LEFT SYMMETRIC ALGEBRAS OVER A REAL REDUCTIVE LIE ALGEBRA

Akira Mizuhara

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Abstract.

Let A be a left symmetric algebra over a real Lie algebra . A symmetric bilinear form $\langle\ ,\ \rangle$ of A is called *of Hessian type* if the following equality holds:

$$\langle xy, z\rangle + \langle y, xz\rangle = \langle yx, z\rangle + \langle x, yz\rangle \quad (x, y, z \in A).$$

H. Shima studied the structures of left symmetric algebras with a positive definite symmetric bilinear form of Hessian type ([S]).

Denote by h the bilinear form defined by

$$h(x, y) = \operatorname{Tr} R(xy),$$

where $R(x)$ denotes the right multiplication on A by an element x. h is a symmetric bilinear form of Hessian type.
It is called *the canonical 2-form on A.* A left symmetric algebra $A = (\ , e, h)$ over with a right identity e and the non degenerate canonical 2-form h is called *regular.*

In this paper, we shall study the structures of a regular algebra over a real reductive Lie algebra and related topics.

1 Preliminaries.

[A] Let be a real Lie algebra of dimension n with a binomial product $\cdot$. The algebra $A = (\ , \cdot)$ is called *an algebra over* if

$$[a, b] = ab - ba \quad (a, b \in\ ).$$

For an algebra A over , denote by R a trilinear mapping $A \times A \times A$ into A defined by

$$R(a, b, c) = a(bc) - b(ac) - [a, b]\, c$$

and call it *the curvature of A*.

An algebra A over with vanishing curvature is called *left symmetric.*

For an algebra A over , a symmetric bilinear form h on A is called *of Hessian type* ([S]) if the following equality holds:

$$h(ab, c) + h(b, ac) = h(ba, c) + h(a, bc) \quad (a, b, c \in A).$$

An algebra (A, h) over with a form h of Hessian type is called *projectively flat* if the following equality holds:

$$R(a, b, c) = -h(b, c)a + h(a, c)b \quad (a, b, c \in A).$$

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[B] Let E^n be a real affine space of dimension n and $\rho = (\varphi, \pi)$ a Lie homomorphism of into the Lie algebra $\mathrm{aff}(E)$ of all infinitesimal affine transformations on E, where $\varphi(a)$ (resp. $\pi(a)$) denotes the linear (resp. translation) part of $\rho(a)$.

A Lie homomorphism ρ is called *an admissible affine representation of in E* if π is a linear isomorphism of onto E.

Let A be a left symmetric algebra over . Denote by $L(a)$ (resp. $R(a)$) the left (resp. right) multiplication of A by an element a. Then the mapping $\tilde{L}$ of into $\mathrm{aff}(A)$ defined by

$$\tilde{L}(a) = (L(a), a)$$

is an admissible affine representation of in A, which is called *the left affine representation of a left symmetric algebra A over* .

We can prove the following (cf. [S]):

Lemma 1. *Let $\rho = (\varphi, \pi)$ be an admissible affine representation of in E. Define a binomial product in by the formula*

$$ab = \pi^{-1}(\varphi(a)\pi(b)) \quad (a, b \in \ ).$$

Then the algebra $A = (\ , \rho)$ with the above multiplication is a left symmetric algebra over .

[C] For an element $a = (a_{ij}, a_i)$ of $\mathrm{aff}(E)$, denote by $\overline{a}$ a vector field on $E(x_1, x_2, \ldots, x_n)$ defined by

$$\overline{a} = -\sum (a_{ij}x_j + a_i)\frac{\partial}{\partial x_i}.$$

For an affine representation $\rho = (\varphi, \pi)$ of in E, denote by $F_\rho(x)$ (resp. $F_\varphi(x)$) a polynomial on $E(x_1, \ldots, x_n)$ defined by

$$F_\rho(x)\omega_0 = \overline{\rho(a_1)} \wedge \overline{\rho(a_2)} \wedge \cdots \wedge \overline{\rho(a_n)} \quad \left(\text{resp. } F_\varphi(x)\omega_0 = \overline{\varphi(a_1)} \wedge \overline{\varphi(a_2)} \wedge \cdots \wedge \overline{\varphi(a_n)}\right),$$

where $\{a_i\}$ is a base of and ω_0 denotes the tensor field on $E(x_1, \ldots, x_n)$ defined by

$$\omega_0 = \left(\frac{\partial}{\partial x_1}\right) \wedge \left(\frac{\partial}{\partial x_2}\right) \wedge \cdots \wedge \left(\frac{\partial}{\partial x_n}\right).$$

The polynomial $F_\rho(x)$ (resp. $F_\varphi(x)$) is uniquely determined by $(\ , \rho)$ (resp. $(\ , \varphi)$) up to a constant multiple.

Denote this polynomial by $F_\rho = |\rho(\ )|$ (resp. $F_\varphi = |\varphi(\ )|$) and call it *the polynomial for $(\ , \rho)$ (resp. $(\ , \varphi)$).*

Let A be a left symmetric algebra over , and $(y_1, y_2, \ldots, y_n)$ the affine coordinate system of A with respect to a base $\{a_i\}$ of A. The polynomial $F(y) = \left|\tilde{L}(\ )\right|$ for the left affine representation $(\ , \tilde{L})$ of A is called *the polynomial for A.*

Lemma 2. *Let $\rho = (\varphi, \pi)$ be an affine representation of in E, and F_ρ (resp. F_φ) the polynomial for $(\ , \rho)$ (resp. $(\ , \varphi)$). Then we have ($a \in$)*

$$L_{\overline{\rho(a)}}F_\rho = \chi(a)F_\rho \quad \left(\textit{resp. } L_{\overline{\varphi(a)}}F_\varphi = \chi(a)F_\varphi\right),$$

where $L_{\overline{X}}$ denotes the Lie differentiation with respect to a vector field $\overline{X}$ and $\chi(a)$ denotes an infinitesimal character on defined by

$$\chi(a) = \mathrm{Tr}\,\mathrm{ad}\, a - \mathrm{Tr}\,\rho(a).$$

In fact, on the one hand, we have

$$\begin{aligned} L_{\overline{\rho(a)}}\left(\overline{\rho(a_1)} \wedge \cdots \wedge \overline{\rho(a_n)}\right) &= \sum \overline{\rho(a_1)} \wedge \cdots \wedge \overline{\rho\left([a, a_i]\right)} \wedge \cdots \wedge \overline{\rho(a_n)} \\ &= (\mathrm{Tr}\,\mathrm{ad}\, a)\, \overline{\rho(a_1)} \wedge \cdots \wedge \overline{\rho(a_n)}. \end{aligned}$$

On the other hand, we have

$$L_{\overline{\rho(a)}}(F_\rho \omega_0) = (L_{\overline{\rho(a)}} F_\rho)\omega_0 + F_\rho(\mathrm{Tr}\,\rho(a))\omega_0.$$

Hence we obtain the desired result. □

For a left symmetric algebra A over , we have

$$L(a) - R(a) = \mathrm{ad}\, a \quad (a \in \ ).$$

Thus we have the following.

Corollary. *Let $F = \left|\tilde{L}(\ )\right|$ be the polynomial for a left symmetric algebra A over . Then we have*

$$L_{\overline{\tilde{L}(a)}} F = -(\mathrm{Tr}\, R(a))F \quad (a \in \ ).$$

Remark. Let A be a left symmetric algebra over and D the characteristic polynomial of A ([H]).
Then D coincides with the polynomial $\left|\tilde{L}(\ )\right|$ of an algebra A over , up to a constant multiple.

In fact, let $\{a_i\}$ be a base of A. Then, for an element $y = \sum y_i a_i$ of A, we have

$$\tilde{L}(a_k)y = a_k y + a_k = \text{the } k\text{-th row of the matrix } (R(y) + I),$$

where I denotes the unit matrix.
Therefore we have

$$-\overline{\tilde{L}(a_k)} = \text{the } k\text{-th row of the matrix } \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n}\right)(R(y) + I).$$

Thus we have $D(y) = \left|\tilde{L}(\ )\right|$, up to a constant multiple.

[D] Let $\rho = (\varphi, \pi)$ be an admissible affine representation of in E. A point P of E is called *a fixed point of* $(\ , \rho)$ if P satisfies the following:

$$\rho(a)P = \rho(a)P + \pi(a) = 0 \quad (a \in \ ).$$

We can easily prove the following.

Lemma 3. *Let $A = (\ , \rho)$ be a left symmetric algebra over corresponding to an admissible affine representation $\rho = (\varphi, \pi)$ of in E. Then the following statements are mutually equivalent:*

(1) A has a right identity e,

(2) $(\ , \rho)$ has a fixed point P.

In fact, if A has a right identity e, then $P = \pi(-e)$ is a fixed point of $(\ , \rho)$, and conversely.

□

Let $\ = \ \oplus \ $ be a real reductive Lie algebra where (resp.) denotes the center (resp. the semi simple ideal) of .

Lemma 4. *Let ρ be an admissible affine representation of a real reductive Lie algebra $\ = \ \oplus \ $ in E. Assume that* $\deg(\varphi| \ ) = \dim \ $. *Then there exists a fixed point of* $(\ , \rho)$.

In fact, since π is a φ-cocycle, that is,

$$\varphi(a)\pi(b) - \varphi(b)\pi(a) = \pi([a, b]) \quad (a, b \in \ ),$$

there exists a point P of E ([J]) such that

$$\pi(s) = \varphi(s)P \quad (s \in \ ).$$

Moreover, since is the center of , for any element c in and s in , we have

$$\varphi(s)(\pi(c) - \varphi(c)P) = \varphi(s)\pi(c) - \varphi(c)\pi(s) = \pi([s, c]) = 0.$$

Since $\deg(\varphi| \ ) = \dim E$, $-P$ is a fixed point of $(\ , \rho)$. □

[E] Let $\rho = (\varphi, \pi)$ be an admissible affine representation of in E. Denote by $A = (\ , \rho)$ the left symmetric algebra over corresponding to ρ, F_ρ (resp. F_φ) the polynomial for $(\ , \rho)$ (resp. $(\ , \varphi)$). Denote by Ω_ρ a subset of E defined by

$$\Omega_\rho = \{x \in E \, ; \, F_\rho(x) \neq 0\}.$$

Put

$$_0 = \left\{ a \in \ ; \, L_{\overline{\rho(a)}} F = 0 \right\}.$$

Then $_0$ is an ideal of containing $[\ , \ ]$.

An algebra A is called *complete* if one of the following equivalent conditions holds ([K1], [K2], [Se]):

(1) $\ = \ _0$,

(2) $\Omega_\rho = E$,

(3) F_ρ is a non zero constant.

In the sequel, we assume that $\quad \neq \ {}_0$.
Under the assumption, there exists an element ξ of such that $\quad = \ {}_0 \oplus R\{\xi\}$ (as a linear space).

Denote by g a tensor field on Ω_ρ of type $(0,2)$ defined by

$$g_{ij} = \left(\frac{\partial^2}{\partial x_i \partial x_j}\right) \log |F_\rho| \, .$$

Lemma 5. *For $a, b \in \ $, we have*

$$g\left(\overline{\rho(a)}, \overline{\rho(b)}\right) = \frac{1}{F_\rho}\left(L_{\overline{\rho(a)\rho(b)}} F_\rho\right).$$

Proof. For the sake of simplicity, put

$$F_\rho = F, \quad \frac{\partial F}{\partial x_i} = F_{,i}, \quad \left(\frac{\partial^2}{\partial x_i \partial x_j}\right) F = F_{,i,j}.$$

Set

$$\overline{\rho(a)} = -\sum (a_{ij}x_j + a_i)\frac{\partial}{\partial x_i} = \sum A^i \frac{\partial}{\partial x_i},$$
$$\overline{\rho(b)} = -\sum (b_{ij}x_j + b_i)\frac{\partial}{\partial x_i} = \sum B^i \frac{\partial}{\partial x_i}.$$

Then, by Lemma 2, we have

$$\chi(a)F = \sum A^i F_{,i}, \quad \chi(b)F = \sum B^i F_{,i}.$$

Therefore we have

$$\chi(a)\chi(b)F = L_{\overline{\rho(b)}}(L_{\overline{\rho(a)}}F) = -L_{\overline{\rho(a)\rho(b)}}F + \sum B^j A^i F_{,i,j}.$$

Hence we obtain

$$g\left(\overline{\rho(a)}, \overline{\rho(b)}\right) = \left(\frac{1}{F}\right)^2 \sum A^i B^j \left(F_{,i,j}F - F_{,i}F_{,j}\right) = \frac{1}{F} L_{\overline{\rho(a)\rho(b)}} F.$$

□

Now, denote by $\langle \ , \ \rangle$ a symmetric bilinear form on defined by

$$\langle a, b\rangle = g\left(\overline{\rho(a)}, \overline{\rho(b)}\right)\Big|_{x=0} \quad (a, b \in \quad),$$

and call it *the 2-form defined by $F = F_\rho$*.

Lemma 6. *The symmetric bilinear form $\langle\ ,\ \rangle$ on defined by F_ρ is of Hessian type on $A = (\ \ , \rho)$.*

Proof. Denote by π' an endomorphism of $\mathrm{aff}(E)$ defined by

$$\pi'(\rho(a)) = (0, \pi(a)).$$

Then it is clear that

(1) $\pi'(\rho(ab)) = \pi'(\rho(a)\rho(b))$,

(2) $L_{\overline{\rho(a)\rho(b)}} F\Big|_0 = L_{\overline{\pi'(\rho(a)\rho(b))}} F\Big|_0$.

Therefore we have, for $a, b, c \in \ \ $,

$$\begin{aligned}
&\langle ab, c\rangle + \langle b, ac\rangle - \langle ba, c\rangle - \langle a, bc\rangle \\
&\quad = \langle c, ab\rangle + \langle b, ac\rangle - \langle c, ba\rangle - \langle a, bc\rangle \\
&\quad = \left(L_{\overline{\rho(c)\rho(ab)}} + L_{\overline{\rho(b)\rho(ac)}} - L_{\overline{\rho(c)\rho(ba)}} - L_{\overline{\rho(a)\rho(bc)}}\right) F\Big|_0 .
\end{aligned}$$

On the one hand, the π-component of

$$\rho(c)\rho(ab) + \rho(b)\rho(ac) - \rho(c)\rho(ba) - \rho(a)\rho(bc)$$

is equal to

$$\pi\left([c, [a, b]]\right).$$

On the other hand, it is clear that $[c, [a, b]] \in \ {}_0$. Thus, by (2), we obtain the desired result. □

A symmetric bilinear form h on A of Hessian type defined by

$$h(a, b) = \operatorname{Tr} R(ab) \quad (a, b \in A)$$

is called *the canonical 2-form on A*.

Denote by $\langle\ ,\ \rangle$ the 2-form on A defined by the polynomial $F = \left|\tilde{L}(\ \ )\right|$ for a left symmetric algebra A. Then, by Lemma 2, Corollary and the above equalities (1) and (2), $\langle\ ,\ \rangle$ is the canonical 2-form on A. A is called *non degenerate* if the canonical 2-form h of A is non degenerate.

Lemma 7. *Let B be an ideal of A. Then $h|B$ is the canonical 2-form on B.*

In fact, for $b \in B$, we have $\operatorname{Tr} R(b) = \operatorname{Tr}\left(R(b)|B\right)$. Therefore it is clear that

$$h(b, b') = \operatorname{Tr} R(bb') = \operatorname{Tr}\left(R(bb')|B\right).$$

□

We can easily prove the following.

Lemma 8. *Let B be an ideal of A, and h a symmetric bilinear form on A of Hessian type. Then the orthogonal complement $B^\perp$ of B with respect to h is a subalgebra of A.*

[F] Let A be a left symmetric algebra over a Lie algebra . A is called *regular* if

(1) A is non degenerate,

(2) A has a right identity.

Lemma 9. *Let $A = (\ , e, h)$ be a regular algebra over a real Lie algebra with a right identity e and the non degenerate canonical 2-form h, and B an ideal of A. If B is regular, then there exists a regular subalgebra $\overline{B}$ of A such that*

(1) $A = B \oplus \overline{B}$, semidirect with $B\overline{B} = 0$,

(2) $B \perp \overline{B}$ with respect to h,

(3) $L(c)|B \ \ (c \in B)$ is a derivation of the algebra B.

Proof. Put $\overline{B} = B^{\perp}$. Then $\overline{B}$ is a subalgebra of A, by Lemma 8. Denote by h_1 the canonical 2-form on B. Then, since $h_1 = h|B$ by Lemma 7 and h_1 is non degenerate by the assumption, we have $B \cap \overline{B} = \{0\}$. Therefore $A = B \oplus \overline{B}$, semi direct sum.

Next denote by e_1 a right identity of B. Then $b(e - e_1) = 0 \ \ (b \in B)$. Thus $e_2 = e - e_1$ is an element of $\overline{B}$. Moreover, for $c \in \overline{B}$, we have

$$ce_2 = ce - ce_1 = c - ce_1.$$

Therefore e_2 is a right identity of $\overline{B}$ and $\overline{B}e_1 = 0$.

For $b, b' \in B$ and $c \in \overline{B}$, by the following equalities:

$$(bc)e_1 - b(ce_1) = (cb)e_1 - c(be_1), \quad \overline{B}e_1 = 0, \quad c(bb') = (cb)b' + b(cb'),$$

we obtain

(1) $bc = 0$, i.e. $B\overline{B} = 0$ and

(2) $L(c)|B \ \ (c \in \overline{B})$ is a derivation of the algebra B.

Moreover (1) implies that $B \perp \overline{B}$ and $h|\overline{B}$ coincides with the non degenerate canonical 2-form h_2 of $\overline{B}$. Thus $\overline{B}$ is a regular subalgebra of A. □

Let B (resp. $\overline{B}$) be a left symmetric algebra over (resp. $\overline{\ }$) and

$$D : \overline{\ } \longrightarrow Der\, B$$

a Lie homomorphism of $\overline{\ }$ into the derivation algebra $Der\, B$ of an algebra B. *The semi direct sum $B \oplus_D \overline{B}$ of B by $\overline{B}$ that is determined by D* is, by definition, an algebra over the direct sum $B \oplus \overline{B}$ of two vector spaces with the following multiplication:

$$(b, c)(b', c') = (bb' + D(c)b', cc').$$

It is clear that $B \oplus_D \overline{B}$ is a left symmetric algebra over a Lie algebra $\oplus_D \overline{\ }$ of the semi direct sum of by $\overline{\ }$ that is determined by D. Thus we can construct a semi direct sum $B \oplus_D \overline{B}$ with $B\overline{B} = 0$.

The following lemma can be easily proved.

Lemma 10. *Let $B \oplus_D \overline{B}$ be the semi direct sum of B by $\overline{B}$ that is determined by D. If both $B = (\ , e_1, h_1)$ and $\overline{B} = (\overline{\ }, e_2, h_2)$ are regular, then $B \oplus_D \overline{B}$ is regular with a right identity $e = e_1 + e_2$ and the non degenerate canonical 2-form $h = h_1 + h_2$.*

2 Let $= \oplus$ be a real reductive Lie algebra of dimension n where (resp.) denotes the center (resp. semi simple ideal) of with $\neq \{0\}$, $\rho = (\varphi, \pi)$ an admissible affine representation of in a real affine space E^n of dimension n.

First we shall prove the following proposition.

Proposition 1. *Let ρ be an admissible affine representation of a real reductive Lie algebra $= \oplus$ with $\neq \{0\}$ in E^n, and A a left symmetric algebra over corresponding to $\rho = (\varphi, \pi)$. Assume that*

$$m = \deg(\varphi| \ ) < n = \dim \ .$$

Then there exist an ideal B of A of dimension m whose underlying Lie algebra contains and a commutative subalgebra $\overline{B}$ of A satisfying the following conditions:

(1) $A = B \oplus \overline{B}$, semi direct sum with $B\overline{B} = 0$,

(2) B has a right identity.

Proof. By the assumption, there exist $\varphi(\ )$-invariant subspaces E_1 and E_2 such that

$$\dim E_1 = m, \quad \dim E_2 = n - m, \quad \varphi(\ )|E_2 = 0, \quad E = E_1 \oplus E_2.$$

Put $= \pi^{-1}(E_1)$, $\overline{\ } = \pi^{-1}(E_2)$.
With respect to the decomposition $E = E_1 \oplus E_2$, $\varphi(s)$ $(s \in \ )$ can be expressed as follows:

$$\varphi(s) = \begin{bmatrix} s_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\big[\rho(\ ), \rho(\ )\big] = \rho(\ )$, we have

$$\rho(s) = \begin{bmatrix} s_{11} & 0 & s_{13} \\ 0 & 0 & 0 \end{bmatrix}.$$

This implies that is contained in .

Moreover, by a direct computation, ρ can be expressed as follows:

$$\rho(b) = \begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & 0 & 0 \end{bmatrix} \ (b \in \ ), \quad \rho(c) = \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & c_{23} \end{bmatrix} \ (c \in \overline{\ }).$$

Therefore we obtain

(1) $\rho' = \rho|$ is an admissible affine representation of a Lie algebra in E_1, and the algebra B corresponding to ρ' is an ideal of A,

(2) $\rho'' = \rho|\overline{\ }$ is an admissible affine representation of a commutative Lie algebra $\overline{\ }$ in E_2, and the algebra $\overline{B}$ corresponding to ρ'' is a commutative subalgebra of A,

(3) $B\overline{B} = 0$.

Moreover, by Lemma 3, 4, B has a right identity. This completes the proof. □

Corollary. *If A is non degenerate, then A, B and $\overline{B}$ are regular.*

Proof. Since $A = B \oplus \overline{B}$, semi direct with $B\overline{B} = 0$, we have

(1) $B \perp \overline{B}$ with respect to h, and

(2) both $h|B$ and $h|\overline{B}$ are non degenerate.

Since B has a right identity e_1 and the canonical 2-form h_1 is non degenerate, by the above condition (2) and Lemma 7, (B, e_1, h_1) is regular. Moreover, since $B\overline{B} = 0$, the canonical 2-form h_2 of $\overline{B}$ coincides with $h|\overline{B}$. Thus, by (2), $\overline{B}$ is a commutative, semi simple associative algebra i.e. $(\overline{B}, e_2, h_2)$ is a regular algebra with an identity e_2. Hence, by Lemma 10, A is also regular. $\square$

Let ρ be an admissible affine representation of a real reductive Lie algebra $\quad = \quad \oplus$ with $\quad \neq \{0\}$ in E^n, and A a left symmetric algebra over $\quad$ corresponding to $\rho = (\varphi, \pi)$.
Assume that

$$\deg(\varphi| \quad) = \dim \quad . \tag{$*$}$$

Proposition 2. *Under the assumption* $(*)$, *let* B *be a non commutative minimal ideal of* A, *then there exists a subalgebra* $\overline{B}$ *of* A *such that*

(1) $A = B \oplus \overline{B}$, *semi direct sum with* $B\overline{B} = 0$,

(2) B *(resp.* $\overline{B}$*) has a right identity.*

Proof. Denote by $\quad$ the underlying Lie algebra of B. Then, since $\quad$ is non commutative, $_1 = \quad \cap \quad$ is an ideal $(\neq \{0\})$ of $\quad$. Put $E_1 = \pi(\quad)$. Since E_1 is $\varphi(\quad)$-invariant, there exists a $\varphi(\quad)$-invariant subspace E_2, complementary to E_1.

First we shall show that $\deg(\varphi(\ _1)|E_1) = \dim E_1$.

In fact, let E_{12} be a maximal $\varphi(\ _1)$-invariant subspace of E_1 satisfying $\varphi(\ _1)|E_{12} = 0$. Denote by E_{11} a $\varphi(\ _1)$-invariant subspace of E_1, complementary to E_{12}. With respect to the decomposition $E_{11} \oplus E_{12} \oplus E_2$, $\varphi(s)$ $(s \in \ _1)$ can be expressed as follows:

$$\varphi(s) = \begin{bmatrix} s_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Set $\rho(x) = (x_{ij})_{1 \leq i \leq 3,\, 1 \leq j \leq 4}$ $(x \in \quad)$. Then, since B is an ideal of A and $\left[\rho(\ _1), \rho(\quad)\right] \subset \rho(\ _1)$, ρ can be expressed as follows:

$$\rho(s) = \begin{bmatrix} s_{11} & 0 & 0 & s_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (s \in \ _1),$$

$$\rho(b) = \begin{bmatrix} b_{11} & 0 & 0 & b_{14} \\ 0 & b_{22} & b_{23} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (b \in \pi^{-1}(E_{11})),$$

$$\rho(b') = \begin{bmatrix} b'_{11} & 0 & 0 & 0 \\ 0 & b'_{22} & b'_{23} & b'_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (b' \in \pi^{-1}(E_{12})),$$

$$\rho(c) = \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & 0 \\ 0 & c_{32} & c_{33} & c_{34} \end{bmatrix} \quad (c \in \pi^{-1}(E_2)).$$

Moreover, by comparing the 4-th row of $[\rho(b), \rho(c)]$ (resp. $[\rho(b), \rho(b')]$), we get

$$b_{22}, b_{23} = 0 \quad (b \in \pi^{-1}(E_{11})).$$

Thus $\pi^{-1}(E_{11})$ has a structure of a non commutative ideal of A. But this is a contradiction. Hence we get

$$\deg(\varphi(\quad_1)|E_1) = \dim E_1.$$

Now, with respect to the decomposition $E_1 \oplus E_2$, ρ can be expressed as follows:

$$\rho(b) = \begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & 0 & 0 \end{bmatrix} \ (b \in \quad), \quad \rho(c) = \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & c_{23} \end{bmatrix} \ (c \in \overline{\quad} = \pi^{-1}(E_2)).$$

Similarly as in the proof of Proposition 1, we have the following:

(1) $\overline{\quad}$ is a Lie subalgebra of ,

(2) $\rho'' = \rho|E_2$ is an admissible affine representation of $\overline{\quad}$ in E_2, and the algebra $\overline{B}$ corresponding to ρ'' is a subalgebra of A,

(3) $A = B \oplus \overline{B}$, semi direct sum with $B\overline{B} = 0$,

(4) B (resp. $\overline{B}$) has a right identity.

This completes the proof. □

We can easily prove the following.

Corollary. *If A is non degenerate, then A, B and $\overline{B}$ are regular and $B \perp \overline{B}$ with respect to h.*

Proposition 3. *Let ρ be an admissible affine representation of a real reductive Lie algebra $\ = \ \oplus \ $ with $\ \neq \{0\}$ in E^n satisfying the condition $(*)$, and A the corresponding algebra over . Let B be a minimal ideal of A.*

Assume that (A, e, h) is regular and indecomposable. Then B is non commutative.

Proof. Assume that B is commutative. Denote by the underlying Lie algebra of B and by N the radical of B. Then there exists a commutative associative semi simple subalgebra S of B such that $B = N \oplus S$, semi direct sum.

We shall investigate the following three cases separately.

(1) $B = S$. In this case, by Lemma 9, the subalgebra $B^{\perp}$ with respect to h satisfies the following conditions:

(1) $A = B \oplus B^{\perp}$, semi direct sum with $BB^{\perp} = 0$,

(2) $L(c)|B$ $(c \in B^{\perp})$ is a derivation of B.

Since B is a commutative semi simple algebra, $Der\, B = \{0\}$. Therefore, by the above condition (2), we have $B^{\perp}B = 0$. But this contradicts to the assumption.

(2) $B = N$. In this case, for any element x of A, we have $h(x, B) = \operatorname{Tr} R(xB) = 0$. Thus we have

$$h(x, B) = 0 \quad (x \in A).$$

This contradicts to the assumption that A is non degenerate.

(3) $N \neq \{0\}$ and $S \neq \{0\}$. Since S is a commutative semi simple algebra, S is expressed as $S = \bigoplus S_i$, where S_i $(1 \leq i \leq r)$ is a commutative simple ideal of S with the identity e_i. Put $B_i = Be_i$ $(i = 1, 2, \ldots, r)$. Since the underlying Lie algebra of B is contained in the center , we have $[L(b), L(x)] = 0$ $(b \in \ , x \in \ )$. Therefore it is clear that B_i is an ideal of A. Since B is a minimal ideal of A, S has to be simple.

Now, since is contained in the center of , the restriction $L(s)|B$ to B of $L(s)$ $(s \in \ )$ induces a Lie homomorphism $L|B$ of a semi simple Lie algebra into a Lie algebra $[B]$ consisting of all linear endomorphisms X of B satisfying $[X, L(b)] = 0$ $(b \in B)$.

By the Lemma below, $[B]$ is a solvable Lie subalgebra of (B). Thus we have $L|B = 0$. But this contradicts to the assumption $(*)$.

This completes the proof of Proposition 3. □

Lemma. *Let $B = N \oplus S$ be a commutative associative algebra over the real with a unit element e, where N (resp. S) denotes the radical (resp. a simple subalgebra) of B. Then the Lie algebra $[B]$ defined above is solvable.*

Proof of Lemma.
(a) $S = R\{e\}$. In this case, since N is nilpotent, there exists a base $\{x_1, x_2, \ldots, x_m\}$ of N such that, with respect to the base $\{x_1, x_2, \ldots, x_m, e\}$ of B, $L(x_i)$ is expressed as a matrix of the following form:

$$L(x_i) = \begin{array}{c} \left[\begin{array}{cccc} & & & 0 \\ & * & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \\ & 0 & & \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ i\text{-th} \\ \\ \end{array} \qquad (1 \leq i \leq m).$$

Let $X = (x_{ij})_{1 \leq i,j \leq m+1}$ be an element of $[B]$. By a direct computation, we have

$$x_{11} = x_{22} = \cdots = x_{m+1\, m+1}, \quad x_{ij} = 0 \ (i > j).$$

Therefore the Lie algebra $[B]$ is solvable.
(b) $S = R\{e, f\}$ with $f^2 = -e$. Denote by $L(x)|N$ the restriction to N of the left (= right) multiplication of B by an element x. Then we have $(L(f)|N)^2 = -\mathrm{id}$.
Therefore $\dim N$ is even. Moreover, since N is nilpotent, there exists a base $\{x_i, y_i\}_{1 \leq i \leq m}$ of N such that, with respect to the base $\{x_1, y_1, \ldots, x_m, y_m, f, e\}$ of B, $L(x_i)$ and $L(y_i)$ are expressed as matrices of the following form:

$$L(x_i) = \left[\begin{array}{cccc} & & & 0 \\ & * & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & J \\ & 0 & & \end{array}\right] \begin{array}{l} \\ \\ \\ i\text{-th} \\ \\ \end{array}, \qquad L(y_i) = \left[\begin{array}{cccc} & & & 0 \\ & * & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & E \\ & 0 & & \end{array}\right] \begin{array}{l} \\ \\ \\ i\text{-th} \\ \\ \end{array}$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $X = (x_{ij})_{1 \leq i,j \leq m+1}$ be an element of $[B]$, where $x_{ij} \in \ (2, R)$.

By a direct computation, we have

$$x_{11} = x_{22} = \cdots = x_{m+1\, m+1} = \alpha E + \beta J \ (\alpha, \beta \in R), \quad x_{ij} = 0 \ (i > j).$$

Therefore the Lie algebra $[B]$ is solvable. This completes the proof of Lemma. □

By Propositions 1, 2 and 3, we obtain the following theorem.

Theorem 1. *Let $A = (\ , e, h)$ be a regular algebra over a real reductive Lie algebra $\ = \ \oplus \ $, and B a minimal ideal of A. Then,*

(1) B is regular,

(2) there exists a regular subalgebra $\overline{B}$ of A such that $A = B \oplus \overline{B}$, semi direct with $B\overline{B} = 0$ and $B \perp \overline{B}$ with respect to the canonical 2-form h of A.

Proof. Let A be a left symmetric algebra over $\ = \ \oplus \ $ corresponding to an admissible affine representation $\rho = (\varphi, \pi)$ in E.
1. Assume that $\ = \{0\}$. Then A is a commutative associative semi simple algebra, and there is nothing to prove.
2. Assume next that $\ \neq \{0\}$. Then, by Propositions 1, 2, 3 and Corollaries, we obtain the theorem. □

Let $B_i = (\ _i, e_i, h_i)$ $(i = 1, 2)$ be a regular algebra over a real reductive Lie algebra $\ _i$. Denote by D a Lie homomorphism of $\ _2$ into the derivation algebra $Der \ _1$ of $\ _1$, and by $\ = \ _1 \oplus_D \ _2$ the semi direct sum of $\ _1$ by $\ _2$ that is determined by D. Then $\ $ is reductive if and only if there exists a Lie homomorphism φ of $\ _2$ into $\ _1$ such that

$$D(c) = \operatorname{ad} \varphi(c) \quad (c \in \ _2).$$

Moreover $D(c) = \operatorname{ad} \varphi(c)$ is a derivation of B_1 is equivalent to

$$L(b)R(\varphi(c)) = R(\varphi(c))L(b) \ \ (b \in B_1).$$

Therefore, by Lemma 10, we obtain the following theorem.

Theorem 2. *Let $B_i = (\ _i, e_i, h_i)$ $(i = 1, 2)$ be a regular algebra over a real reductive Lie algebra $\ _i$, and φ a Lie homomorphism of $\ _2$ into $\ _1$ satisfying the following condition:*

$$[L(b), R(\varphi(c))] = 0 \quad (b \in \ _1, c \in \ _2),$$

where $D(c) = \operatorname{ad} \varphi(c)$. Then the semi direct sum $A = B_1 \oplus_D B_2$ is a regular algebra over a real reductive Lie algebra $\ = \ _1 \oplus_D \ _2$ with a right identity $e = e_1 + e_2$ and the canonical 2-form $h = h_1 + h_2$.

3 In this section, we shall give a remark and some examples.

First, let A be a left symmetric algebra over a real Lie algebra $\ $. Assume that A has a right identity e. Denote by $\ _0$ the linear subspace of $\ $ defined by

$$\ _0 = \{a \in \ ; \operatorname{Tr} R(a) = 0\}.$$

Then we have the following direct sum decomposition as a linear space: $\ = \ _0 \oplus R\{e\}$.

Put, for $a, b \in \ $,

$$ab = a * b + h_0(a, b)e,$$

where $a * b$ denotes the $\ _0$-component of ab. Then, by a direct computation, we have

(1) $A_0 = (\ _0, *)$ is an algebra over $\ _0$ satisfying

$$R_0(a,b,c) = -h_0(b,c)a + h_0(a,c)b, \tag{1}$$

where R_0 denotes the curvature of A_0,

(2) h_0 is a symmetric bilinear form on A_0 of Hessian type satisfying

$$h_0(a,b) = \frac{1}{n}h(a,b) \quad (a,b \in A_0),$$

where h denotes the canonical 2-form on A,

(3) $D = \operatorname{ad} e$ is a derivation of the Lie algebra $\ _0$ satisfying

$$h_0(a, D(b)) + h_0(D(a), b) = 0, \tag{2}$$

$$D(a * b) = D(a) * b + a * D(b). \tag{3}$$

Conversely let $A_0 = (\ _0, *)$ be an algebra over a real Lie algebra $\ _0$ with a symmetric bilinear form h_0 of Hessian type and a derivation D of $\ _0$ satisfying the above conditions (1), (2) and (3). Then we enlarge $\ _0$ as follows:

$$\ = \ _0 \oplus R\{e\} \ \text{ with } \ [e,a] = D(a) \ \ (a \in \ _0).$$

Moreover we define a multiplication in $\ $ by

$$ab = a * b + h_0(a,b)e, \quad ae = a, \quad ea = a + D(a), \quad ee = e \ \ (a, b \in \ _0).$$

We can easily show that $A = (\ , \cdot)$ is a left symmetric algebra over $\ $ with a right identity e. Thus we obtain the following theorem.

Theorem 3. *Let $A = (\ , \cdot)$ be a left symmetric algebra over $\ $ with a right identity e. Then there exist an ideal $\ _0$ of $\ $ of codimension 1, a structure of an algebra $A_0 = (\ _0, *)$ over $\ _0$, a symmetric bilinear form h_0 on A_0 of Hessian type, and a derivation D of $\ _0$ satisfying the following conditions:*

(1) an algebra A_0 is projectively flat with respect to h_0,

(2) $h_0(D(a), b) + h_0(a, D(b)) = 0$,

(3) $D(a * b) = D(a) * b + a * D(b)$.

Conversely, if an algebra $A_0 = (\ _0, *)$ over a Lie algebra $\ _0$ with a symmetric bilinear form h_0 of Hessian type and a derivation D of $\ _0$ satisfying the above conditions (1), (2) and (3), then we can construct a structure of a left symmetric algebra $A = (\ , \cdot)$ over an enlarged Lie algebra

$$\ = \ _0 \oplus R\{e\} \ \textit{with} \ [e,x] = D(x) \ \ (x \in \ _0)$$

having a right identity e.

This is a slight modification of a theorem about a left symmetric algebra with identity ([N.P], [M,2]).

Let $A = (\ \ ,\cdot)$ be a left symmetric algebra over   with a right identity e. Suppose that   has a non trivial center, that is, there exists an element a_0 of   such that $e + a_0$ is in the center of . We define a multiplication in $\ = \ _0 \oplus R\{e\}$ by

$$ab = a * b + h_0(a,b)e_t, \quad ae_t = a, \quad e_t a = a + (1-t)D(a), \quad e_t e_t = e_t,$$

where $e_t = e + ta_0$ $(t \in R)$ and $a, b \in \ _0$.
Then $A_t = (\ \ ,\cdot)$ is a left symmetric algebra over   with a right identity e_t, for any t. For $t = 1$, A_1 is a left symmetric algebra over   with an identity e_1.

We shall give two examples.

Example 1. Let A_t be a left symmetric algebra over $\ (2,R)$ whose multiplication table is as follows:

$\{a,b,c,e\}$ is a base of $\ (2,R)$ and $t \in R$.

A_t	a	b	c	e
a	$e - tc$	b	$-c$	$a - tc$
b	$-b$	0	$\frac{1}{2}(e + a - tc)$	$\frac{1}{2}(te + ta + 2b - t^2c)$
c	c	$\frac{1}{2}(e - a - tc)$	0	c
e	$a - tc$	$\frac{1}{2}(te + ta + 2b - t^2c)$	c	$e + tc$

Then $e_t = e - tc$ is a right identity of A_t. Moreover, $e_0 = e$ is an identity of A_0.

For non zero s, t, we have an algebraic isomorphism σ of A_s onto A_t defined by

$$\sigma(e) = e, \quad \sigma(a) = a, \quad \sigma(b) = \frac{s}{t}b \quad \text{and} \quad \sigma(c) = \frac{t}{s}c.$$

Example 2. Let $A_{(s,t)}$ be a left symmetric algebra whose multiplication table is as follows:

$\{e,a,b\}$ is a base and $(s,t) \in R^2$.

$A_{(s,t)}$	e	a	b
e	$e + s(s+1)a + 2t(s+1)b$	$(1+s)a + 2tb$	b
a	$(1+s)a + 2tb$	a	b
b	b	0	0

By a direct computation, it is easily proved that

(1) $e - sa - 2tb$ is a right identity of $A_{(s,t)}$,

(2) $e - sa - 2tb$ is an identity if and only if $(s,t) = (0,0)$,

(3) if $s \neq s'$, then there does not exist an algebraic isomorphism of $A_{(s,t)}$ onto $A_{(s',t')}$.

Next we shall give examples of left symmetric algebras over a real reductive Lie algebra.

Example 3. Denote by D the adjoint representation of $\ (2,R)$. Then we can construct the semi direct sum $A = \ (2,R) \oplus_D \ (2,R)$ of the associative algebra $\ (2,R)$ by the same associative algebra $\ (2,R)$ that is determined by D.

A has an identity and the canonical 2-form h on A is non degenerate.

Example 4. Let   be a Lie subalgebra of $\ (6,R)$ generated by $\{a \otimes E_3\,;\, a \in \ (2,R)\}$ and the set C of matrices of degree 6 defined below:

(1) $C = \left\{ \begin{bmatrix} E_2 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & E_2 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & E_2 \end{bmatrix} \right\},$

(2) $C = \left\{ \begin{bmatrix} E_2 & 0 & 0 \\ & E_2 & 0 \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & E_2 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & E_2 \end{bmatrix} \right\},$

(3) $C = \left\{ \begin{bmatrix} E_2 & 0 & 0 \\ & E_2 & 0 \\ & & E_2 \end{bmatrix}, \begin{bmatrix} 0 & E_2 & 0 \\ & 0 & E_2 \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & E_2 \\ & 0 & 0 \\ & & 0 \end{bmatrix} \right\}.$

The polynomial $|\ |$ is expressed as follows:

(1) $(x_1x_6 - x_2x_5)(x_2x_3 - x_1x_4)(x_4x_5 - x_3x_6)$,

(2) $(x_1x_4 - x_2x_3)(x_3x_6 - x_4x_5)^2$,

(3) $(x_3x_6 - x_4x_5)^3$.

By a direct computation, we have the following.

(1) The algebra corresponding to (1) is simple with non degenerate canonical 2-form,

(2) The algebra corresponding to (2) is simple with degenerate canonical 2-form and trivial radical,

(3) The algebra corresponding to (3) is simple with degenerate canonical 2-form and non trivial radical.

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