A NONLINEAR STRONG ERGODIC THEOREM FOR FAMILIES OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS WITH COMPACT DOMAINS

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Abstract. In this paper, we prove a nonlinear strong ergodic theorem for families of asymptotically nonexpansive mappings from a compact convex subset of a strictly convex Banach space into itself.

1. Introduction

The first nonlinear ergodic theorem for nonexpansive mappings with bounded domains in a Hilbert space was proved by Baillon [3]. Baillon and Brezis [4] also proved the following nonlinear ergodic theorem for nonexpansive semigroups in a Hilbert space : Let C be a nonempty closed convex subset of a Hilbert space and let $\mathcal{S} = \{S(t) \mid t \geq 0\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Then, for every $x \in C$, $\frac{1}{t}\int_0^t S(\tau)x d\tau$ converges weakly to some $y \in F(\mathcal{S})$. Hirano and Takahashi [8] extended Baillon and Brezis's theorem to an asymptotically nonexpansive semigroup. Hirano and Takahashi's theorem was extended to a uniformly convex Banach space whose norm is Fréchet differentiable by Tan and Xu [11]. On the other hand, Atsushiba and Takahashi [2] obtained a nonlinear ergodic theorem for nonexpansive semigroups with compact domains in a Banach space which generalizes Dafermos and Slemrod's result [7] : Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \mid t \geq 0\}$ be a nonexpansive semigroup on C. Then, for every $x \in C$, $\frac{1}{t}\int_0^t S(\tau + h)x d\tau$ converges strongly to some $y \in F(\mathcal{S})$ uniformly in $h \geq 0$.

In this paper, we extend Atsushiba and Takahashi's theorem to an asymptotically nonexpansive semigroup by using the methods employed in Atsushiba and Takahashi [1, 2], Bruck [5, 6] and Shioji and Takahashi [10].

2. Preliminaries and Lemmas

Throughout this paper, a Banach space is real and we denote by $\mathbf{N}$ and R^+, the set of all positive integers and the set of all nonnegative real numbers, respectively. We denote by Δ^n the set $\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$ for $n \in \mathbf{N}$. Let E be a Banach space and let $r > 0$. We denote by $D_r(x)$ the open ball in E with center x and radius r. For a subset C of E, we denote by $co\, C$ the convex hull of C. E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let C be a subset of E, let T be a mapping from C into itself and let $\varepsilon > 0$. By $F_\varepsilon(T)$, we mean the set $\{x \in C \mid \|x - Tx\| \leq \varepsilon\}$. Let $K > 0$. We denote by $Lip(C, K)$, the set of all mappings from C into itself satisfying $\|Tx - Ty\| \leq K\|x - y\|$ for each $x, y \in C$. We denote by Γ the set of

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all strictly increasing, continuous convex functions $\gamma{:}R^+ \longmapsto R^+$ with $\gamma(0) = 0$. Let C be a nonempty subset of E. C is said to satisfy the convex approximation property if for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that $coM \subset co_m M + D_\varepsilon(0)$ for every subset M of C, where $co_m M = \{\sum_{i=1}^m \lambda_i x_i \,|\, x_i \in M,\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1\}$.
A family $\mathcal{S} = \{S(t) \,|\, t \geq 0\}$ is said to be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) \,|\, t \geq 0\}$ if

(i) for each $t \geq 0$, $S(t)$ is a mapping from C into itself and $\|S(t)x - S(t)y\| \leq k(t)\|x - y\|$ for each $x, y \in C$;
(ii) $S(t+s)x = S(t)S(s)x$ for each $t, s \geq 0$ and $x \in C$;
(iii) $S(0)x = x$ for each $x \in C$;
(iv) for each $x \in C$, the mapping $t \mapsto S(t)x$ is continuous.
(v) $t \mapsto k(t)$ is continuous mapping from the set of nonnegative real numbers into itself;
(vi) $\limsup_{t\to\infty} k(t) \leq 1$.

$\mathcal{S}$ is said to be a nonexpansive semigroup on C if $k(t) = 1$ for all $t \geq 0$. We denote by $F(\mathcal{S})$, the set of common fixed points of $\mathcal{S} = \{S(t) \,|\, t \geq 0\}$, i.e., $\cap_{t\geq 0}\{x \in C \,|\, S(t)x = x\}$. The following lemmas was obtained by Bruck [5, 6].

Lemma 2.1. Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, there exists $\gamma \in \Gamma$ such that for each $K > 0$ and $T \in Lip(C, K)$,

$$\|T(\lambda x + (1-\lambda)y) - (\lambda Tx + (1-\lambda)Ty)\| \leq K\gamma^{-1}\Big(\|x-y\| - \frac{1}{K}\|Tx - Ty\|\Big)$$

holds for every $x, y \in C$ and $\lambda \in [0, 1]$.

Lemma 2.2. Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, for each $p \in \mathbf{N}$, there exists $\gamma_p \in \Gamma$ such that for each $K > 0$ and $T \in Lip(C, K)$,

$$\Bigg\| T\Big(\sum_{i=1}^p \lambda_i x_i\Big) - \sum_{i=1}^p \lambda_i T x_i \Bigg\| \leq K\gamma_p^{-1}\Big(\max_{1\leq i,j\leq p}\Big\{\|x_i - x_j\| - \frac{1}{K}\|Tx_i - Tx_j\|\Big\}\Big)$$

holds for every $x_1, x_2, \ldots, x_p$ in C and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Delta^p$.

Following ideas in Atsushiba and Takahashi [1, 2], we can show the following lemma.

Lemma 2.3. Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \,|\, t \geq 0\}$ be an asymptotically nonexpansive semigroup on C. Let $x \in C$ and $t > 0$. Then, for each $\varepsilon > 0$, there exist $l_0 = l_0(t, \varepsilon) \geq 0$ and $m_0 = m_0(t, \varepsilon) \geq 0$ such that

$$\Bigg\| \frac{1}{t}\int_0^t S(l+m+\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\,d\tau\Big) \Bigg\| < \varepsilon$$

for every $l \geq l_0$ and $m \geq m_0$.

Proof. Let $x \in C$, $t > 0$ and $\varepsilon > 0$. Let $\{k(t) \,|\, t \geq 0\}$ be Lipschitz constants of $\mathcal{S}$. Put $\sup\{k(t) \,|\, t \geq 0\} = M_0$. Since $\{k(t) \,|\, t \geq 0\}$ is bounded, $M_0 < \infty$ holds. ¿From the assumption of $\mathcal{S}$, we have

$$\begin{aligned} &\Bigg\| \frac{1}{t}\int_0^t S(l+m+\tau)x\,d\tau - \frac{1}{n}\sum_{i=1}^n S\Big(l+m+\frac{t}{n}i\Big)x \Bigg\| \\ &\qquad \leq \frac{1}{t}\sum_{i=1}^n \int_{\frac{i-1}{n}t}^{\frac{i}{n}t} \Big\| S(l+m+\tau)x - S\Big(l+m+\frac{t}{n}i\Big)x \Big\|\ d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \frac{M_0}{t}\sum_{i=1}^{n}\int_{\frac{i-1}{n}t}^{\frac{i}{n}t}\Big\| \, S(\tau)x - S\Big(\frac{t}{n}i\Big)x \, \Big\| \, d\tau \\
&\leq \frac{M_0}{t}\sum_{i=1}^{n}\Big\{ M_0 \cdot \frac{t}{n}\Big(\sup_{0\leq \tau \leq \frac{t}{n}} \Big\| \, S(\tau)x - S\Big(\frac{t}{n}\Big)x \, \Big\|\Big)\Big\} \\
&= M_0^2 \cdot \sup_{0\leq \tau \leq \frac{t}{n}} \Big\| \, S(\tau)x - S\Big(\frac{t}{n}\Big)x \, \Big\| \longrightarrow 0\,,
\end{aligned}$$

as $n \to \infty$, uniformly in $l, m \geq 0$. Similarly, we have

$$\Big\| \, S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\, d\tau\Big) - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\Big(m + \frac{t}{n}i\Big)x\Big) \, \Big\| \longrightarrow 0\,,$$

as $n \to \infty$, uniformly in $l, m \geq 0$. So, there exists $N_1 \in \mathbf{N}$ such that

$$\Big\| \, \frac{1}{t}\int_0^t S(l+m+\tau)x\, d\tau - \frac{1}{n}\sum_{i=1}^{n} S\Big(l+m+\frac{t}{n}i\Big)x \, \Big\| < \frac{\varepsilon}{3}$$

and

$$\Big\| \, S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\, d\tau\Big) - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\Big(m+\frac{t}{n}i\Big)x\Big) \, \Big\| < \frac{\varepsilon}{3}$$

for every $n \geq N_1$ and $l, m \geq 0$. Hence we get

$$\begin{aligned}
(1) \qquad & \Big\| \, \frac{1}{t}\int_0^t S(l+m+\tau)x\, d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\, d\tau\Big) \, \Big\| \\
\leq \; & \Big\| \, \frac{1}{t}\int_0^t S(l+m+\tau)x\, d\tau - \frac{1}{n}\sum_{i=1}^{n} S\Big(l+m+\frac{t}{n}i\Big)x \, \Big\| \\
& + \Big\| \, \frac{1}{n}\sum_{i=1}^{n} S\Big(l+m+\frac{t}{n}i\Big)x - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\Big(m+\frac{t}{n}i\Big)x\Big) \, \Big\| \\
& + \Big\| \, S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\Big(m+\frac{t}{n}i\Big)x\Big) - S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\, d\tau\Big) \, \Big\| \\
\leq \; & \frac{2}{3}\varepsilon + \Big\| \, \frac{1}{n}\sum_{i=1}^{n} S\Big(l+m+\frac{t}{n}i\Big)x - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\Big(m+\frac{t}{n}i\Big)x\Big) \, \Big\|
\end{aligned}$$

for every $n \geq N_1$ and $l, m \geq 0$. Fix $n \in \mathbf{N}$ with $n \geq N_1$. Without loss of generality, we assume that $k(l) > 0$ for all $l \in R^+$. ¿From Lemma 2.2, there exists $\gamma_n \in \Gamma$ such that

$$\begin{aligned}
(2) \qquad & \Big\| \, \frac{1}{n}\sum_{i=1}^{n} S\Big(l+m+\frac{t}{n}i\Big)x - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\Big(m+\frac{t}{n}i\Big)x\Big) \, \Big\| \\
& \leq \; k(l)\gamma_n^{-1}\Big(\max_{1\leq i,j\leq n}\Big\{\Big\| \, S\Big(m+\frac{t}{n}i\Big)x - S\Big(m+\frac{t}{n}j\Big)x \, \Big\| \\
& \qquad - \frac{1}{k(l)}\Big\| \, S\Big(l+m+\frac{t}{n}i\Big)x - S\Big(l+m+\frac{t}{n}j\Big)x \, \Big\|\Big\}\Big)
\end{aligned}$$

for every $l, m \geq 0$. ¿From $\gamma_n \in \Gamma$, there exists $\delta > 0$ such that

$$(3) \qquad k(l)\gamma_n^{-1}(\delta) < \frac{\varepsilon}{3}$$

for every $l \geq 0$. For $1 \leq i, j \leq n$, we set $r_{i,j} = \inf_{m \geq 0} \left\| S\Big(m + \frac{t}{n}i\Big)x - S\Big(m + \frac{t}{n}j\Big)x \right\|$. There exists $m_1 \geq 0$ such that $\left\| S\Big(m_1 + \frac{t}{n}i\Big)x - S\Big(m_1 + \frac{t}{n}j\Big)x \right\| < r_{i,j} + \frac{\delta}{4}$. By $\limsup_{l \to \infty} k(l) \leq 1$, there exists $l_1 > 0$ such that

$$k(l) \leq \frac{r_{i,j} + \frac{\delta}{2}}{\|S(m_1 + \frac{t}{n}i)x - S(m_1 + \frac{t}{n}j)x\| + \frac{\delta}{4}}$$

for every $l \geq l_1$. So, we have

$$\begin{aligned} &\left\| S\Big(l + m_1 + \frac{t}{n}i\Big)x - S\Big(l + m_1 + \frac{t}{n}j\Big)x \right\| \\ &\quad \leq \quad k(l) \left\| S\Big(m_1 + \frac{t}{n}i\Big)x - S\Big(m_1 + \frac{t}{n}j\Big)x \right\| \leq r_{i,j} + \frac{\delta}{2} \end{aligned}$$

for every $l \geq l_1$. Put $m_2 = m_2(i,j) = l_1 + m_1$. Then, there holds

$$\left\| S\Big(m + \frac{t}{n}i\Big)x - S\Big(m + \frac{t}{n}j\Big)x \right\| \leq r_{i,j} + \frac{\delta}{2}$$

for every $m \geq m_2$. Similarly, there exists $l_2 = l_2(i,j) \geq 0$ such that

$$r_{i,j} - \frac{\delta}{2} \leq \frac{1}{k(l)} \left\| S\Big(l + m + \frac{t}{n}i\Big)x - S\Big(l + m + \frac{t}{n}j\Big)x \right\|$$

for every $l \geq l_2$ and $m \geq m_2$. Let

$$l_0 = \max\{l_2(i,j) \,|\, 1 \leq i,j \leq n\} \text{ and } m_0 = \max\{m_2(i,j) \,|\, 1 \leq i,j \leq n\}.$$

Then, we have

$$\begin{aligned} (4) \qquad 0 \leq \max_{1 \leq i,j \leq n} \Big\{ &\left\| S\Big(m + \frac{t}{n}i\Big)x - S\Big(m + \frac{t}{n}j\Big)x \right\| \\ &- \frac{1}{k(l)} \left\| S\Big(l + m + \frac{t}{n}i\Big)x - S\Big(l + m + \frac{t}{n}j\Big)x \right\| \Big\} \leq \delta \end{aligned}$$

for every $l \geq l_0$ and $m \geq m_0$. So, it follows from (1), (2), (3) and (4) that

$$\begin{aligned} &\left\| \frac{1}{t} \int_0^t S(l + m + \tau)x \, d\tau - S(l)\left(\frac{1}{t} \int_0^t S(m + \tau)x \, d\tau \right) \right\| \\ &\quad \leq \quad \frac{2}{3}\varepsilon + k(l)\gamma_n^{-1}(\delta) \leq \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for every $l \geq l_0$ and $m \geq m_0$. □

The following lemma was obtained by Atsushiba and Takahashi [1].

Lemma 2.4. Let C be a nonempty compact subset of a Banach space. Then, C satisfies the convex approximation property.

The following lemmas were obtained by Nakajo and Takahashi [9].

Lemma 2.5. Let C be a nonempty compact convex subset of a strictly convex Banach space. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$ holds for every $T \in Lip(C, 1 + \delta)$, where $\overline{co}\, A$ is the closure of the convex hull of A.

Lemma 2.6. Let C be a nonempty closed bounded convex subset of a Banach space. Let $\gamma \in \Gamma$, $L \geq 1$ and $T \in Lip(C, L)$ such that

$$\|T(\lambda x + (1 - \lambda)y) - (\lambda Tx + (1 - \lambda)Ty)\| \leq L\gamma^{-1}\Big(\|x - y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all $x, y \in C$ and $\lambda \in [0,1]$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $\frac{1}{n}\sum_{i=1}^{n}\|x_{i+1} - Tx_i\| \le a_n$ and $\frac{1}{n}\sum_{i=1}^{n}\|y_{i+1} - Ty_i\| \le a_n$ for all $n \in \mathbf{N}$, where $\{a_n\}$ is a sequence in R^+. Then, for each $n \in \mathbf{N}$ and $\lambda \in [0,1]$,

$$\frac{1}{n}\sum_{i=1}^{n}\|\lambda x_{i+1} + (1-\lambda)y_{i+1} - T(\lambda x_i + (1-\lambda)y_i)\| \le L\gamma^{-1}\Big(\frac{R}{n} + (L-1)R + 2a_n\Big) + a_n\,,$$

where $R = diam\, C$.

Lemma 2.7. Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \in \mathbf{N}$ such that for every $T \in Lip(C, 1+\delta)$ and $\{x_n\}$ in C satisfying $\|x_{n+1} - Tx_n\| \le \delta$ for all $n \in \mathbf{N} \cup \{0\}$, there holds $\frac{1}{n}\sum_{i=0}^{n-1} x_i \in F_\varepsilon(T)$ for every $n \ge N_0$.

Lemma 2.8. Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, for each $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \in \mathbf{N}$ such that for every $l \in \mathbf{N}$ and mapping T from C into itself satisfying $T^l \in Lip(C, 1+\delta)$, there holds

$$\left\| \frac{1}{m}\sum_{i=0}^{m-1} T^i x - T^l\Big(\frac{1}{m}\sum_{i=0}^{m-1} T^i x\Big) \right\| \le \varepsilon$$

for all $m \in \mathbf{N}$ with $m - 1 \ge lN_0$ and $x \in C$.

As in the proof of [10], we have the following lemma. However, for the sake of completeness, we give the proof.

Corollary 2.9. Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \,|\, t \ge 0\}$ be an asymptotically nonexpansive semigroup on C. Then,

$$\limsup_{l\to\infty}\limsup_{t\to\infty}\sup_{x\in C}\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| = 0.$$

Proof. Let $\{k(t) \,|\, t \ge 0\}$ be Lipschitz constants of $\mathcal{S}$. Let $\varepsilon > 0$. There exist $\delta > 0$ and $N_0 \in \mathbf{N}$ which satisfy the condition in Lemma 2.8. ¿From $\limsup_{l\to\infty} k(l) \le 1$, there exists $l_0 \ge 0$ such that $k(l) < 1 + \delta$ for every $l \ge l_0$. Let $l > l_0$. Then, there exists $t_l > 0$ such that $\frac{1}{N_0} \ge \frac{l}{t}$ for all $t \ge t_l$. Let $t \ge t_l$. For each $n \in \mathbf{N}$, let j_n be the nonnegative integer which satisfies $t \cdot \frac{j_n}{n} \le l < t \cdot \frac{j_n + 1}{n}$. Then, $n \ge j_n N_0$ for every $n \in \mathbf{N}$ and by $l > l_0$, there exists $n_0 \in \mathbf{N}$ such that $t \cdot \frac{j_n}{n} \ge l_0$ for all $n \ge n_0$. Hence, from Lemma 2.8 we get

$$\left\| \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x - S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x\Big) \right\| < \varepsilon$$

for every $n \ge n_0$ and $x \in C$. So, we have

$$\begin{aligned} &\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \\ &\quad \le \left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x \right\| \end{aligned}$$

$$\begin{aligned}
& + \left\| \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x - S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x\Big) \right\| \\
& + \left\| S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x\Big) - S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \\
& + \left\| S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \\
\leq \;& (2+\delta)\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x \right\| \\
& + \varepsilon + \left\| S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\|
\end{aligned}$$

for every $n \geq n_0$ and $x \in C$. Tending n to infinity, we get

$$\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \leq \varepsilon$$

for every $x \in C$. So,we have

$$\limsup_{l\to\infty}\limsup_{t\to\infty}\sup_{x\in C}\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion. □

Remark. We can obtain $F(\mathcal{S}) \neq \emptyset$. In fact, let $x \in C$ and put $x_t = \dfrac{1}{t}\displaystyle\int_0^t S(\tau)x\,d\tau$ for every $t > 0$. Since C is compact, there exists a subnet $\{x_{t_\alpha}\}$ of $\{x_t\}$ such that x_{t_α} converges strongly to some x_0 in C. So, we have

$$\begin{aligned}
0 &= \limsup_{l\to\infty}\limsup_{t\to\infty}\|x_t - S(l)x_t\| \\
&= \limsup_{l\to\infty}\limsup_{\alpha}\|x_{t_\alpha} - S(l)x_{t_\alpha}\| = \limsup_{l\to\infty}\|x_0 - S(l)x_0\|
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_0 - S(s)x_0\| &\leq \limsup_{l\to\infty}\|x_0 - S(l)x_0\| + \limsup_{l\to\infty}\|S(l)x_0 - S(s)x_0\| \\
&\leq 0 + k(s)\cdot 0 = 0
\end{aligned}$$

for every $s \geq 0$. Therefore $x_0 \in F(\mathcal{S})$.

3. Strong ergodic theorem

The following is crucial to prove our theorem.

Lemma 3.1. Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \mid t \geq 0\}$ be an asymptotically nonexpansive semigroup on C. Let $x \in C$. Then, there exists a net $\{i_t\}_{t\geq 0} \subset R^+$ such that $\displaystyle\lim_{t\to\infty}\left\| \frac{1}{t}\int_0^t S(\tau + i_t)x\,d\tau - z \right\|$ exists for every $z \in F(\mathcal{S})$.

Proof. We use the methods employed in Atsushiba and Takahashi [1, 2]. ¿From Lemma 2.3, there exist nets $\{i_t\}_{t\geq 0}$ in R^+ and $\{l_t\}_{t\geq 0}$ in R^+ such that

$$\left\| \frac{1}{t}\int_0^t S(l+i+\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(i+\tau)x\,d\tau\Big) \right\| < \frac{1}{t} \tag{5}$$

for every $t > 0$, $i \ge i_t$ and $l \ge l_t$. Let $z \in F(\mathcal{S})$. For every $s, t > 0$, consider

$$\begin{aligned} I &= \left\| \frac{1}{s}\int_0^s S(i_s + i_t + \tau)x\,d\tau - z \right\| \\ &= \left\| \frac{1}{s}\int_0^s \Big(\frac{1}{t}\int_0^t S(\tau+\sigma+i_s+i_t)x\,d\sigma\Big)\,d\tau \right. \\ &\quad \left. + \frac{1}{st}\int_0^t (t-\tau)\{S(\tau+i_s+i_t)x - S(s+\tau+i_s+i_t)x\}\,d\tau - z \right\|, \\ I_1 &= \left\| \frac{1}{st}\int_0^t (t-\tau)\{S(\tau+i_s+i_t)x - S(s+\tau+i_s+i_t)x\}\,d\tau \right\|, \\ I_2 &= \left\| \frac{1}{s}\int_0^s \Big(\frac{1}{t}\int_0^t S(\tau+\sigma+i_s+i_t)x\,d\sigma\Big)\,d\tau \right. \\ &\quad \left. - \frac{1}{s}\int_0^s S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big)\,d\tau \right\| \end{aligned}$$

and

$$I_3 = \left\| \frac{1}{s}\int_0^s S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big)\,d\tau - z \right\|.$$

Then, we have $I \le I_1 + I_2 + I_3$. Fix $t > 0$ and put $R = diam\,C$. We have

$$I_1 \le \frac{1}{st}\int_0^t (t-\tau)R\,d\tau = \frac{t}{2s}R$$

for every $s > 0$. It follows from (5) that

$$\begin{aligned} I_2 &\le \frac{1}{s}\int_0^s \left\| \frac{1}{t}\int_0^t S(\tau+\sigma+i_s+i_t)x\,d\sigma - S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big) \right\| d\tau \\ &\le \frac{1}{s}\int_0^s \frac{1}{t}\,d\tau = \frac{1}{t} \end{aligned}$$

for every $s > 0$ with $i_s \ge l_t$. By $z \in F(\mathcal{S})$, we obtain

$$\begin{aligned} I_3 &\le \frac{1}{s}\int_0^s \left\| S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big) - z \right\| d\tau \\ &\le \frac{1}{s}\int_0^s k(\tau+i_s)\left\| \frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma - z \right\| d\tau \\ &= \left\{ \frac{1}{s}\int_0^s k(\tau+i_s)\,d\tau \right\} \cdot \left\| \frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma - z \right\| \end{aligned}$$

for every $s > 0$, where $\{k(t) \mid t \ge 0\}$ is Lipschitz constants of $\mathcal{S}$. Therefore, since $\lim_{s\to\infty} I_1 = 0$ and $\{k(t) \mid t \ge 0\}$ is Lipschitz constants of $\mathcal{S}$, we have

$$\begin{aligned} &\limsup_{s\to\infty} \left\| \frac{1}{s}\int_0^s S(\tau+i_s)x\,d\tau - z \right\| \\ &\quad = \limsup_{s\to\infty} \left\| \frac{1}{s}\int_0^s S(\tau+i_s+i_t)x\,d\tau - z \right\| \\ &\quad = \limsup_{s\to\infty} I \le \limsup_{s\to\infty}(I_1+I_2+I_3) \\ &\quad \le \frac{1}{t} + \left\| \frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma - z \right\| \cdot \limsup_{s\to\infty} \frac{1}{s}\int_0^s k(\tau+i_s)\,d\tau \end{aligned}$$

$$\leq \quad \frac{1}{t} + \left\| \frac{1}{t} \int_0^t S(\sigma + i_t)x \, d\sigma - z \right\|$$

for every $t > 0$. So, we get

$$\limsup_{s \to \infty} \left\| \frac{1}{s} \int_0^s S(\tau + i_s)x \, d\tau - z \right\| \leq \liminf_{t \to \infty} \left\| \frac{1}{t} \int_0^t S(\sigma + i_t)x \, d\sigma - z \right\|$$

Hence, $\lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t S(\tau + i_t)x \, d\tau - z \right\|$ exists. □

Remark. In Lemma 3.1, take a net $\{i'_t\}_{t \geq 0}$ in R^+ such that $i'_t \geq i_t$ for every $t \geq 0$. Then, we can get

$$\lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t S(\tau + i_t)x \, d\tau - z \right\| = \lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t S(\tau + i'_t)x \, d\tau - z \right\|$$

for every $z \in F(\mathcal{S})$.

Theorem 3.2. Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \,|\, t \geq 0\}$ be an asymptotically nonexpansive semigroup on C. Let $x \in C$. Then, $\frac{1}{t} \int_0^t S(\tau + h)x \, d\tau$ converges strongly to a common fixed point of $\mathcal{S}$ uniformly in $h \geq 0$. In this case, if $Qx = \lim_{t \to \infty} \frac{1}{t} \int_0^t S(\tau)x \, d\tau$ for every $x \in C$, then Q is a nonexpansive mapping from C onto $F(\mathcal{S})$ such that $QS(t) = S(t)Q = Q$ for every $t \geq 0$ and $Qx \in \overline{co}\{S(t)x \,|\, t \geq 0\}$ for every $x \in C$.

Proof. ¿From Lemma 3.1, there exists a net $\{i_t\}_{t \geq 0}$ in R^+ such that

$$\lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t S(\tau + i_t)x \, d\tau - z \right\| \tag{6}$$

exists for every $z \in F(\mathcal{S})$. Set $\Phi_t = \frac{1}{t} \int_0^t S(\tau + i_t)x \, d\tau$. As in the Remark of Corollary 2.9, there exists a subnet $\{\Phi_{t_\alpha}\}$ of $\{\Phi_t\}$ such that Φ_{t_α} converges strongly to a common fixed point y_0 of $\mathcal{S}$. So it follows from (6) that

$$\lim_{t \to \infty} \|\Phi_t - y_0\| = \lim_{\alpha} \|\Phi_{t_\alpha} - y_0\| = 0.$$

This implies that $\Phi_t \longrightarrow y_0$. Next we prove that $\frac{1}{t} \int_0^t S(\tau + i_t + h)x \, d\tau$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in $h \geq 0$. Take a net $\{i'_t\}_{t \geq 0}$ in R^+ such that $i'_t \geq i_t$ for every $t \geq 0$. Then, from Remark of Lemma 3.1, we have $\frac{1}{t} \int_0^t S(\tau + i'_t)x \, d\tau \longrightarrow y_0 \in F(\mathcal{S})$. Since $\{i'_t\}_{t \geq 0}$ is any net in R^+ such that $i'_t \geq i_t$ for every $t \geq 0$, it follows that $\frac{1}{t} \int_0^t S(\tau + i_t + h)x \, d\tau$ converges strongly to y_0 uniformly in $h \geq 0$. Let $\varepsilon > 0$. Then, there exists $t_0 \geq 0$ such that

$$\left\| \frac{1}{t} \int_0^t S(\tau + i_t + h)x \, d\tau - y_0 \right\| < \varepsilon$$

for every $t \geq t_0$ and $h \geq 0$. So, we have

$$\begin{aligned} &\left\| \frac{1}{t} \int_0^t S(\tau + h)x \, d\tau - y_0 \right\| \\ &\quad = \quad \left\| \frac{1}{t} \int_0^t \Big(\frac{1}{s} \int_0^s S(\tau + h + \sigma)x \, d\sigma \Big) \, d\tau \right. \end{aligned}$$

$$
\begin{aligned}
& + \frac{1}{ts}\int_0^s (s-\tau)\{S(\tau+h)x - S(t+\tau+h)x\}\,d\tau - y_0 \Big\| \\
\le\ & \frac{1}{t}\left\| \int_0^t \left\{ \frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0 \right\} d\tau \right\| \\
& + \frac{1}{ts}\int_0^s (s-\tau)\|S(\tau+h)x - S(t+\tau+h)x\|\,d\tau \\
=\ & \frac{1}{t}\Big\| \int_0^{i_s} \left\{ \frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0 \right\} d\tau \\
& + \int_{i_s}^t \left\{ \frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0 \right\} d\tau \Big\| \\
& + \frac{1}{ts}\int_0^s (s-\tau)\|S(\tau+h)x - S(t+\tau+h)x\|\,d\tau \\
\le\ & \frac{1}{t}\int_0^{i_s} \left\| \frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0 \right\| d\tau \\
& + \frac{1}{t}\int_0^{t-i_s} \left\| \frac{1}{s}\int_0^s S(\tau+i_s+h+\sigma)x\,d\sigma - y_0 \right\| d\tau \\
& + \frac{1}{ts}\int_0^s (s-\tau)\|S(\tau+h)x - S(t+\tau+h)x\|\,d\tau \\
\le\ & \frac{i_s}{t}R + \frac{t-i_s}{t}\varepsilon + \frac{s}{2t}R
\end{aligned}
$$

for every $s \ge t_0$, $t \ge i_s$ and $h \ge 0$, where $R = diam\, C$. Since $\varepsilon > 0$ is arbitrary, it follows that $\frac{1}{t}\int_0^t S(\tau+h)x\,d\tau$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in $h \ge 0$. If $Qx = \lim_{t\to\infty} \frac{1}{t}\int_0^t S(\tau)x\,d\tau$ for every $x \in C$, then Q is a nonexpansive mapping from C onto $F(\mathcal{S})$. In fact, let $\{k(t) \mid t \ge 0\}$ be Lipschitz constants of $\mathcal{S}$. Then, we get

$$
\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - \frac{1}{t}\int_0^t S(\tau)y\,d\tau \right\| \le \|x-y\| \cdot \frac{1}{t}\int_0^t k(\tau)\,d\tau\,,
$$

which implies $\|Qx - Qy\| \le \|x-y\|$ for every $x, y \in C$. Moreover, we have $QS(t) = S(t)Q = Q$ for every $t \ge 0$ and $Qx \in \overline{co}\{S(t)x \mid t \ge 0\}$ for every $x \in C$. □

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