_x(t,x)\, dt dx \leq\ C_m\, \nu^{-1-\epsilon}. \tag{4.1}$$

First, we take the positive integer p such that $\frac{1}{p+1} < x_m \leq \frac{1}{p}$ and set $\epsilon = \frac{1}{5}$. Next, we take ν sufficiently large such that $\nu^{-\frac{1}{5}} < \frac{1}{p} - \frac{1}{p+1} (= a_p - a_{p+1})$. Next, we take the positive integer N such that $\frac{1}{N} \leq \nu^{-\frac{1}{5}} < \frac{1}{N-1}$. Then, from (4.1), we have

$$\sum_{k=N}^{\infty} \nu^{-\frac{1}{5}} \cdot a_{k+4} \cdot \left(\frac{a_k - a_{k+1}}{2^2}\right) \cdot (a_{k+2} - a_{k+3}) \leq\ C_m\, \nu^{-1-\frac{1}{5}}.$$

Hence we have

$$\sum_{k=N}^{\infty} \frac{(a_k - a_{k+1})(a_{k+2} - a_{k+3}) a_{k+4}}{2^2} \leq\ C_m\, \nu^{-1}. \tag{4.2}$$

The lefthand side of (4.2)$= \frac{1}{4N(N+1)(N+2)(N+3)}$. Therefore we have:

$$\frac{1}{4N(N+1)(N+2)(N+3)} \leq\ C_m\, \nu^{-1} \leq \frac{C_m}{(N-1)^5}. \tag{4.3}$$

Since N can be taken sufficiently large, (4.3) produces a contradiction. □

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