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RELATIVELY SUBPARACOMPACT SPACES

ZHIBIN QU AND YOSHIKAZU YASUI

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ABSTRACT. In this paper, the relative version of subparacompactness is studied. We start with definitions of relative subparacompactness and some examples to show the relationships between those definitions. Then we give some characterizations of relative subparacompactness. Finally, we talk about some basic properties of relative subparacompactness and the relationships between relative subparacompactness and some other known relative topological properties.

1. INTRODUCTION

Throughout this paper, all spaces are *regular* and T_1; all mappings are *continious* and *onto*. N will denote the set of all *natural numbers* and Y will always be a subspace of X. You may refer to [4] and [3] for undefined notations and terminologies.

Let $\mathcal{V}$ and $\mathcal{U}$ be families of subsets of a space X, we say that $\mathcal{V}$ is a *partial refinement* of $\mathcal{U}$, if for any V in $\mathcal{V}$ there is a U in $\mathcal{U}$ such that $V \subset U$; moreover, if in addition $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ is also satisfied, we will say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$. Let $\mathcal{U}$ and X be the same as above, Y a subspace of X, we say $\mathcal{U}$ is *discrete at Y in X*, if for each point y in Y there is an open in X neighbourhood of y intersects at most one member of $\mathcal{U}$. $\mathcal{U}$ is *locally finite at Y in X* may be defined in a similar way. Having the definitions above, we will have *σ-discrete at Y in X, σ-locally finite at Y in X* naturally.

Definition 1.1. We say Y is 1-subparacompact in X, if for any open cover $\mathcal{U}$ of X, there is a σ-discrete in X closed in X partial refinement $\mathcal{F}$ of $\mathcal{U}$, such that $\bigcup \mathcal{F} \supset Y$.

Definition 1.2. We say Y is 2-subparacompact in X, if for any open cover $\mathcal{U}$ of X, there is a σ-discrete at Y in X closed in X partial refinement $\mathcal{F}$ of $\mathcal{U}$, such that $\bigcup \mathcal{F} \supset Y$.

Definition 1.3. We say Y is 1^*-subparacompact in X, if for any open cover $\mathcal{U}$ of X, there is a σ-discrete at Y in X closed in X refinement $\mathcal{F}$ with $\bigcup \mathcal{F} = X$.

Remark 1.4. *When Y is equal to X, the three relative versions of subparacompactness above obviously coincide with the original version.*

Proposition 1.5. *Let X be a regular space and Y a subspace of X, then*

(1) *Y is 1-subparacompact in X $\iff$ for any open cover $\mathcal{U}$ of X there is a σ-discrete in X closed in Y partial refinement $\mathcal{F}$ of $\mathcal{U}$, such that $\bigcup \mathcal{F} = Y$;*

(2) *Y is 2-subparacompact in X $\iff$ for any open cover $\mathcal{U}$ of X, there is a σ-discrete in Y closed in Y partial refinement $\mathcal{F}$ of $\mathcal{U}$, such that $\bigcup \mathcal{F} = Y$.*

Proposition 1.6. *If Y is 1-subparacompact in X and $Z \subset Y$, then Z is 1-subparacompact in X.*

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Proposition 1.7. *Let X, Y and Z be subspaces of a space W with $Z \subset Y \subset X \subset W$ and Y 2-subparacompact in X, then Z is 2-subparacompact in W.*

As to the relationship among the three relative versions of subparacompactness, we have the following diagram:

$$\text{1-subparacompact} \Longrightarrow \text{2-subparacompact} \Longleftarrow \text{1}^*\text{-subparacompact}$$

The following two examples show that neither of the implication relations above can be reversed; What's more, there is not any implication relation between 1-subparacompactness and 1^*-subparacompactness.

Example 1.8. Let X be the set of all countable ordinals with the natural order topology and Y the subset of all the isolated points in it. Then Y is 2-subparacompact in X and at the same time 1^*-subparacompact in X, but not 1-subparacompact in X.

Proof. That Y is 2-subparacompct and 1^*-subparacompact in X is obvious, so we will only show that Y is not 1-subparacompact in X.

Suppose that Y is 1-subparacompact in X, then by Theorem 2.1 on page 743, for the open cover $\mathcal{U} = \{[0, \alpha] : \alpha \in X\}$ of X, there is a sequence $\{\mathcal{G}_n : n \in N\}$ of open (in X) refinements of $\mathcal{U}$ satisfying the condition (6) of Theorem 2.1. Then, there should be an uncountable subset Y_1 of Y and a natural number $n \in N$ such that, for all $\alpha \in Y_1$, $Ord(\alpha, \mathcal{G}_n) = 1$. Consequently, we will have a sequence $\{\alpha_i : i \in N\}$ of distinct points in Y_1, such that, each $G \in \mathcal{G}_n$ contains at most one point of $\{\alpha_i : i \in N\}$. But this will contradict with the fact that X is countably compact and $\{\mathcal{G}_n\}$ is a cover of X. □

Example 1.9. Let X be the space $(\omega_2 \times \omega_2) \setminus \{(0,0)\}$ with the topology generated by the family of subsets, $\{H_\alpha \setminus F : \alpha \in \omega_2 \setminus \{0\}, F$ is a finite subset of $X\} \bigcup \{V_\alpha \setminus F : \alpha \in \omega_2 \setminus \{0\}$, F is a finite subset of $X\}$. Where, $H_\alpha = \omega_2 \times \{\alpha\}$ and $V_\alpha = \{\alpha\} \times \omega_2$ are defined for all $\alpha \in (0, \omega_2)$.

Let Y be the subspace $((\{0\} \times \omega_2) \bigcup (\omega_2 \times \{0\})) \bigcap X$. Then, Y is 1-subparacompact in X and of course 2-subparacompact in X, but not 1^*-subparacompact in X.

Proof. We will only show that Y is not 1^*-subparacompact in X. Suppose for the open cover $\mathcal{U} = \{V_\alpha, H_\alpha : \alpha \in (0, \omega_2)\}$ of X, there is a refinement $\mathcal{P} = \bigcup_1^\infty \mathcal{P}_n$ such that $\mathcal{P}_n$ is discrete (or even locally finite) at Y in X. Let $\mathcal{H}_n = \{P \in \mathcal{P}_n : P \subset H_\alpha$ for some $\alpha \in (0, \omega_2)\}$, $\mathcal{V}_n = \{P \in \mathcal{P}_n : P \subset V_\alpha$ for some $\alpha \in (0, \omega_2)\}$, $A_n = \bigcup \mathcal{H}_n$, $B_n = \bigcup \mathcal{V}_n$, $A = \bigcup_1^\infty A_n$ and $B = \bigcup_1^\infty B_n$.

Obviously, for any $\alpha \in (0, \omega_2)$, both $A \bigcap V_\alpha$ and $B \bigcap H_\alpha$ are countable. It is then not difficult to see that $A \bigcup B \neq X$.

□

2. Main results

Theorem 2.1. *Let Y be a subspace of a space X, then the following are equivalent:*

1. *Y is 1-subparacompact in X;*
2. *For any open cover $\mathcal{U}$ of X, there is a σ-locally finite in X closed in X partial refinement $\mathcal{F}$ of $\mathcal{U}$, such that $\bigcup \mathcal{F} \supset Y$;*
3. *For any open cover $\mathcal{U}$ of X, there is a σ-closure preserving closed in X partial refinement $\mathcal{F}$ of $\mathcal{U}$, such that $\bigcup \mathcal{F} \supset Y$;*
4. *For any open cover $\mathcal{U}$ of X, there is a σ-cushioned partial refinement $\mathcal{F}$ of $\mathcal{U}$, such that $\bigcup \mathcal{F} \supset Y$;*
5. *For any open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{G}_n\}_1^\infty$ of open refinements such that for any $y \in Y$, there is some $n \in N$ with $St(y, \mathcal{G}_n) \subset U$ for some $U \in \mathcal{U}$;*

(6) *For any open cover* $\mathcal{U}$ *of* X*, there is a sequence* $\{\mathcal{G}_n\}_1^\infty$ *of open refinements such that for any* $y \in Y$*, there is some* $n \in N$ *with* $Ord(y, \mathcal{G}_n) = 1$.

Proof. (6) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. We will show (1) $\Rightarrow$ (6), (5) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) consequently.

(1) $\Rightarrow$ (6): Let $\mathcal{U}$ be any open cover of X, by (1), there is a closed in X partial refinement $\mathcal{F} = \bigcup_1^\infty \mathcal{F}_n$ such that $\bigcup \mathcal{F} \supset Y$ and $\mathcal{F}_n$ is discrete in X for all $n \in N$. For each $n \in N$, put $E_n = \bigcup \mathcal{F}_n$. For each $F \in \mathcal{F}_n$, pick $U(F) \in \mathcal{U}$ with $F \subset U(F)$. Let $G(F) = U(F) \setminus (E_n \setminus F)$, and $\mathcal{G}_n = \{G(F) : F \in \mathcal{F}_n\} \bigcup \{U \setminus E_n : U \in \mathcal{U}\}$. It's routine to check that $\{\mathcal{G}_n\}_1^\infty$ is the desired sequence of open refinements in (6).

(5) $\Rightarrow$ (4): Suppose $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is an open cover of X, and $\{\mathcal{G}_n\}_1^\infty$ is the sequence of open refinements as described in (5). For each $n \in N$ and each $\alpha \in \Lambda$, define $C(\alpha, n) = \{x \in X : St(x, \mathcal{G}_n) \subset U_\alpha\}$ and $\mathcal{C}_n = \{C(\alpha, n) : \alpha \in \Lambda\}$

Then $\mathcal{C} = \bigcup_1^\infty \mathcal{C}_n$ is a partial refinement of $\mathcal{U}$, satisfying $\bigcup \mathcal{C} \supset Y$.

To see $\mathcal{C}_n$ is cushioned in $\mathcal{U}$ for all $n \in N$, suppose there is some $\Lambda^* \subset \Lambda$ and some point $z \in (X \setminus \bigcup_{\alpha \in \Lambda^*} U_\alpha)$, then as for any $\alpha \in \Lambda^*$ and any $y \in C(\alpha, n)$, we have $St(y, \mathcal{G}_n) \subset U_\alpha$, we will have $z \notin St(y, \mathcal{G}_n)$. Consequently, $y \notin St(z, \mathcal{G}_n)$. We have shown $z \in St(z, \mathcal{G}_n) \subset X \setminus \bigcup_{\alpha \in \Lambda^*} C(\alpha, n)$. The fact that $St(z, \mathcal{G}_n)$ is a neighbourhood of z finishes our proof.

(4) $\Rightarrow$ (1):

Notation: for any $n, k \in N$ and any sequence $s = (i_1, i_2, \ldots, i_k) \in N^k$, denote by $s \oplus n$ the sequence $(i_1, i_2, \ldots, i_k, n) \in N^{k+1}$.

Suppose (4) is true and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is an open cover of X with Λ well-ordered.

For each $k \in N$ and each $s \in N^k$, we define (by induction on k) an open refinement $\mathcal{G}(s)$ of $\mathcal{U}$ and a corresponding σ-cushioned partial refinement $\mathcal{F}(s)$ of $\mathcal{G}(s)$ with the condition that $\bigcup \mathcal{F}(s) \supset Y$ as follows:

(i) For each $t \in N$ (as a sequence of length 1), Let $V_\alpha(t) = W_\alpha(t) = U_\alpha$ for all $\alpha \in \Lambda$. Define $\mathcal{G}(t) = \{V_\alpha(t) : \alpha \in \Lambda\} \bigcup \{W_\alpha(t) : \alpha \in \Lambda\}$.

(ii) Assume $\mathcal{G}(s)$, an open refinement of $\mathcal{U}$, has been defined for $s \in N^k$ where $\mathcal{G}(s)$ has the form $\mathcal{G}(s) = \{V_\alpha(s) : \alpha \in \Lambda\} \bigcup \{W_\alpha(s) : \alpha \in \Lambda\}$. Where, $V_\alpha(s) \bigcup W_\alpha(s)$(denoted as $G_\alpha(s)$) $\subset U_\alpha$.

Let $\mathcal{F}(s)$ be a σ-cushioned partial refinement of $\mathcal{G}(s)$ such that $\bigcup \mathcal{F}(s) \supset Y$, where $\mathcal{F}(s)$ has the form

$$\mathcal{F}(s) = \{H_\alpha(s \oplus n) : \alpha \in \Lambda, n \in N\}$$
$$\bigcup \{K_\alpha(s \oplus n) : \alpha \in \Lambda, n \in N\},$$

where, for each $n \in N$, $\{H_\alpha(s \oplus n) : \alpha \in \Lambda\}$ is cushioned in $\{V_\alpha(s) : \alpha \in \Lambda\}$ and $\{K_\alpha(s \oplus n) : \alpha \in \Lambda\}$ is cushioned in $\{W_\alpha(s) : \alpha \in \Lambda\}$.

(iii) To complete the induction, define $V_\alpha(s \oplus n) = G_\alpha(s) \setminus Cl_X(\bigcup_{\beta \neq \alpha}(H_\beta(s \oplus n) \cup K_\beta(s \oplus n)))$, $W_\alpha(s \oplus n) = G_\alpha(s) \bigcap (\cup_{\beta > \alpha} G_\beta(s)) \setminus Cl_X(\bigcup_{\beta < \alpha}(H_\beta(s \oplus n) \cup K_\beta(s \oplus n)))$ and $\mathcal{G}(s \oplus n) = \{V_\alpha(s \oplus n) : \alpha \in \Lambda\} \bigcup \{W_\alpha(s \oplus n) : \alpha \in \Lambda\}$. Then, $\mathcal{G}(s \oplus n)$ covers X. In fact, for each $x \in X$, if $x \notin \bigcup_{\alpha \in \Lambda} V_\alpha(s \oplus n)$, we can pick the smallest $\gamma \in \Lambda$ such that $x \in G_\gamma(s)$. Thereby, $x \notin Cl_X(\bigcup_{\beta < \gamma}(H_\beta(s \oplus n) \bigcup K_\beta(s \oplus n)))$ and $x \in Cl_X(\bigcup_{\beta > \gamma}(H_\beta(s \oplus n) \bigcup K_\beta(s \oplus n)))$ imply that $x \in W_\gamma(s \oplus n)$.

Finally, for $s \in N^k, n \in N$, and $\alpha \in \Lambda$, let $T_\alpha(s \oplus n) = Cl_X(H_\alpha(s \oplus n)) \setminus \bigcup_{\beta \neq \alpha} V_\beta(s)$.

Then, $\{T_\alpha(s \oplus n) : \alpha \in \Lambda\}$ is a discrete collection of closed subsets in X and $T_\alpha(s \oplus n) \subset U_\alpha$ for all $\alpha \in \Lambda$. So we are through if we show $\mathcal{T} = \{T_\alpha(s \oplus n) : \alpha \in \Lambda, n \in N, s \in \bigcup_{k=1}^\infty N^k\}$ covers Y.

To show that, let y be any point in Y and $\delta = min\{\beta \in \Lambda : y \in H_\beta(s) \bigcup K_\beta(s), s \in \bigcup_{k=2}^\infty N^k\}$. Then there is some $t \in \bigcup_{k=1}^\infty N^k$ and some $n \in N$, such that $y \in H_\delta(t \oplus$

$n)\bigcup K_\delta(t\oplus n)$. Consequently we will have $y\notin V_\alpha(t\oplus n)$ (for $\alpha\neq\delta$) and $y\notin W_\alpha(t\oplus n)$ (for $\alpha>\delta$). Let $m\in N,\sigma\in\Lambda$ (the existence of such m and σ is trivial), such that $y\in H_\sigma(t\oplus n\oplus m)\bigcup K_\sigma(t\oplus n\oplus m)\subset V_\sigma(t\oplus n)\bigcup W_\sigma(t\oplus n)$. Then, $y\in H_\delta(t\oplus n\oplus m)\bigcup K_\delta(t\oplus n\oplus m)$ and $y\notin W_\delta(t\oplus n\oplus m)$. In a similar way, we can also show $y\notin V_\alpha(t\oplus n\oplus m)$ for all $\alpha\neq\delta$, and and $y\in H_\delta(t\oplus n\oplus m\oplus k)\bigcup K_\delta(t\oplus n\oplus m\oplus k)$ (for some $k\in N$). Therefore,

$$y\in H_\delta(t\oplus n\oplus m\oplus k)\setminus\bigcup_{\beta\neq\delta}V_\beta(t\oplus n\oplus m)\subset T_\delta(t\oplus n\oplus m\oplus k)$$

□

Theorem 2.2. *Let Y be a subspace of a space X, then the following are equivalent:*

(1) *Y is 2-subparacompact in X;*
(2) *For any open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{G}_n\}_1^\infty$ of open in X partial refinements such that $\bigcup\mathcal{G}_n\supset Y$, and for any $y\in Y$, there is some $n\in N$ with $Ord(y,\mathcal{G}_n)=1$.*

To prove the above theorem, we use the following easy propositions, the proof of which is omitted.

Proposition 2.3. *The condition (2) of Theorem 2.2 is equivalent to the following condition:*

(2′): *for every open cover of X, there is a sequence $\{\mathcal{G}_n\}_{n=1}^\infty$ of open in Y partial refinements such that $\bigcup\mathcal{G}_n=Y$ for all n, and for each $y\in Y$ there is some $n\in N$ with $Ord(y,\mathcal{G}_n)=1$.*

Proof of Theorem 2.2. (1) ⇒ (2): For any open cover $\mathcal{U}$ of X, by Proposition 1.5, we will have a σ-discrete in Y closed in Y partial refinement $\mathcal{F}$ of $\mathcal{U}$, with $\bigcup\mathcal{F}=Y$. In a similar way to that in the proof of the implication (1) ⇒ (6) of Theorem 2.1, we can get a sequence of open in Y covers of Y satisfying the condition (2′) of Proposition 2.3.

2) ⇒ (1): Let $\mathcal{U}$ be any open cover of X, by Proposition 2.3, there is a sequence $\{\mathcal{G}_n\}_{n=1}^\infty$ of open in Y covers of Y satisfying the condition (2′) of Proposition 2.3. For each $n\in N$, let $Y_n=\{y\in Y:Ord(y,\mathcal{G}_n)=1\}$. Obviously, $\bigcup_{n=1}^\infty Y_n=Y$.

For every $y\in Y_n$, let $G(y)$ be the unique open set in $\mathcal{G}_n$ with $y\in G(y)$. Let $\mathcal{F}_n=\{G(y)\bigcap Y_n:y\in Y_n\}$, then $\mathcal{F}_n$ is a closed and discrete collection in the space Y. And $\mathcal{F}=\bigcup_{n=1}^\infty\mathcal{F}_n$ is the desired collection satisfying the condition described in (2) of Proposition 1.5. □

3. Applications

The condition (3) of Theorem 2.1 yields the following

Theorem 3.1. *If Y is 1-subparacompact in X and $f:X\longrightarrow X^*$ is closed and onto with $f(Y)=Y^*$, then Y^* is 1-subparacompact in X^*.*

Theorem 3.2. *If $f:X\longrightarrow X^*$ is perfect onto and Y^* is 1-subparacompact in X^* with $Y=f^{-1}(Y^*)$, then Y is 1-subparacompact in X.*

Proof. The proof of this theorem is routine, so we omit it.

□

Theorem 3.3. *Countable union of 1-subparacompact in X subspaces is 1-subparacompact in X.*

Proof. Omitted. □

Theorem 3.4. *Let X be a space and Y a F_σ-set in X (we may write Y as $Y=\bigcup\{Y_n:n\in N\}$, where Y_n is closed in X), then the following are equivalent:*

(1) *Y is 1-subparacompact in X;*
(2) *Y is 2-subparacompact in X;*
(3) *Y_n is 1-subparacompact in X for all $n \in N$;*
(4) *Y_n is 2-subparacompact in X for all $n \in N$;*
(5) *Y is subparacompact.*

Proof. Obvious. Check, *e.g.*, the following loop:
(1) $\to$ (3) $\to$ (5) $\to$ (2) $\to$ (4) $\to$ (1). □

Let Y be a subspace of a space X, recall that Y is compact in X, if every open cover of X has a finite subfamily cover Y; Y is said to be Lindelöf in X, if for any open cover of X there is a countable subfamily cover Y. Then clearly, both Y is compact in X and Y is Lindelöf in X imply that Y is 1-subparacompact in X.

Recall that a subspace Y of a space X is said to be α-paracompact in X if every cover of Y by open subsets of X has a partial refinement by open subsets of X, locally finite in X, which covers Y [2].

Theorem 3.5. [6, 7] *Let X be a regular space and E a subspace of X. Then the following conditions are equivalent:*

a) *E is α-paracompact in X.*
b) 1) *Every cover $\mathcal{U}$ of E by open subsets of X has a refinement $\mathcal{V}$ by open subsets of X, σ-locally finite in X, which covers E, and*
   2) *Every open subset U of X with $E \subset U$ has an open subset V such that $E \subset V \subset \overline{V} \subset U$.*
c) *Every cover $\mathcal{U}$ of E by open subsets of X has a refinement $\mathcal{A} = \{A_s\}_{s\in S}$ by arbitrary sets of X, locally finite in X, such that $E \subset (\bigcup_{s\in S} A_s)^0$ (denotes the interior of $\bigcup_{s\in S} A_s$).*
d) *Every cover $\mathcal{U}$ of E by open subsets of X has a refinement $\mathcal{F} = \{F_j\}_{j\in J}$ by closed subsets of X, locally finite in X, such that $E \subset (\bigcup_{j\in J} F_j)^0$.*

The above Theorem and Theorem 2.1 yield

Theorem 3.6. *Let X be a regular space and Y a subspace of X. If Y is α-paracompact in X, then Y is 1-subparacompact in X.*

Let Y be a subset of X. in [1], the following definitions were given:

Definition 3.7. [1] A symmetric d on (Y, X) is a nonnegative real valued function d defined on $X \times X$ which satisfies the following two conditions for all x in X and y in Y:

s1) $d(x, y) = 0$ if and only if $x = y$;
s2) $d(x, y) = d(y, x)$.

Definition 3.8. [1] A symmetric d on (Y, X) defines Y in X if the following three conditions are satisfied:

o1) for every closed subset P of X, and each $y \in Y \setminus P$, $d(y, P) > 0$;
o2) if $A \subset Y$ and $d(x, A) > 0$ for each $x \in X \setminus A$, then A is closed in X; and
o3) for every closed subset P of X, and for each point x in $X \setminus P$, $d(x, P \bigcap Y) > 0$.

Moreover, we say a *symmetric d* on (Y, X) *properly defines* Y in X, or that it is a *proper symmetric* on (Y, X) [1], if d satisfies condition o1), o2), o3), and the next condition

o4) If $y \in Y$ and $A \subset X$ with $d(y, A) > 0$, then y is not in the closure of A.

Definition 3.9. [1] A symmetric ρ on (Y, X) is called a metric on (Y, X), if $\rho(y, z) \leq \rho(y, x) + \rho(x, z)$, for every y, z in Y and x in X. A subspace Y of a space X is metrizable in X, if there is a symmetric ρ on (Y, X) which defines Y in X and is a metric on (Y, X).

We say that Y is *properly metrizable in* X [1], if there is a metric ρ on (Y,X) properly defining Y in X.

Theorem 3.10. *Let X be a regular space and Y a subspace of X. If Y is properly metrizable in X, then Y is 2-subparacompact in X.*

Proof. Recall that Y is *strictly Aull-paracompact in* X, if for every family γ of open subsets of X with $Y \subset \bigcup \gamma$ there is a family μ of open subsets of X locally finite and σ-discrete in X at all points of Y and also satisfying the following conditions: $Y \subset \bigcup \mu$, and μ partially refines γ.[1].

Then this theorem follows directly from Theorem 7 of [1] which says: If Y is properly metrizable in X then Y is stictly Aull-paracompact in X. □

Definition 3.11. [1] A symmetric ρ on (Y,X) will be called a 2-metric, or an Aull-metric, on (Y,X), if whenever any two out of the three points in the triangle inequality are in Y, the inequality holds. We shall say that Y is 2-metrizable, or Aull-metrizable,in X, if there is an Aull-metric ρ on (Y,X), which satisfies the conditions o1), o2), and o3).

As Y is 2-metrizable in X implies that Y is metrizable in X, we have the following corollary:

Corollary 3.12. *If Y is properly 2-metrizable (defined similarly to that of properly metrizable) in X, then Y is 2-subparacompact in X.*

Definition 3.13. [1] We say that ρ is a 1-metric on X, if ρ is a symmetric on X, and whenever x, y, z are three points in X, at least one of which belongs to Y, then the triangle inequality holds: $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

Definition 3.14. [1] A symmetric d on X strictly defines Y in X, or strictly symmetrizes Y in X, if conditions o1) and o3) are satisfied, as well as the next condition o$^{\#}$2), which strengthens condition o2):

o$^{\#}$2) If A is a subset of X concentrated on Y (*i.e.*, $A \subset \overline{A \bigcap Y}$) such that $d(x,A) > 0$, for each $x \in X \setminus A$, then A is closed in X.

Theorem 3.15. [1] *If X is regular, and ρ is a 1-metric on (Y,X) strictly defining Y in X, then ρ metrizes the subspace $\overline{Y}$, that is, the restriction of ρ to $\overline{Y}$ generates the original topology of $\overline{Y}$.*

The above theorem and Proposition 1.6 yield

Theorem 3.16. *If X is regular, and Y is strictly 1-metrizable in X (i.e., there is ρ, a 1-metric on (Y,X) strictly defining Y in X), then Y is 1-subparacompact in X.*

4. Open questions

The characterization of 1-subparacompactness, Theorem 2.1 is beautiful. Unfortunately, we don't know whether the analogous characterization of 2-subparacompactness is true.

Questions 4.1. *Let X be a regular space and Y its subspace . Are the following equivalent?:*

(1) *Y is 2-subparacompact in X.*
(2) *For any open cover of X, there is a partial refinement $\mathcal{F}$ of closed subsets of X which is σ-locally-finite at Y in X and satisfies the condition: $\bigcup \mathcal{F} \supset Y$.*

Having Theorem 3.1 and Theorem 3.2, it's then natural to ask the following questions:

Questions 4.2. *Is 2-subparacompactness preserved under closed mappings?*

Questions 4.3. *Is 2-subparacompactness an inverse invariant of perfect mappings?*

Suppose X is a space and Y a subspace of X. The following definitions are due to Gordienko [5]. The subspace Y is said to be 1-paracompact in X provided every open cover of X has an open refinement locally finite at Y in X. The subspace Y is 2-paracompact in X provided every open cover of X has an open partial refinement cover Y and locally finite at Y in X. The subspace Y is 3-paracompact in X provided every open cover of X has a partial refinement cover Y, consisting of sets open in Y and locally finite in Y.

Then we have the following question:

Questions 4.4. *Let X be a regular space and Y its subspace. Then*

(1) *If Y is 1-paracompact in X, is Y 1-subparacompact in X?*
(2) *If Y is 2-paracompact (or 3-paracompact) in X, is Y 2-subparacompact in X?*

Remark 4.5. *It is easy to see that Y is 2-paracompact in X does not imply that Y is 1-subparacompact. (see e.g., Example 1.8)*

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Zhibin QU: Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka, 582-8582, Japan

Yoshikazu YASUI: Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka, 582-8582, Japan