FUZZY SUBALGEBRAS OF BCK/BCI-ALGEBRAS REDEFINED

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ABSTRACT. A new fuzzification of subalgebras in BCK/BCI-algebras is considered, and its several properties are investigated.

1. Introduction

The notion of BCK-algebras was proposed by Imai and Iséki in 1966. In the same year, Iséki [1] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. After the introduction of the concept of fuzzy sets by Zadeh [3], several researches were conducted on the generalization of the notion of fuzzy sets. In this paper we introduce the notion of fuzzy dot subalgebras of BCK/BCI-algebras as a generalization of a fuzzy subalgebra, and then we investigate several basic properties which are related to fuzzy dot subalgebras. We state a condition for a fuzzy subset to be a fuzzy dot subalgebra. We investigate how to deal with the homomorphic image and inverse image of fuzzy dot subalgebras. We give a relation between a fuzzy dot subalgebra in BCK/BCI-algebras and a fuzzy dot subalgebra in the product algebra of BCK/BCI-algebras, and consider the projections of fuzzy dot subalgebras in product algebras.

2. Preliminaries

We review some definitions and properties that will be useful in our results. For more details, we refer to [1] and the book "BCK-algebras" [2].

By a *BCI-algebra* we mean an algebra (X, *, 0) of type (2,0) satisfying the following conditions:

- (I) ((x * y) * (x * z)) * (z * y) = 0,
- (II) (x * (x * y)) * y = 0,
- (III) x * x = 0,
- (IV) x * y = 0 and y * x = 0 imply x = y.
- A BCI-algebra X satisfying the additional condition:
- (V) 0 * x = 0 for all $x \in X$

is called a *BCK-algebra*. In any BCK/BCI-algebra X one can define a partial order " \leq " by putting $x \leq y$ if and only if x * y = 0.

A BCK/BCI-algebra X has the following properties:

- (2.1) x * 0 = x,
- $(2.2) \quad (x*y)*z = (x*z)*y,$
- (2.3) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$,
- (2.4) $(x * z) * (y * z) \le x * y$

for all $x, y, z \in X$. A BCI-algebra X is said to be *medial* if x * (x * y) = y for all $x, y \in X$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$

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whenever $x, y \in S$. A map f from a BCK/BCI-algebra X to a BCK/BCI-algebra Y is called a *homomorphism* if f(x * y) = f(x) * f(y) for all $x, y \in X$.

We now review some fuzzy logic concepts. A fuzzy subset of a set X is a function $\mu: X \to [0, 1]$. For any fuzzy subsets μ and ν of a set X, we define

$$\mu \subseteq \nu \Leftrightarrow \mu(x) \le \nu(x) \quad \forall x \in X,$$

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad \forall x \in X.$$

Let $f: X \to Y$ be a function from a set X to a set Y and let μ be a fuzzy subset of X. The fuzzy subset ν of Y defined by

$$\nu(y) := \begin{cases} \sup_{\substack{x \in f^{-1}(y) \\ 0}} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \ \forall y \in Y, \\ 0 & \text{otherwise,} \end{cases}$$

is called the *image* of μ under f, denoted by $f[\mu]$. If ν is a fuzzy subset of Y, the fuzzy subset μ of X given by $\mu(x) = \nu(f(x))$ for all $x \in X$ is called the *preimage* of ν under f and is denoted by $f^{-1}[\nu]$.

A fuzzy relation μ on a set X is a fuzzy subset of $X \times X$, that is, a map $\mu : X \times X \to [0, 1]$. A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if $\mu(x * y) \ge \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3. Fuzzy Subalgebras Redefined

We begin with the following definition which is a new fuzzification of subalgebra.

Definition 3.1. A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy dot subal*gebra of X if $\mu(x * y) \ge \mu(x) \cdot \mu(y)$ for all $x, y \in X$.

Example 3.2. Consider a BCK-algebra $X = \{0, a, b, c\}$ having the following Cayley table

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a fuzzy subset μ of X by $\mu(0) = \mu(a) = \mu(c) = 0.6$, and $\mu(b) = 0.3$. Then μ is a fuzzy dot subalgebra of X. Also a fuzzy subset ν of X defined by $\nu(0) = 0.38$, $\nu(a) = 0.4$, $\nu(b) = 0.6$, and $\nu(c) = 0.3$ is a fuzzy dot subalgebra of X which is not a fuzzy subalgebra of X since

$$\nu(a * b) = \nu(0) = 0.38 < 0.4 = \min\{\nu(a), \nu(b)\}.$$

Note that every fuzzy subalgebra of a BCK/BCI-algebra is a fuzzy dot subalgebra of a BCK/BCI-algebra, but the converse is not true (see Example 3.2).

Proposition 3.3. (i) If μ is a fuzzy dot subalgebra of a BCK/BCI-algebra X, then $\mu(0) \ge (\mu(x))^2$ for all $x \in X$.

(ii) If μ is a fuzzy dot subalgebra of a BCI-algebra X, then

$$\mu(0^n * x) \ge (\mu(x))^{2n+1}$$

for all $x \in X$ and $n \in \mathbb{N}$ where $0^n * x = 0 * (0 * (\cdots (0 * x) \cdots))$ in which 0 occurs n times.

Proof. Since x * x = 0 for all $x \in X$, it follows that

$$\mu(0) = \mu(x * x) \ge \mu(x) \cdot \mu(x) = (\mu(x))^2$$

for all $x \in X$. The proof of second part is by induction on n. For n = 1, we have $\mu(0 * x) \ge \mu(0) \cdot \mu(x) \ge (\mu(x))^3$ for all $x \in X$. Assume that $\mu(0^k * x) \ge (\mu(x))^{2k+1}$ for all $x \in X$. Then

$$\begin{array}{rcl} \mu(0^{k+1}*x) & = & \mu(0*(0^k*x)) \ge \mu(0) \cdot \mu(0^k*x) \\ & \ge & (\mu(x))^2 \cdot (\mu(x))^{2k+1} = (\mu(x))^{2(k+1)+1} \end{array}$$

Hence $\mu(0^n * x) \ge (\mu(x))^{2n+1}$ for all $x \in X$ and $n \in \mathbb{N}$.

Proposition 3.4. Let μ be a fuzzy dot subalgebra of a BCK/BCI-algebra X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} (\mu(x_n))^2 = 1$, then $\mu(0) = 1$.

Proof. According to Proposition 3.3, $\mu(0) \ge (\mu(x_n))^2$ for each $n \in \mathbb{N}$. Since $1 \ge \mu(0) \ge \lim_{n \to \infty} (\mu(x_n))^2 = 1$, it follows that $\mu(0) = 1$.

Theorem 3.5. If μ and ν are fuzzy dot subalgebras of a BCK/BCI-algebra X, then so is $\mu \cap \nu$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} (\mu \cap \nu)(x * y) &= \min\{\mu(x * y), \nu(x * y)\} \\ &\geq \min\{\mu(x) \cdot \mu(y), \nu(x) \cdot \nu(y)\} \\ &\geq \left(\min\{\mu(x), \nu(x)\}\right) \cdot \left(\min\{\mu(y), \nu(y)\}\right) \\ &= \left((\mu \cap \nu)(x)\right) \cdot \left((\mu \cap \nu)(y)\right). \end{aligned}$$

Hence $\mu \cap \nu$ is a fuzzy dot subalgebra of X.

Note that a fuzzy subset μ of a BCK/BCI-algbera X is a fuzzy subalgebra of X if and only if a nonempty level subset

$$U(\mu; t) := \{ x \in X \mid \mu(x) \ge t \}$$

is a subalgebra of X for every $t \in [0, 1]$. But, we know that if μ is a fuzzy dot subalgebra of X, then there exists $t \in [0, 1]$ such that

$$U(\mu; t) := \{ x \in X \mid \mu(x) \ge t \}$$

is not a subalgbra of X as seen in the following example.

Example 3.6. Consider a BCI-algebra $X = \{0, a, b, c\}$ possessing the following Cayley table

*	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

Define a fuzzy subset μ of X by $\mu(0) = 0.37$, $\mu(a) = 0.4$, $\mu(b) = 0.5$, and $\mu(c) = 0.3$. Then μ is a fuzzy dot subalgebra of X, and $U(\mu; 0.35) = \{0, a, b\}$ is not a subalgebra of X.

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Theorem 3.7. If μ is a fuzzy dot subalgebra of a BCK/BCI-algebra X, then

$$U(\mu; 1) := \{ x \in X \mid \mu(x) = 1 \}$$

is either empty or a subalgebra of X.

Proof. If x and y belong to $U(\mu; 1)$, then $\mu(x * y) \ge \mu(x) \cdot \mu(y) = 1$. Hence $\mu(x * y) = 1$ which implies $x * y \in U(\mu; 1)$. Consequently, $U(\mu; 1)$ is a subalgebra of X.

Theorem 3.8. Let X be a medial BCI-algebra and let μ be a fuzzy subset of X such that

 $\mu(0 * x) \ge \mu(x) \text{ and } \mu(x * (0 * y)) \ge \mu(x) \cdot \mu(y)$

for all $x, y \in X$. Then μ is a fuzzy dot subalgebra of X.

Proof. Since X is medial, we have 0 * (0 * y) = y for all $y \in X$. Hence

$$\mu(x * y) = \mu(x * (0 * (0 * y))) \ge \mu(x) \cdot \mu(0 * y) \ge \mu(x) \cdot \mu(y)$$

for all $x, y \in X$. Therefore μ is a fuzzy dot subalgebra of X.

Theorem 3.9. Every fuzzy subset μ of a medial BCI-algebra X with $\mu(0) = 1$ is a fuzzy dot subalgebra of X.

Proof. Since 0 * (0 * y) = y for all $y \in X$, it follows that

$$\begin{array}{lll} \mu(x*y) &=& \mu(x*(0*(0*y))) \\ &\geq& \mu(x)\cdot\mu(0*(0*y)) \\ &\geq& \mu(x)\cdot\mu(0)\cdot\mu(0)\cdot\mu(y) \\ &=& \mu(x)\cdot\mu(y) \end{array}$$

for all $x, y \in X$. Hence μ is a fuzzy dot subalgebra of X.

Theorem 3.10. Let $g: X \to Y$ be a homomorphism of BCK/BCI-algebras. If ν is a fuzzy dot subalgebra of Y, then the preimage $g^{-1}[\nu]$ of ν under g is a fuzzy dot subalgebra of X.

Proof. For any $x_1, x_2 \in X$, we have

$$\begin{array}{rcl} g^{-1}[\nu](x_1 \ast x_2) &=& \nu(g(x_1 \ast x_2)) = \nu(g(x_1) \ast g(x_2)) \\ &\geq& \nu(g(x_1)) \cdot \nu(g(x_2)) = g^{-1}[\nu](x_1) \cdot g^{-1}[\nu](x_2). \end{array}$$

Thus $g^{-1}[\nu]$ is a fuzzy dot subalgebra of X.

Theorem 3.11. Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If μ is a fuzzy dot subalgebra of X, then the image $f[\mu]$ of μ under f is a fuzzy dot subalgebra of Y.

Proof. For any $y_1, y_2 \in Y$, let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$, and $A_{12} = f^{-1}(y_1 * y_2)$. Consider the set

$$A_1 * A_2 := \{ x \in X \mid x = a_1 * a_2 \text{ for some } a_1 \in A_1 \text{ and } a_2 \in A_2 \}$$

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that

$$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2,$$

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that is, $x \in f^{-1}(y_1 * y_2) = A_{12}$. Hence $A_1 * A_2 \subseteq A_{12}$. It follows that

$$f[\mu](y_1 * y_2) = \sup_{\substack{x \in f^{-1}(y_1 * y_2) \\ \ge \\ x \in A_1 * A_2 \\ \ge \\ x_1 \in A_1, x_2 \in A_2}} \mu(x) = \sup_{\substack{x \in A_{12} \\ x_1 \in A_1, x_2 \in A_2}} \mu(x)$$

Since $\cdot : [0,1] \times [0,1] \to [0,1]$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\tilde{x_1} \ge \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\tilde{x_2} \ge \sup_{x_2 \in A_2} \mu(x_2) - \delta$, then $\tilde{x_1} \cdot \tilde{x_2} \ge \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) - \varepsilon$. Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\mu(a_1) \ge \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\mu(a_2) \ge \sup_{x_2 \in A_2} \mu(x_2) - \delta$. Then

Then

$$\mu(a_1) \cdot \mu(a_2) \ge \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) - \varepsilon$$

Consequently,

$$f[\mu](y_1 * y_2) \geq \sup_{\substack{x_1 \in A_1, x_2 \in A_2 \\ x_1 \in A_1}} \mu(x_1) \cdot \mu(x_2)$$

$$\geq \sup_{\substack{x_1 \in A_1 \\ x_2 \in A_2}} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2)$$

$$= f[\mu](y_1) \cdot f[\mu](y_2),$$

and hence $f[\mu]$ is a fuzzy dot subalgebra of Y.

Theorem 3.12. For a fuzzy subset σ of a BCK/BCI-algbera X, let μ_{σ} be a fuzzy subset of $X \times X$ defined by $\mu_{\sigma}(x, y) = \sigma(x) \cdot \sigma(y)$ for all $x, y \in X$. Then σ is a fuzzy dot subalgebra of X if and only if μ_{σ} is a fuzzy dot subalgebra of $X \times X$.

Proof. Assume that σ is a fuzzy dot subalgebra of X. For any $x_1, x_2, y_1, y_2 \in X$, we have

$$\mu_{\sigma}((x_{1}, y_{1}) * (x_{2}, y_{2})) = \mu_{\sigma}(x_{1} * x_{2}, y_{1} * y_{2}) = \sigma(x_{1} * x_{2}) \cdot \sigma(y_{1} * y_{2}) \geq \left(\sigma(x_{1}) \cdot \sigma(x_{2})\right) \cdot \left(\sigma(y_{1}) \cdot \sigma(y_{2})\right) = \left(\sigma(x_{1}) \cdot \sigma(y_{1})\right) \cdot \left(\sigma(x_{2}) \cdot \sigma(y_{2})\right) = \mu_{\sigma}(x_{1}, y_{1}) \cdot \mu_{\sigma}(x_{2}, y_{2}),$$

and so μ_{σ} is a fuzzy dot subalgebra of $X \times X$.

Conversely, suppose that μ_{σ} is a fuzzy dot subalgebra of $X \times X$ and let $x, y \in X$. Then

$$\begin{aligned} (\sigma(x*y))^2 &= & \mu_{\sigma}(x*y,x*y) = \mu_{\sigma}((x,x)*(y,y)) \\ &\geq & \mu_{\sigma}(x,x) \cdot \mu_{\sigma}(y,y) = (\sigma(x) \cdot \sigma(y))^2, \end{aligned}$$

and so $\sigma(x * y) \geq \sigma(x) \cdot \sigma(y)$, that is, σ is a fuzzy dot subalgebra of X.

Theorem 3.13. Let μ be a fuzzy relation on a BCK/BCI-algebra X satisfying the inequality $\mu(x, y) \leq \mu(x, 0)$ for all $x, y \in X$. Given $z \in X$, let σ_z be a fuzzy subset of X defined by $\sigma_z(x) = \mu(x, z)$ for all $x \in X$. If μ is a fuzzy dot subalgebra of $X \times X$, then σ_z is a fuzzy dot subalgebra of X for all $z \in X$.

Proof. Let $z, x, y \in X$. Then

$$\begin{aligned} \sigma_z(x*y) &= & \mu(x*y,z) = \mu(x*y,z*0) \\ &= & \mu((x,z)*(y,0)) \ge \mu(x,z) \cdot \mu(y,0) \\ &\ge & \mu(x,z) \cdot \mu(y,z) = \sigma_z(x) \cdot \sigma_z(y), \end{aligned}$$

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completing the proof.

Theorem 3.14. Let μ be a fuzzy relation on a BCK/BCI-algebra X and let σ_{μ} be a fuzzy subset of X given by $\sigma_{\mu}(x) = \inf_{y \in X} \mu(x, y) \cdot \mu(y, x)$ for all $x \in X$. If μ is a fuzzy dot subalgebra of $X \times X$ satisfying the equality $\mu(x,0) = 1 = \mu(0,x)$ for all $x \in X$, then σ_{μ} is a fuzzy dot subalgebra of X.

Proof. For any $x, y, z \in X$, we have

$$\begin{array}{lll} \mu(x*y,z) &=& \mu(x*y,z*0) = \mu((x,z)*(y,0)) \\ &\geq& \mu(x,z)\cdot \mu(y,0) = \mu(x,z) \end{array}$$

and

$$\begin{array}{lll} \mu(z,x*y) &=& \mu(z*0,x*y) = \mu((z,x)*(0,y)) \\ &\geq& \mu(z,x)\cdot \mu(0,y) = \mu(z,x). \end{array}$$

It follows that

$$\begin{array}{lll} \mu(x*y,z)\cdot\mu(z,x*y) & \geq & \mu(x,z)\cdot\mu(z,x) \\ & \geq & \left(\mu(x,z)\cdot\mu(z,x)\right)\cdot\left(\mu(y,z)\cdot\mu(z,y)\right) \end{array}$$

so that

$$\begin{aligned} {}_{\mu}(x*y) &= \inf_{z \in X} \mu(x*y,z) \cdot \mu(z,x*y) \\ &\geq \left(\inf_{z \in X} \mu(x,z) \cdot \mu(z,x) \right) \cdot \left(\inf_{z \in X} \mu(y,z) \cdot \mu(z,y) \right) \\ &= \sigma_{\mu}(x) \cdot \sigma_{\mu}(y). \end{aligned}$$

This completes the proof.

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Theorem 3.15. Let μ and ν be fuzzy dot subalgebras of BCK/BCI-algebras X and Y respectively. Then the cross product $\mu \times \nu$ of μ and ν defined by $(\mu \times \nu)(x,y) = \mu(x) \cdot \nu(y)$ for all $(x, y) \in X \times Y$ is a fuzzy dot subalgebra of $X \times Y$.

Proof. The proof is straightforward.

Theorem 3.16. Let X and Y be BCK/BCI-algberas and let
$$\mu$$
 be a fuzzy dot subalgebra of the product algebra $X \times Y$. Then the fuzzy subset $P_X[\mu]$ (resp. $P_Y[\mu]$) of X (resp. Y) defined by

$$P_X[\mu](x) = \mu(x,0) \ (resp. \ P_Y[\mu](y) = \mu(0,y))$$

for all $x \in X$ (resp. $y \in Y$) is a fuzzy dot subalgebra of X (resp. Y).

Proof. For any $x_1, x_2 \in X$, we have

$$P_X[\mu](x_1 * x_2) = \mu(x_1 * x_2, 0) = \mu(x_1 * x_2, 0 * 0)$$

= $\mu((x_1, 0) * (x_2, 0)) \ge \mu(x_1, 0) \cdot \mu(x_2, 0)$
= $P_X[\mu](x_1) \cdot P_X[\mu](x_2).$

Hence $P_X[\mu]$ is a fuzzy dot subalghera of X. Similarly, we can prove that $P_Y[\mu]$ is a fuzzy dot subalgbera of Y. \mathcal{O}

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