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SOME INEQUALITIES FOR SUMS OF MATRICES

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Abstract. A group of inequalities for sums of Hermitian nonnegative definite matrices are derived by using some generalized matrix versions of the Cauchy-Schwarz and Frucht-Kantorovich inequalities.

In the recent decade, the well-known Cauchy-Schwarz and Frucht-Kantorovich inequalities have been extended to various generalized versions for matrices. Two remarkable cases were given by Marshall and Olkin (1990) as follows:

$$(X^*AX)^{-1} \leq X^*A^{-1}X \leq \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}(X^*AX)^{-1}, \tag{1}$$

where A is an $n \times n$ Hermitian positive definite matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n > 0$, while the $n \times p$ complex matrix X and its conjugate transpose of X^* satisfy $X^*X = I_p$. The Löwner partial ordering $A \geq B$ means that $A-B$ is Hermitian nonnegative definite. After that, some other extensions of (1) are also presented in the literature (see, e.g. [1, 3, 5, 6, 10, 12]). A generalization of (1) related to Moore-Penrose inverses of matrices is

Lemma 1. *Let A be an $n \times n$ nonnull Hermitian nonnegative definite matrix with rank $r \leq n$ and r positive eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0$, and let X be an $n \times p$ complex matrix. Then*

$$X^*P_AX(X^*AX)^+X^*P_AX \leq X^*A^+X \leq \frac{(\lambda_1+\lambda_r)^2}{4\lambda_1\lambda_r}X^*P_AX(X^*AX)^+X^*P_AX, \tag{2}$$

where $P_A = AA^+$ is the orthogonal projector on the range (column space) of A.

The left-hand inequality in (2) was first given by Baksalary and Puntanen (1991), the right-hand inequality in (2) was recently established by Drury et al (2000, Theorem 1). The left-hand side of (2) was later extended to a more general form by Pečarić et al (1996) as follows:

Lemma 2. *Let A be an $n \times n$ nonnegative definite matrix, and let P and Q be $n \times p$ and $n \times q$ matrices, respectively. Then*

$$Q^*AQ \geq Q^*AP(P^*AP)^+P^*AQ, \tag{3}$$

and

$$\operatorname{rank}[\,Q^*AQ - Q^*AP(P^*AP)^+P^*AQ\,] = \operatorname{rank}[\,AP,\ AQ\,] - \operatorname{rank}(AP). \tag{4}$$

Moreover, the following three statements are equivalent:

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(a) *The equality in* (3) *holds.*
(b) $\operatorname{range}(AQ) \subseteq \operatorname{range}(AP)$, *i.e., there is* Z *such that* $APZ = AQ$.
(c) $AQ = AP(P^*AP)^+P^*AQ$.

Here we give a direct proof for (3). As $A \geq 0$, we have

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} \geq 0,$$

and further

$$\begin{bmatrix} I_n & 0 \\ 0 & P \end{bmatrix}^* \begin{bmatrix} A & A \\ A & A \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} A & AP \\ P^*A & P^*AP \end{bmatrix} \geq 0.$$

In that case, the Schur complement $A - AP(P^*AP)^+P^*A$ of P^*AP in the block matrix is also nonnegative definite, i.e., $A - AP(P^*AP)^+P^*A \geq 0$. Consequently

$$Q^*[\, A - AP(P^*AP)^+P^*A\,]Q = Q^*AQ - Q^*AP(P^*AP)^+P^*AQ \geq 0,$$

as required for (3). Next let

$$M = \begin{bmatrix} P^*AP & P^*AQ \\ Q^*AP & Q^*AQ \end{bmatrix},$$

and note that $\operatorname{range}(P^*AQ) \subseteq \operatorname{range}(P^*AP)$. Then

$$\operatorname{rank}(M) = \operatorname{rank}(PAP^*) + \operatorname{rank}[\, Q^*AQ - Q^*AP(PAP^*)^+P^*AQ\,], \tag{5}$$

cf. Marsaglia and Styan (1974). On the other hand, $M = [\, A^{1/2}P,\ A^{1/2}Q\,]^*[\, A^{1/2}P,\ A^{1/2}Q\,]$. Thus

$$\operatorname{rank}(M) = \operatorname{rank}[\, A^{1/2}P,\ A^{1/2}Q\,] = \operatorname{rank}[\, AP,\ AQ\,]. \tag{6}$$

Combine (5) and (6) to yield

$$\operatorname{rank}[\, Q^*AQ - Q^*AP(P^*AP)^+P^*AQ\,] = \operatorname{rank}[\, AP,\ AQ\,] - \operatorname{rank}(AP).$$

Thus $Q^*AP(PAP^*)^+P^*AQ = Q^*AQ$ holds if and only if $\operatorname{range}(AQ) \subseteq \operatorname{range}(AP)$. Another method to show (3) is simpler: Note that

$$[\, AQ - AP(P^*AP)^+P^*AQ\,]^*A^+[\, AQ - AP(P^*AP)^+P^*AQ\,] \geq 0.$$

Expanding it and simplying, we also get (3). Clearly the equality in (3) holds if and only if $AQ = AP(P^*AP)^+P^*AQ$.

Since P and Q are arbitrarily given in (3), one can derive from (3) various consequences. One special case of (3) used in the sequel is

$$(NY)^*A(NY) \geq (X^*ANY)^*(X^*AX)^+(X^*ANY), \tag{7}$$

with equality if and only if $\operatorname{range}(ANY) \subseteq \operatorname{range}(AX)$.

The inequalities in (2), (3) and (7) can be used to establish various inequalities for sums of Hermitian nonnegative definite matrices. Our work in the note is to present some general inequalities for sums of Hermitian nonnegative definite matrices, and to give various special cases.

Theorem 3. *Let $A_1,\ A_2,\ \ldots,\ A_k$ be $n\times n$ Hermitian nonnegative definite matrices, and let $N_1,\ N_2,\ \ldots,\ N_k$ be $n\times p$ complex matrices. Then*

$$\sum_{i=1}^{k} N_i^* A_i N_i \geq \left(\sum_{i=1}^{k} A_i N_i\right)^* \left(\sum_{i=1}^{k} A_i\right)^+ \left(\sum_{i=1}^{k} A_i N_i\right), \tag{8}$$

with equality if and only if there is Z such that $A_i Z = A_i N_i,\ i=1,\ \ldots,\ k$. Furthermore, let $X_1,\ X_2,\ \ldots,\ X_k$ be $n\times q$ complex matrices. Then

$$\sum_{i=1}^{k} N_i^* A_i N_i \geq \left(\sum_{i=1}^{k} X_i^* A_i N_i\right)^* \left(\sum_{i=1}^{k} X_i^* A_i X_i\right)^+ \left(\sum_{i=1}^{k} X_i^* A_i N_i\right), \tag{9}$$

with equality if and only if there is Z such that $(A_i X_i)Z = A_i N_i,\ i=1,\ \ldots,\ k$.

Proof. Letting

$$A = \mathrm{diag}(\,A_1,\ A_2,\ \ldots,\ A_k\,), \qquad N = \mathrm{diag}(\,N_1,\ N_2,\ \ldots,\ N_k\,), \tag{10}$$

$$X^* = [\,I_n,\ I_n,\ \ldots,\ I_n\,], \qquad Y^* = [\,I_p,\ I_p,\ \ldots,\ I_p\,] \tag{11}$$

in (7), we see that

$$(NY)^* A (NY) = N_1^* A_1 N_1 + N_2^* A_2 N_2 + \ldots + N_k^* A_k N_k, \tag{12}$$

$$X^* ANY = A_1 N_1 + A_2 N_2 + \ldots + A_k N_k, \quad X^* AX = A_1 + A_2 + \ldots + A_k. \tag{13}$$

Putting (12) and (13) in (7) we obtain (8). Consequently, we let A, N and Y as in (10) and (11), and set $X^* = [\,X_1^*,\ X_2^*,\ \ldots,\ X_k^*\,]$. Then

$$X^* AX = X_1^* A_1 X_1 + X_2^* A_2 X_2 + \ldots + X_k^* A_k X_k, \tag{14}$$

$$X^* ANY = X_1^* A_1 N_1 + X_2^* A_2 N_2 + \ldots + X_k^* A_k N_k. \tag{15}$$

In that case, putting (12), (14) and (15) in (7) yields (9). $\square$

Take the replacements $N_i \longleftrightarrow X_i,\ i=1,\ \ldots,\ k$ in (9), we see that

$$\sum_{i=1}^{k} X_i^* A_i X_i \geq \left(\sum_{i=1}^{k} N_i^* A_i X_i\right)^* \left(\sum_{i=1}^{k} N_i^* A_i N_i\right)^+ \left(\sum_{i=1}^{k} N_i^* A_i X_i\right), \tag{16}$$

with equality if and only if there is Z such that $(A_i N_i)Z = A_i X_i,\ i=1,\ \ldots,\ k$. This inequality could be regarded as a dual inequality of (9) by taking Moore-Penrose inverses of the both sides of (9) as conventional matrix inverse operations. The dual inequality of (8) is

$$\sum_{i=1}^{k} A_i \geq \left(\sum_{i=1}^{k} A_i N_i\right) \left(\sum_{i=1}^{k} N_i^* A_i N_i\right)^+ \left(\sum_{i=1}^{k} A_i N_i\right)^*, \tag{17}$$

with equality if and only if there is Z such that $A_i N_i Z = A_i,\ i=1,\ \ldots,\ k$.

Various special cases can be derived from (8) and (9) and their dual forms (16) and (17). Let $N_i = A_i,\ i=1,\ \ldots,\ k$ in (8) and (17). Then we see that

$$\sum_{i=1}^{k} A_i^3 \geq \left(\sum_{i=1}^{k} A_i^2\right)^* \left(\sum_{i=1}^{k} A_i\right)^+ \left(\sum_{i=1}^{k} A_i^2\right), \tag{18}$$

with equality if and only if there is Z such that $A_iZ = A_i^2,\ i = 1,\ ...,\ k$; and

$$\sum_{i=1}^{k} A_i \geq \left(\sum_{i=1}^{k} A_i^2\right)^* \left(\sum_{i=1}^{k} A_i^3\right)^+ \left(\sum_{i=1}^{k} A_i^2\right), \tag{19}$$

with equality if and only if there is Z such that $A_i^2Z = A_i,\ i = 1,\ ...,\ k$.

Let $N_i = A_i^t,\ i = 1,\ ...,\ k$ in (8). Then we have

$$\sum_{i=1}^{k} A_i^{2t+1} \geq \left(\sum_{i=1}^{k} A_i^{t+1}\right)^* \left(\sum_{i=1}^{k} A_i\right)^+ \left(\sum_{i=1}^{k} A_i^{t+1}\right), \tag{20}$$

with equality if and only if there is Z such that $A_iZ = A_i^{t+1},\ i = 1,\ ...,\ k$. Its dual inequality by (17) is

$$\sum_{i=1}^{k} A_i \geq \left(\sum_{i=1}^{k} A_i^{t+1}\right)^* \left(\sum_{i=1}^{k} A_i^{2t+1}\right)^+ \left(\sum_{i=1}^{k} A_i^{t+1}\right), \tag{21}$$

with equality if and only if there is Z such that $A_i^{t+1}Z = A_i,\ i = 1,\ ...,\ k$.

If A_i is Hermitian positive definite and $N_i = A_i^{-1}B_i,\ i = 1,\ ...,\ k$, then (8) becomes

$$\sum_{i=1}^{k} B_i^* A_i^{-1} B_i \geq \left(\sum_{i=1}^{k} B_i\right)^* \left(\sum_{i=1}^{k} A_i\right)^{-1} \left(\sum_{i=1}^{k} B_i\right), \tag{22}$$

with equality if and only if $A_1^{-1}B_1 = ... = A_k^{-1}B_k$. Inequality (22) was recently established by Nakamoto and Takahashi (1999, Theorem 5), see also Zhang (1999, Theorem 6.3). Its dual inequality by (17) is

$$\sum_{i=1}^{k} A_i \geq \left(\sum_{i=1}^{k} B_i\right) \left(\sum_{i=1}^{k} B_i^* A_i^{-1} B_i\right)^+ \left(\sum_{i=1}^{k} B_i\right)^*, \tag{23}$$

with equality if and only if there is Z such that $B_iZ = A_i\ i = 1,\ ...,\ k$.

Setting $N_i = A_i^+,\ i = 1,\ ...,\ k$, in (8) yields

$$\sum_{i=1}^{k} A_i^+ \geq \left(\sum_{i=1}^{k} P_{A_i}\right) \left(\sum_{i=1}^{k} A_i\right)^+ \left(\sum_{i=1}^{k} P_{A_i}\right), \tag{24}$$

with equality if and only if there is Z such that $A_iZ = P_{A_i},\ i = 1,\ ...,\ k$.

Let $N_i = A_i^+X_i,\ i = 1,\ ...,\ k$ in (9), we have

$$\sum_{i=1}^{k} X_i^* A_i^+ X_i \geq \left(\sum_{i=1}^{k} X_i^* P_{A_i} X_i\right) \left(\sum_{i=1}^{k} X_i^* A_i X_i\right)^+ \left(\sum_{i=1}^{k} X_i^* P_{A_i} X_i\right), \tag{25}$$

with equality if and only if there is Z such that $(A_iX_i)Z = A_iA_i^+X_i,\ i = 1,\ ...,\ k$.

If we set $X_i = \sqrt{w_i} I_n,\ i = 1,\ ...,\ k$ in (25) with $\sum_{i=1}^k w_i = 1$, then

$$\sum_{i=1}^{k} w_i A_i^+ \geq \left(\sum_{i=1}^{k} w_i P_{A_i}\right) \left(\sum_{i=1}^{k} w_i A_i\right)^+ \left(\sum_{i=1}^{k} w_i P_{A_i}\right), \tag{26}$$

with equality if and only if there is Z such that $A_iZ = A_iA_i^+$, $i = 1, \ldots, k$. In particular,

$$w_1A_1^{-1} + w_2A_2^{-1} + \ldots + w_kA_k^{-1} \geq (w_1A_1 + w_2A_2 + \ldots + w_kA_k)^{-1}, \tag{27}$$

with equality if and only if $A_1 = \ldots = A_k$. This inequality is well known (see, e.g., [2, 4, 9]).

When all A_i's are positive definite in (25), we have

$$\sum_{i=1}^{k} X_i^*A_i^{-1}X_i \geq \left(\sum_{i=1}^{k} X_i^*X_i\right)\left(\sum_{i=1}^{k} X_i^*A_iX_i\right)^{+}\left(\sum_{i=1}^{k} X_i^*X_i\right), \tag{28}$$

with equality if and only if there is Z such that $(A_iX_i)Z = X_i$, $i = 1, \ldots, k$. Equivalently, (28) can be written as

$$\sum_{i=1}^{k} X_i^*A_iX_i \geq \left(\sum_{i=1}^{k} X_i^*X_i\right)\left(\sum_{i=1}^{k} X_i^*A_i^{-1}X_i\right)^{+}\left(\sum_{i=1}^{k} X_i^*X_i\right), \tag{29}$$

with equality if and only if there is Z such that $X_iZ = A_iX_i$, $i = 1, \ldots, k$. This result was first established by Kiefer (1959, Lemma 3.2).

Another inequality for the sum $\sum_{i=1}^{k} A_i^+$ can be derived from the right-hand side of (2).

Theorem 4. *Let $A_1, A_2, \ldots, A_k$ be $n \times n$ nonnull Hermitian nonnegative definite matrices. Then*

$$\sum_{i=1}^{k} A_i^+ \leq \frac{(m+M)^2}{4mM}\left(\sum_{i=1}^{k} P_{A_i}\right)\left(\sum_{i=1}^{k} A_i\right)^{+}\left(\sum_{i=1}^{k} P_{A_i}\right), \tag{30}$$

where M and m are, respectively, the maximum and minimum positive eigenvalues of $A_1, A_2, \ldots, A_k$.

In fact, letting $A = \mathrm{diag}(A_1, A_2, \ldots, A_k)$ and $X = [I_n, I_n, \ldots, I_n]$, we have $X^*P_AX = P_{A_1} + P_{A_2} + \ldots + P_{A_k}$, $X^*AX = A_1 + A_2 + \ldots + A_k$, and $X^*A^+X = A_1^+ + A_2^+ + \ldots + A_k^+$. In that case, the right-hand side of (2) becomes (30).

Combining (24) and (30), we can get a two-side inequality for the sum $\sum_{i=1}^{k} A_i^+$

$$S\left(\sum_{i=1}^{k} A_i\right)^{+}S \;\leq\; \sum_{i=1}^{k} A_i^+ \;\leq\; \frac{(m+M)^2}{4mM}S\left(\sum_{i=1}^{k} A_i\right)^{+}S, \tag{31}$$

where $S = \sum_{i=1}^{k} P_{A_i}$. Replacing A_i by A_i^+, $i = 1, \ldots, k$ in (31), we also have

$$S\left(\sum_{i=1}^{k} A_i^+\right)^{+}S \;\leq\; \sum_{i=1}^{k} A_i \;\leq\; \frac{(m+M)^2}{4mM}S\left(\sum_{i=1}^{k} A_i^+\right)^{+}S. \tag{32}$$

We call the first inequality in (31) the Cauchy-Schwarz part, and the second the Frucht-Kantorovich part.

Notice that $S = \sum_{i=1}^{k} P_{A_i}$ is Hermitian. It follows that $SS^+ = S^+S$. On the other hand, it is easy to verify that for nonnegative definite matrices $A_1, A_2, \ldots, A_k$

$$\mathrm{range}\left(\sum_{i=1}^{k} P_{A_i}\right) = \mathrm{range}\left(\sum_{i=1}^{k} A_i\right) = \mathrm{range}\left(\sum_{i=1}^{k} A_i^+\right).$$

Thus

$$SS^{+}\left(\sum_{i=1}^{k} A_i\right) = \left(\sum_{i=1}^{k} A_i\right) S^{+}S = \sum_{i=1}^{k} A_i, \quad SS^{+}\left(\sum_{i=1}^{k} A_i^{+}\right) = \left(\sum_{i=1}^{k} A_i^{+}\right) S^{+}S = \sum_{i=1}^{k} A_i.$$

Now pre- and post-multiplying $SS^{+} = SS^{+}$ to the inequalities (31) and (32), we get two equivalent inequalities as follows

$$\frac{4mM}{(m+M)^2}\sum_{i=1}^{k} S^{+}A_iS^{+} \le \left(\sum_{i=1}^{k} A_i^{+}\right)^{+} \le \sum_{i=1}^{k} S^{+}A_iS^{+}, \tag{33}$$

$$\frac{4mM}{(m+M)^2}\sum_{i=1}^{k} S^{+}A_i^{+}S^{+} \le \left(\sum_{i=1}^{k} A_i\right)^{+} \le \sum_{i=1}^{k} S^{+}A_i^{+}S^{+}. \tag{34}$$

If $A_1,\ A_2,\ \ldots,\ A_k$ are Hermitian positive definite, then (31) reduces to

$$k^2\left(\sum_{i=1}^{k} A_i\right)^{-1} \le \sum_{i=1}^{k} A_i^{-1} \le k^2\frac{(m+M)^2}{4mM}\left(\sum_{i=1}^{k} A_i\right)^{-1}. \tag{35}$$

In particular, when $k = 2$, (35) becomes

$$4(\,A+B\,)^{-1} \le A^{-1} + B^{-1} \le \frac{(m+M)^2}{mM}(\,A+B\,)^{-1}, \tag{36}$$

or equivalently,

$$4A(\,A+B\,)^{-1}B \le A + B \le \frac{(m+M)^2}{mM}A(\,A+B\,)^{-1}B, \tag{37}$$

where M and m are, respectively, the maximum and minimum positive eigenvalues of A and B. This result was presented by Zhang (1999, Sec. 6.2).

The product $A(\,A+B\,)^{-1}B$ is well known in the literature (see, e.g., Rao and Mitra (1971)) as the parallel sum of A and B. Thus (37) are in fact two inequalities for sum and parallel sum of two Hermitian positive definite matrices.

Some related work can further be considered, for example, how to find, just as what have been done in (31) and (32), the Frucht-Kantorovich counterparts for the inequalities (8) and (9), or in particular, for (18)–(23) and (25)–(28).

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