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TWO-QUEUE AND TWO-SERVER MODEL WITH A HYSTERETIC CONTROL SERVICE POLICY

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ABSTRACT. This paper investigates a queueing system consisting of two-parallel queues and two servers. The service policy is a hysteretic control one such that a set of two forward thresholds (F_1, F_2) and a set of two reverse thresholds (R_1, R_2) are set up in one of two queues, say, the second queue, and at each epoch of service completion, the server decides which queue is to be served next according to the control level the number of customers in the second queue reaches. The arrival process for each queue is Poisson, and the service times are exponentially distributed with different means. We derive the generating functions of the stationary joint queue-length distribution, and then obtain the mean queue length and the mean waiting time for each queue.

1. Introduction

Threshold-based service policies have been applied by many authors to queueing systems with a single-queue as policies to control service rate, number of servers or vacation, and proved to be optimal to some queueing systems(see Larsen and Agrawa(1983), Lin and Kumar(1984), Morrison(1990), Igaki(1992), Mishimura and Jian(1995), Ibe and Keilson(1995), Liu and Golubchik(1999)). Especially, in order to avoid oscillation in a simple threshold-based system, or reduce the non-negligible server setup and removal costs, Ibe and Keilson(1995), and Liu and Golubchik(1999) consider a threshold-based control service with hysteresis for multi-server queues. Recently, such threshold-based service policies have been considered by some researchers for polling systems consisting of two queues and a single server(see Lee and Sengupta(1993), Boxma and Down(1997), Feng, Kowada and Adachi(1998, 1999)). On the other hand, the queueing systems with two classes of customers, multiple servers and non-threshold service policies have been extensively studied(see Cohen(1982), Mitrani and King(1981), Falin et al.(1994), Gail et al.(1988) and (1992)). In this paper we apply the hysteresis threshold-based service policy proposed by Ibe and Keilson(1995) to a system with two queues and two homogeneous servers. The two queues with infinite buffer capacities are denoted by Q_1 and Q_2. A set of two forward thresholds (F_1, F_2) and a set of two reverse thresholds (R_1, R_2) are set up in the second queue. Without loss of generality, we assume that $R_1 \leq F_1 < R_2 \leq F_2$. The two servers serve the two queues according to the schedule described as follows.

1. At each epoch of service completion in Q_1, (i) when the two servers all serve in Q_1, if the number of customers in Q_2 exceeds the threshold F_1, the server switches its service to Q_2, otherwise it continues to serve the customers in Q_1; (ii) when the two servers server respectively in Q_1 and Q_2, if the number of customers in Q_2 exceeds the threshold F_2, the server switches its service to Q_2, otherwise it continues to serve the customers in Q_1,

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2. At each epoch of service completion in Q_2, (i) when the two servers all serve in Q_2, if the number of customers in Q_2 drops below the threshold R_2, the server switches its service to Q_1, otherwise it continues to serve the customers in Q_2; (ii) when the two servers server respectively in Q_1 and Q_2, if the number of customers in Q_2 drops below the threshold R_1, the server switches its service to Q_1, otherwise it continues to serve the customers in Q_2,

3. The server does not idle if there are customers present at either queue. That is, if there are no customers waiting in the present queue at an epoch of service completion, the server switches its service to the other queue. The service discipline is first-come-first-served within each queue and non-preemptive.

Since by choosing the threshold values, one can easily assign a higher priority to some a queue, the threshold service schedule is a very flexible control policy. In modern communication network systems which employ the fixed packet sizes of ATM technology, the different types of traffics require the different demand of service. Sometimes these requirements vary according to the system state, and not anyone of these traffics has absolute priority. As it has been seen, the hysteretic threshold-based control service policy in the above provides such a priority scheduling strategy. Q_2 has a priority over Q_1 in the segment (F_2, ∞), Q_1 has a priority over Q_2 in the segment $(0, R_1]$, and the segment $(R_1, F_2]$ is a non-priority part in the sense that servers do not switch their service to another queue when the length of Q_2 is in this segment. For this system, we derive the generating functions of the stationary joint queue-length distributions by the variable elimination approach. We then obtain the mean queue length and the mean waiting time for each queue.

The organization of the paper is as follows. In Section 2 the model is described in detail, and the system equations of the generating functions of the stationary joint queue-length distribution are established. The solutions of the system equations are derived in Section 3. The mean queue lengths and the waiting times are given in Section 4. Finally, a summary is included in Section 5.

2. The Model and generating function equations

We consider a queueing system consisting of two-parallel queues Q_1 and Q_2 and two equal speed servers. For $i = 1, 2$, the arrival processes in Q_i is a Poisson process with rate λ_i, and the service time distribution at Q_i is exponential with parameter μ_i. Interarrival times and service times are assumed to be mutually independent. The service policy for this system is a hysteretic control one described in the previous section. Write $\lambda \equiv \lambda_1 + \lambda_2$. The traffic load at Q_i is $\rho_i \equiv \lambda_i/\mu_i$ $(i = 1, 2)$. The ergodicity condition of the system is satisfied if and only if the total traffic load $\rho \equiv \rho_1 + \rho_2 < 2$. Throughout the paper, we assume that this condition holds. Let $Q_i(t)$ be the number of customers waiting for service in queue Q_i $(i = 1, 2)$ at time t. Let $I_i(t)$ denote the number of servers serving in queue Q_i $(i = 1, 2)$ at time t. Then $\{(I_1(t), I_2(t), Q_1(t), Q_2(t))\}_{t\geq 0}$ is an irreducible continuous-time Markov chain according to the assumptions of the Poisson arrival processes and the exponentially distributed service times. The Markov chain has the equilibrium probabilities under the ergodicity condition. We denote the equilibrium probabilities by $\{p_{i,j,n,m},\ \ 0 \leq i + j \leq 2; n, m \geq 0\}$, that is,

$$p_{i,j,n,m} = \lim_{t\to\infty} P((I_1(t), I_2(t), Q_1(t), Q_2(t)) = (i, j, n, m)). \tag{2.1}$$

Note that when $i + j < 2$(at least one server is idle), $n = m = 0$. Then we can denote the state $(i, j, 0, 0)$ by (i, j), and write $p_{i,j,0,0}$ by $p_{i,j}$. Also when $i + j = 2$(the two servers all are busy), $j = 2 - i$. Then we can denote the state (i, j, n, m) by (i, n, m), and write $p_{i,j,n,m}$ by $p_{i,n,m}$. We derive the Kolmogrov equations for the equilibrium probabilities as follows:

CASE 1. $i+j<2$,

$$(\lambda_1+\lambda_2+\mu_1)p_{1,0}=\lambda_1 p_{0,0}+2\mu_1 p_{2,0,0}+\mu_2 p_{1,0,0}, \tag{2.2}$$
$$(\lambda_1+\lambda_2+\mu_2)p_{0,1}=\lambda_2 p_{0,0}+\mu_1 p_{1,0,0}+2\mu_2 p_{0,0,0}, \tag{2.3}$$
$$(\lambda_1+\lambda_2)p_{0,0}=\mu_1 p_{1,0}+\mu_2 p_{0,1}, \tag{2.4}$$

CASE 2. $i+j=2$,

(1) $i=0$,

$$(\lambda_1+\lambda_2+2\mu_2)p_{0,n,m}=\lambda_1 p_{0,n-1,m}+\lambda_2 p_{0,n,m-1}+\mu_1 p_{1,n,m+1}\delta_{\{m+1>F_2\}}+2\mu_2 p_{0,n,m+1}\times\delta_{\{m+1>R_2\}}, \quad n,m>0, \tag{2.5}$$
$$(\lambda_1+\lambda_2+2\mu_2)p_{0,n,0}=\lambda_1 p_{0,n-1,0}+\mu_1 p_{1,n,1}\delta_{\{F_2=0\}}+2\mu_2 p_{0,n,1}\delta_{\{R_2=0\}}, \quad n>0, \tag{2.6}$$
$$(\lambda_1+\lambda_2+2\mu_2)p_{0,0,m}=\lambda_2 p_{0,0,m-1}+\mu_1 p_{1,0,m+1}+2\mu_2 p_{0,0,m+1}, \quad m>0, \tag{2.7}$$
$$(\lambda_1+\lambda_2+2\mu_2)p_{0,0,0}=\lambda_2 p_{0,1}+\mu_1 p_{1,0,1}+2\mu_2 p_{0,0,1}, \tag{2.8}$$

(2) $i=1$,

$$(\lambda_1+\lambda_2+\mu_1+\mu_2)p_{1,n,m}=\lambda_1 p_{1,n-1,m}+\lambda_2 p_{1,n,m-1}+\mu_1 p_{1,n+1,m}\delta_{\{m\le F_2\}}+\mu_2 p_{1,n,m+1}\times\delta_{\{m+1>R_1\}}+2\mu_1 p_{2,n,m+1}\delta_{\{m+1>F_1\}}+2\mu_2 p_{0,n+1,m}\delta_{\{m\le R_2\}},$$
$$n,m>0, \tag{2.9}$$
$$(\lambda_1+\lambda_2+\mu_1+\mu_2)p_{1,n,0}=\lambda_1 p_{1,n-1,0}+\mu_1 p_{1,n+1,0}+2\mu_2 p_{0,n+1,0}+\mu_2 p_{1,n,1}\delta_{\{R_1=0\}}+2\mu_1 p_{2,n,1}\delta_{\{F_1=0\}}, \quad n>0, \tag{2.10}$$
$$(\lambda_1+\lambda_2+\mu_1+\mu_2)p_{1,0,m}=\lambda_2 p_{1,0,m-1}+\mu_1 p_{1,1,m}\delta_{\{m\le F_2\}}+\mu_2 p_{1,0,m+1}+2\mu_1 p_{2,0,m+1}+2\mu_2 p_{0,1,m}\delta_{\{m\le R_2\}}, \quad m>0, \tag{2.11}$$
$$(\lambda_1+\lambda_2+\mu_1+\mu_2)p_{1,0,0}=\lambda_1 p_{0,1}+\lambda_2 p_{1,0}+\mu_1 p_{1,1,0}+\mu_2 p_{1,0,1}+2\mu_1 p_{2,0,1}+2\mu_2 p_{0,1,0}, \tag{2.12}$$

(3) $i=2$,

$$(\lambda_1+\lambda_2+2\mu_1)p_{2,n,m}=\lambda_1 p_{2,n-1,m}+\lambda_2 p_{2,n,m-1}+2\mu_1 p_{2,n+1,m}\delta_{\{m\le F_1\}}+\mu_2 p_{1,n+1,m}\delta_{\{m\le R_1\}}, \quad n,m>0, \tag{2.13}$$
$$(\lambda_1+\lambda_2+2\mu_1)p_{2,n,0}=\lambda_1 p_{2,n-1,0}+2\mu_1 p_{2,n+1,0}+\mu_2 p_{1,n+1,0}, \quad n>0, \tag{2.14}$$
$$(\lambda_1+\lambda_2+2\mu_1)p_{2,0,m}=\lambda_2 p_{2,0,m-1}+2\mu_1 p_{2,1,m}\delta_{\{m\le F_1\}}+\mu_2 p_{1,1,m}\delta_{\{m\le R_1\}}, \quad m>0, \tag{2.15}$$
$$(\lambda_1+\lambda_2+2\mu_1)p_{2,0,0}=\lambda_1 p_{1,0}+2\mu_1 p_{2,1,0}+\mu_2 p_{1,1,0}. \tag{2.16}$$

For $|z|\le 1, |w|\le 1$, define respectively two-dimensional generating functions and one-dimensional generating functions as follows

$$\Psi_i(z,w)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{i,n,m}z^n w^m, \qquad i=0,1,2, \tag{2.17}$$

$$\psi_{im}(z)=\sum_{n=0}^{\infty} p_{i,n,m}z^n, \qquad i=0,1,2;\quad 0\le m\le F_2. \tag{2.18}$$

Multiplying both sides of (2.5) to (2.16) by $z^n w^m$ and summing over all $n\ge 0, m\ge 0$ yield

$$\alpha_0(z,w)\Psi_0(z,w)+\mu_1\Psi_1(z,w)=2\mu_2\sum_{m=0}^{R_2}(\psi_{0m}(z)-\psi_{0m}(0))w^m+\mu_1\sum_{m=0}^{F_2}(\psi_{1m}(z)-\psi_{1m}(0))w^m +2\mu_2 p_{0,0,0}+\mu_1 p_{1,0,0}-\lambda_2 w p_{0,1} \tag{2.19}$$

$$\alpha_1(z,w)z\Psi_1(z,w)+2\mu_1 z\Psi_2(z,w)=-2\mu_2 w\sum_{m=0}^{R_2}(\psi_{0m}(z)-\psi_{0m}(0))w^m-\mu_1 w\sum_{m=0}^{F_2}(\psi_{1m}(z) -\psi_{1m}(0))w^m+\mu_2 z\sum_{m=0}^{R_1}(\psi_{1m}(z)-\psi_{1m}(0))w^m+2\mu_1 z\sum_{m=0}^{F_1}(\psi_{2m}(z)-\psi_{2m}(0))w^m +\mu_2 z p_{1,0,0}+2\mu_1 z p_{2,0,0}+\lambda_1 z w p_{0,1}+\lambda_2 z w p_{1,0} \tag{2.20}$$

$$\alpha_2(z,w)z\Psi_2(z,w) = \mu_2 \sum_{m=0}^{R_1}(\psi_{1m}(z)-\psi_{1m}(0))w^m + 2\mu_1 \sum_{m=0}^{F_1}(\psi_{2m}(z)-\psi_{2m}(0))w^m + \lambda_1 z p_{1,0}, \tag{2.21}$$

where $\alpha_0(z,w) = 2\mu_2(1-w) - w(\lambda_1(1-z) + \lambda_2(1-w))$,
$\alpha_1(z,w) = \mu_2(1-w) - w(\lambda_1(1-z) + \lambda_2(1-w) + \mu_1)$,
$\alpha_2(z,w) = \lambda_1(1-z) + \lambda_2(1-w) + 2\mu_1$.

Furthermore, for the functions $\psi_{im}(z) - \psi_{im}(0)$ $(i = 0,1,2)$ we have

$$\beta_0(z)(\psi_{00}(z)-\psi_{00}(0)) = \lambda_1 z p_{0,0,0}, \tag{2.22}$$

$$\beta_0(z)(\psi_{0m}(z) - \psi_{0m}(0)) = \lambda_2(\psi_{0(m-1)}(z) - \psi_{0(m-1)}(0)) + \lambda_1 z p_{0,0,m}, \quad 1 \le m \le R_2 - 1, \tag{2.23}$$

$$\begin{aligned}\beta_1(z)(\psi_{10}(z) - \psi_{10}(0)) &= 2\mu_2(\psi_{00}(z) - \psi_{00}(0)) + \mu_2 z(\psi_{11}(z) - \psi_{11}(0))\delta_{\{R_1=0\}} + 2\mu_1 z \times \\ &(\psi_{21}(z) - \psi_{21}(0))\delta_{\{F_1=0\}} - z(\lambda_1(1-z) + \lambda_2 + \mu_1 + \mu_2)p_{1,0,0} + \mu_2 z p_{1,0,1} \\ &+2\mu_1 z p_{2,0,1} + \lambda_1 z p_{0,1} + \lambda_2 z p_{1,0},\end{aligned} \tag{2.24}$$

$$\begin{aligned}\beta_1(z)(\psi_{1m}(z)-\psi_{1m}(0)) &= 2\mu_2(\psi_{0m}(z)-\psi_{0m}(0))\delta_{\{m\le R_2\}} + \lambda_2 z(\psi_{1(m-1)}(z)-\psi_{1(m-1)}(0)) \\ &+\mu_2 z(\psi_{1(m+1)}(z) - \psi_{1(m+1)}(0))\delta_{\{m+1>R_1\}} + \mu_2 z p_{1,0,m+1} + 2\mu_1 z(\psi_{2(m+1)}(z) \\ &-\psi_{2(m+1)}(0))\delta_{\{m+1>F_1\}} + 2\mu_1 z p_{2,0,m+1} - z(\lambda_1(1-z) + \lambda_2 + \mu_1 + \mu_2)p_{1,0,m} \\ &+\lambda_2 z p_{1,0,m-1}, \quad 1 \le m \le F_2 - 1,\end{aligned} \tag{2.25}$$

$$\beta_2(z)(\psi_{20}(z)-\psi_{20}(0)) = \mu_2(\psi_{10}(z)-\psi_{10}(0)) - z(\lambda_1(1-z)+\lambda_2+2\mu_1)p_{2,0,0} + \lambda_1 z p_{1,0} \tag{2.26}$$

$$\begin{aligned}\beta_2(z)(\psi_{2m}(z)-\psi_{2m}(0)) &= \mu_2(\psi_{1m}(z)-\psi_{1m}(0))\delta_{\{m\le R_1\}} + \lambda_2 z(\psi_{2(m-1)}(z)-\psi_{2(m-1)}(0)) \\ &-z(\lambda_1(1-z) + \lambda_2 + 2\mu_1)p_{2,0,m} + \lambda_2 z p_{2,0,m-1}, \quad 1 \le m \le F_1,\end{aligned} \tag{2.27}$$

$$\beta_3(z)(\psi_{2m}(z) - \psi_{2m}(0)) = \lambda_2(\psi_{2(m-1)}(z) - \psi_{2(m-1)}(0)) + \lambda_1 z p_{2,0,m}, \quad F_1 < m \le F_2. \tag{2.28}$$

where $\beta_0(z) = \lambda_1(1-z) + \lambda_2 + 2\mu_2$, $\beta_1(z) = z(\lambda_1(1-z) + \lambda_2 + \mu_2) - \mu_1(1-z)$,
$\beta_2(z) = z(\lambda_1(1-z) + \lambda_2) - 2\mu_1(1-z)$, $\beta_3(z) = \lambda_1(1-z) + \lambda_2 + 2\mu_1$.

Using the definitions of $\Psi_i(z,w), i = 0,1,2$, the normalizing condition $\sum_{i=0}^{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{i,n,m} + p_{0,0} + p_{0,1} + p_{1,0} = 1$ can be denoted as

$$\Psi_0(1,1) + \Psi_1(1,1) + \Psi_2(1,1) + p_{0,0} + p_{0,1} + p_{1,0} = 1. \tag{2.29}$$

3. Determination of the function equations

In this section, we derive the two-dimensional generating functions $\Psi_i(z,w)$ $(i = 0,1,2)$. The functional equations (2.19), (2.20) and (2.21) show that they can be obtained as soon as the one-dimensional generating functions $\psi_{0m}(z)$ $(0 \le m \le R_2)$, $\psi_{1m}(z)$ $(0 \le m \le F_2)$ and $\psi_{2m}(z)$ $(0 \le m \le F_1)$ are determined. Therefore, the main aim here is to deduce system equations about those one-dimensional generating functions and get their solutions. Since the unique zero $z_0 = 1 + (\lambda_2 + 2\mu_2)/\lambda_1 > 1$ of $\beta_0(z)$ and the unique zero $z_3 = 1 + (\lambda_2 + 2\mu_1)/\lambda_1 > 1$ of $\beta_3(z)$ all are in $|z| > 1$, a simple induction on m from (2.22) and (2.23) yields that

$$\psi_{0m}(z) - \psi_{0m}(0) = \frac{\lambda_1 z}{\beta_0(z)} \sum_{k=0}^{m} (\frac{\lambda_2}{\beta_0(z)})^{m-k} p_{0,0,k}, \quad 0 \le m \le R_2 - 1, \tag{3.1}$$

and similarly from (2.28) yields that

$$\psi_{2m}(z)-\psi_{2m}(0)=\frac{\lambda_2^{m-F_1}}{\beta_3^{m-v}(z)}(\psi_{2F_1}(z)-\psi_{2F_1}(0))+\frac{\lambda_1 z}{\beta_3(z)}\sum_{k=1}^{m-F_1}(\frac{\lambda_2}{\beta_3(z)})^{m-F_1-k}p_{2,0,F_1+k},$$
$$F_1<m\le F_2. \tag{3.2}$$

Furthermore, from (2.15) we have

$$p_{2,0,m}=\left(\frac{\lambda_2}{\beta_3(0)}\right)^{m-F_1}p_{2,0,F_1},\qquad F_1<m\le F_2. \tag{3.3}$$

Substituting (3.3) into (3.2) gives

$$\psi_{2m}(z)-\psi_{2m}(0)=\frac{\lambda_2^{m-F_1}}{\beta_3^{m-F_1}(z)}(\psi_{2F_1}(z)-\psi_{2F_1}(0))+\left(\frac{\lambda_1\lambda_2^{m-F_1}z}{\beta_3^{m-F_1+1}(z)}\sum_{k=1}^{m-F_1}(\frac{\beta_3(z)}{\beta_3(0)})^k\right)p_{2,0,F_1},$$
$$F_1<m\le F_2. \tag{3.4}$$

Then we can rewrite the equations (2.24)-(2.27) by substituting (3.1) into (2.24) and (2.25), and (3.4) into (2.25) when $F_1<m\le F_2$, and rearranging these terms. We have

$$\beta_1(z)(\psi_{10}(z)-\psi_{10}(0))=2\mu_2\frac{\lambda_1 z}{\beta_0(z)}p_{0,0,0}-z(\lambda_1(1-z)+\lambda_2+\mu_1+\mu_2)p_{1,0,0}+\mu_2 zp_{1,0,1}$$
$$+2\mu_1 zp_{2,0,1}+\lambda_1 zp_{0,1}+\lambda_2 zp_{1,0}, \tag{3.5}$$
$$-\lambda_2 z(\psi_{1(m-1)}(z)-\psi_{1(m-1)}(0))+\beta_1(z)(\psi_{1m}(z)-\psi_{1m}(0))-\mu_2 z(\psi_{1(m+1)}(z)-\psi_{1(m+1)}(0))\times$$
$$\delta_{\{m+1>R_1\}}=2\mu_2\frac{\lambda_1 z}{\beta_0(z)}\sum_{k=0}^{m}(\frac{\lambda_2}{\beta_0(z)})^{m-k}p_{0,0,k}\delta_{\{m\le R_2\}}+\lambda_2 zp_{1,0,m-1}-z(\lambda_1(1-$$
$$z)+\lambda_2+\mu_1+\mu_2)p_{1,0,m}+\mu_2 zp_{1,0,m+1}+2\mu_1 zp_{2,0,m+1}\qquad 1\le m\le F_1-1, \tag{3.6}$$
$$-\lambda_2 z(\psi_{1(m-1)}(z)-\psi_{1(m-1)}(0))+\beta_1(z)(\psi_{1m}(z)-\psi_{1m}(0))-\mu_2 z(\psi_{1(m+1)}(z)-\psi_{1(m+1)}(0))$$
$$-2\mu_1 z(\frac{\lambda_2}{\beta_3(z)})^{m+1-F_1}(\psi_{2F_1}(z)-\psi_{2F_1}(0))=2\mu_2\frac{\lambda_1 z}{\beta_0(z)}\sum_{k=0}^{m}(\frac{\lambda_2}{\beta_0(z)})^{m-k}p_{0,0,k}\times$$
$$\delta_{\{m\le R_2\}}+\lambda_2 zp_{1,0,m-1}-z(\lambda_1(1-z)+\lambda_2+\mu_1+\mu_2)p_{1,0,m}+\mu_2 zp_{1,0,m+1}$$
$$+2\mu_1 z\left((\frac{\lambda_2}{\beta_3(0)})^{m-F_1+1}+\frac{\lambda_1\lambda_2^{m+1-F_1}z}{\beta_3^{m-F_1+2}(z)}\sum_{k=1}^{m+1-F_1}(\frac{\beta_3(z)}{\beta_3(0)})^k\right)p_{2,0,F_1},$$
$$F_1\le m\le F_2-1. \tag{3.7}$$
$$-\mu_2(\psi_{10}(z)-\psi_{10}(0))+\beta_2(z)(\psi_{20}(z)-\psi_{20}(0))=-z(\lambda_1(1-z)+\lambda_2+2\mu_1)p_{2,0,0}+\lambda_1 zp_{1,0} \tag{3.8}$$
$$-\mu_2(\psi_{1m}(z)-\psi_{1m}(0))\delta_{\{m\le R_1\}}-\lambda_2 z(\psi_{2(m-1)}(z)-\psi_{2(m-1)}(0))+\beta_2(z)(\psi_{2m}(z)-\psi_{2m}(0))$$
$$=\lambda_2 zp_{2,0,m-1}-z((\lambda_1(1-z)+\lambda_2+2\mu_1)p_{2,0,m},\quad 1\le m\le F_1. \tag{3.9}$$

The functions $\alpha_i(z,w)$ $(i=0,1,2)$ are same as those in Feng et al.(1999), and their zeros for every fixed z with $|z|\le 1$ have been determined. Here we present the proof again to make the paper self contained.

Theorem 1. If $\rho<2$, then for every fixed $|z|\le 1$,

(i) For $i=0,1$, $\alpha_i(z,w)$ has exactly two zeros $y_i(z), w_i(z)$, and $\alpha_2(z,w)$ has exactly one zero $y_2(z)$. These zeros are

$$y_i(z), w_i(z) = \frac{\lambda_1(1-z)+\lambda_2+i\mu_1+(2-i)\mu_2}{2\lambda_2} + \frac{\sqrt{(\lambda_1(1-z)+\lambda_2+i\mu_1+(2-i)\mu_2)^2-4(2-i)\lambda_2\mu_2}}{2\lambda_2}, \quad i=0,1, \tag{3.10}$$

$$y_2(z) = \frac{\lambda_1(1-z)+\lambda_2+2\mu_1}{\lambda_2} \tag{3.11}$$

(ii) The zeros $y_i(z)$ $(i=0,1,2)$ are in $|w|>1$, and $w_i(z)$ $(i=0,1)$ are in $|w|\le 1$. Furthermore, $w_0(z)\ne w_1(z)$, and $w_i(z)\ne 0$ $(i=0,1)$.

(iii) $w_i(z)$ $(i=0,1)$ are analytic in $|z|<1$ and continuous in $|z|\le 1$.

Proof. (i) For $i=0,1$, write the equation $\alpha_i(z,w)=0$ as follows

$$\lambda_2 w^2-(\lambda_1(1-z)+\lambda_2+i\mu_1+(2-i)\mu_2)w+(2-i)\mu_2=0$$

which is a quadratic polynomial in w. The number and form (3.10) of the roots are obvious. The root $y_2(z)$ in (3.11) of $\alpha_2(z,w)=0$ also is obvious.

(ii) Since $|z|\le 1$, we have Re $y_2(z)>1$, which means that for any fixed $|z|\le 1$, $\alpha_2(z,w)$ has no zeros in $|w|\le 1$. Next, let

$$f_i(z,w)=\frac{(2-i)\mu_2}{\lambda_1(1-z)+\lambda_2+i\mu_1+(2-i)\mu_2}, \qquad i=0,1$$

Then $f_i(z,w)$ is LST(Laplace-Stieltjes Transform) of the exponential distribution with the parameter $(2-i)\mu_2$. The equation $\alpha_i(z,w)=0$ can be rewritten as $w-f_i(z,w)=0$. Fix $|z|\le 1$. For w with $|w|=1$, (1) when $i=1$, we have $|f_1(z,w)|<\mu_2/(\mu_1+\mu_2)<1=|w|$. An easy application of Rouché's Theorem shows that $\alpha_1(z,w)$ has exactly one zero in $|w|\le 1$, which is $w_1(z)$, (2) when $i=0$, if $w\ne 1$ or $w=1$ but $z\ne 1$, we also have $|f_0(z,w)|<1=|w|$. Again by Rouché's Theorem, $\alpha_0(z,w)$ has exactly one zero in $|w|\le 1$, which is $w_0(z)$. When $z=1$, $\alpha_0(1,w)=0$ becomes $\lambda_2 w^2-(\lambda_2+2\mu_2)w+2\mu_2=0$. This equation has two zeros $y_0(1)=2\mu_2/\lambda_2$ and $w_0(1)=1$. Note that under the ergodic condition $\rho<2$, the inequality $\lambda_2<2\mu_2$ holds certainly. Then $y_0(1)>1$. Hence $w_0(1)=1$ is the unique zero of $\alpha_0(1,w)$ in $|w|\le 1$. Furthermore, assume that $w_0(z)=w_1(z)\equiv w(z)$ for some $|z|\le 1$. We obtain the equation $\mu_2-(\mu_2-\mu_1)w(z)=0$ from $\alpha_i(z,w(z))=0, i=0,1$. If $\mu_2=\mu_1$, this equation results in that $\mu_2=0$, which is contradictory to that $\mu_2>0$. If $\mu_2\ne\mu_1$, we get $w(z)=\mu_2/(\mu_2-\mu_1)$ which means that $|w(z)|>1$. This is contradictory to that $w(z)$ is a zero in $|w|\le 1$. The conclusion (iii) can be proved by the implicit function theorem. We omit it here. These complete the proof of Theorem 1. □

Since $\alpha_2(z,w)$ has no zeros in $|w|\le 1$ for any fixed $|z|\le 1$, and similarly with z and w interchanged, dividing (2.21) by $\alpha_2(z,w)$ and then substituting $z\Psi_2(z,w)$ into (2.20), we have

$$\alpha_1(z,w)\Psi_1(z,w) = -2\mu_2 z^{-1}w^{R_2}(\psi_{0R_2}(z)-\psi_{0R_2}(0))+\Bigg\{(\mu_2-\frac{2\mu_1\mu_2}{z\alpha_2(z,w)}-\mu_1 z^{-1}w)\times$$

$$\sum_{m=0}^{R_1}(\psi_{1m}(z)-\psi_{1m}(0))w^m-\mu_1 z^{-1}w\sum_{m=R_1+1}^{F_2}(\psi_{1m}(z)-\psi_{1m}(0))w^m\Bigg\}+2\mu_1(1-\frac{2\mu_1}{z\alpha_2(z,w)})\times$$

$$\sum_{m=0}^{F_1}(\psi_{2m}(z)-\psi_{2m}(0))w^m - \frac{2\mu_2\lambda_1 w}{\beta_0(z)}\sum_{k=0}^{R_2-1}\left(\sum_{m=k}^{R_2-1}(\frac{\lambda_2}{\beta_0(z)})^{m-k}w^m\right)p_{0,0,k}+\mu_2 p_{1,0,0}$$
$$+2\mu_1 p_{2,0,0}+\lambda_1 w p_{0,1}+(\lambda_2 w-\tfrac{2\lambda_1\mu_1}{\alpha_2(z,w)})p_{1,0}. \tag{3.12}$$

Substituting (3.1) into (2.19), and expressing it with (3.12) in matrix form, we have

$$\mathcal{N}(z,w)\begin{bmatrix}\Psi_0(z,w)\\ \Psi_1(z,w)\end{bmatrix}=\sum_{i=1}^{2}\mathcal{A}_i(z,w)(\mathbf{\Pi}_i(z)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(z,w)(\psi_{0R_2}(z)-\psi_{0R_2}(0))$$
$$+\mathcal{C}_0(z,w)\mathcal{P}_0+\mathcal{C}(z,w)\mathcal{P}, \tag{3.13}$$

where the vectors $\mathbf{\Pi}_i(z)$ $(i=1,2)$, $\mathbf{a}_0(z,w)$, $\mathcal{P}_0$, $\mathcal{P}$, and the matrices $\mathcal{N}(z,w)$, $\mathcal{A}_i(z,w)$ $(i=1,2)$, $\mathcal{C}_0(z,w)$ and $\mathcal{C}(z,w)$ are defined as follows:

$$\mathbf{\Pi}_1(z)=(\Psi_{10}(z),\cdots,\Psi_{iF_2}(z))^\tau,\quad \mathbf{\Pi}_2(z)=(\Psi_{20}(z),\cdots,\Psi_{2F_1}(z))^\tau,$$
$$\mathcal{P}_0=(p_{0,0,0},p_{0,0,1},\cdots,p_{0,0,R_2}),\quad \mathcal{P}=(p_{1,0,0},p_{2,0,0},p_{0,1},p_{1,0})^\tau$$

where τ is a symbol of transpose, and

$$\mathcal{N}(z,w)=\begin{bmatrix}\alpha_0(z,w) & \mu_1\\ 0 & \alpha_1(z,w)\end{bmatrix},\qquad \mathbf{a}_0(z,w)=\begin{bmatrix}2\mu_2 w^{R_2}\\ -2\mu_2 z^{-1}w^{R_2+1}\end{bmatrix},$$

$$\mathcal{A}_1(z,w)=\left[\begin{matrix}\mu_1 & \cdots & \mu_1 w^{R_1}\\ \mu_2-\frac{2\mu_1\mu_2}{z\alpha_2(z,w)}-\mu_1 z^{-1}w & \cdots & (\mu_2-\frac{2\mu_1\mu_2}{z\alpha_2(z,w)}-\mu_1 z^{-1}w)w^{R_1}\end{matrix}\right.$$
$$\left.\begin{matrix}\mu_1 w^{R_1+1} & \cdots & \mu_1 w^{F_2}\\ -\mu_1 z^{-1}w^{R_1+2} & \cdots & -\mu_1 z^{-1}w^{F_2+1}\end{matrix}\right],$$

$$\mathcal{A}_2(z,w)=\begin{bmatrix}0 & 0 & \cdots & 0\\ 2\mu_1(1-\frac{2\mu_1}{z\alpha_2(z,w)}) & 2\mu_1(1-\frac{2\mu_1}{z\alpha_2(z,w)})w & \cdots & 2\mu_1(1-\frac{2\mu_1}{z\alpha_2(z,w)})w^{F_1}\end{bmatrix},$$

$$\mathcal{C}_0(z,w)=2\mu_2\begin{bmatrix}1+\sum_{m=0}^{R_2-1}(\frac{\lambda_2}{\beta_0(z)})^m w^m & \sum_{m=1}^{R_2-1}(\frac{\lambda_2}{\beta_0(z)})^{m-1}w^m & \cdots & w^{R_2} & 0\\ \frac{-\lambda_1}{\beta_0(z)}\sum_{m=0}^{R_2-1}(\frac{\lambda_2}{\beta_0(z)})^m w^m & \frac{-\lambda_1}{\beta_0(z)}\sum_{m=1}^{R_2-1}(\frac{\lambda_2}{\beta_0(z)})^{m-1}w^m & \cdots & \frac{-\lambda_1}{\beta_0(z)}w^{R_2} & 0\end{bmatrix}.$$

$$\mathcal{C}(z,w)=\begin{bmatrix}\mu_1 & 0 & -\lambda_2 w & 0\\ \mu_2 & 2\mu_1 & \lambda_1 w & \lambda_2 w-\frac{2\lambda_1\mu_1}{\alpha_2(z,w)}\end{bmatrix}.$$

Then we have

$$\begin{bmatrix}\Psi_0(z,w)\\ \Psi_1(z,w)\end{bmatrix}=\mathcal{N}^{-1}(z,w)\{\sum_{i=1}^{2}\mathcal{A}_i(z,w)(\mathbf{\Pi}_i(z)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(z,w)(\psi_{0R_2}(z)-\psi_{0R_2}(0))$$
$$+\mathcal{C}_0(z,w)\mathcal{P}_0+\mathcal{C}(z,w)\mathcal{P}\}$$

$$=\frac{\begin{bmatrix}\alpha_1(z,w) & -\mu_1\\ 0 & \alpha_0(z,w)]\end{bmatrix}\{\sum_{i=0}^{2}\mathcal{A}_i(z,w)(\mathbf{\Pi}_i(z)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(z,w)(\psi_{0R_2}(z)-\psi_{0R_2}(0)) +\mathcal{C}_0(z,w)\mathcal{P}_0+\mathcal{C}(z,w)\mathcal{P}\}}{\alpha_0(z,w)\alpha_1(z,w)}. \tag{3.14}$$

Since $(\Psi_0(z,w),\Psi_1(z,w))^\tau$ is analytic in $\{(z,w):|z|<1,|w|<1\}$, and continuous in the $\{(z,w):|z|\le 1,|w|\le 1\}$, the numerator of the right-hand side of (3.14) must vanish at the zero $w_i(z)$ of $\alpha_i(z,w)$ for $i=0,1$. We get the matrix equations

$$\begin{bmatrix}\alpha_1(z,w_0(z)) & -\mu_1\\ 0 & \alpha_0(z,w_0(z))]\end{bmatrix}\{\sum_{i=0}^{2}\mathcal{A}_i(z,w_0(z))(\mathbf{\Pi}_i(z)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(z,w_0(z))(\psi_{0R_2}(z)$$

$$-\psi_{0R_2}(0)) + \mathcal{C}_0(z,w_0(z))\mathcal{P}_0 + \mathcal{C}(z,w_0(z))\mathcal{P}\} = 0, \tag{3.16}$$

$$\begin{bmatrix} \alpha_1(z,w_1(z)) & -\mu_1 \\ 0 & \alpha_0(z,w_1(z))] \end{bmatrix} \{\sum_{i=0}^{2} \mathcal{A}_i(z,w_1(z))(\mathbf{\Pi}_i(z) - \mathbf{\Pi}_i(0)) + \mathbf{a}_0(z,w_1(z))(\psi_{0R_2}(z) - \psi_{0R_2}(0)) + \mathcal{C}_0(z,w_1(z))\mathcal{P}_0 + \mathcal{C}(z,w_1(z))\mathcal{P}\} = 0. \tag{3.17}$$

Note that $\alpha_i(z,w_i(z)) = 0$ for $i = 0,1$ and $\alpha_i(z,w_j(z)) \neq 0$ for $i \neq j; i,j = 0,1$. We can only obtain the following two linearly independent equations from (3.16) and (3.17).

$$\sum_{i=1}^{2} \mathcal{B}_i(z)(\mathbf{\Pi}_i(z) - \mathbf{\Pi}_i(0)) + \mathbf{b}_0(z)(\psi_{0R_2}(z) - \psi_{0R_2}(0)) + \mathcal{D}_0(z)\mathcal{P}_0 + \mathcal{D}(z)\mathcal{P} = 0 \tag{3.18}$$

where the matrices $\mathcal{B}_i(z)$ $(i = 1,2)$, $\mathcal{D}_0(z)$, $\mathcal{D}(z)$ and the vector $\mathbf{b}_0(z)$ are deduced as follows

$$\mathcal{B}_1(z) = \begin{bmatrix} \mu_1[z\alpha_1(z,w_0(z)) - (\mu_2 z - \frac{2\mu_1\mu_2}{\alpha_2(z,w_0(z))} - \mu_1 w_0(z))] & \cdots & \mu_1[z\alpha_1(z,w_0(z)) - (\mu_2 z - \frac{2\mu_1\mu_2}{\alpha_2(z,w_0(z))} - \mu_1 w_0(z))]w_0^{R_1}(z) & \mu_1[z\alpha_1(z,w_0(z)) + \mu_1 w_0(z)]w_0^{R_1+1}(z) & \cdots & \mu_1[z\alpha_1(z,w_0(z)) + \mu_1 w_0(z)]w_0^{F_1}(z)) \\ \mu_2 z - \frac{2\mu_1\mu_2}{\alpha_2(z,w_1(z))} - \mu_1 w_1(z) & \cdots & [\mu_2 z - \frac{2\mu_1\mu_2}{\alpha_2(z,w_1(z))} - \mu_1 w_1(z)]w_1^{R_1}(z) & -\mu_1 w_1^{R_1+2}(z) & \cdots & -\mu_1 w_1^{F_1+1}(z) \end{bmatrix},$$

$$\mathcal{B}_2(z) = 2\mu_1 \begin{bmatrix} -\mu_1[z - \frac{2\mu_1}{\alpha_2(z,w_0(z))}] & \cdots & -\mu_1[z - \frac{2\mu_1}{\alpha_2(z,w_0(z))}]w_0^{F_1}(z) \\ z - \frac{2\mu_1}{\alpha_2(z,w_1(z))} & \cdots & [z - \frac{2\mu_1}{\alpha_2(z,w_1(z))}]w_1^{F_1}(z) \end{bmatrix},$$

$$\mathcal{D}_0(z) = 2\mu_2 z \begin{bmatrix} \alpha_1(z,w_0(z)) + (\alpha_1(z,w_0(z)) + \frac{\lambda_1\mu_1 w_0(z)}{\beta_0(z)})\sum_{m=0}^{R_2-1}(\frac{\lambda_2}{\beta_0(z)})^m w_0^m(z) & \cdots & (\alpha_1(z,w_0(z)) + \frac{\lambda_1\mu_1 w_0(z)}{\beta_0(z)})w_0^{R_2-1}(z) & 0 \\ -\frac{\lambda_1\mu_1}{\beta_0(z)})\sum_{m=0}^{R_2-1}(\frac{\lambda_2}{\beta_0(z)})^m w_1^{m+1}(z) & \cdots & -\frac{\lambda_1\mu_1}{\beta_0(z)})w_1^{R_2}(z) & 0 \end{bmatrix},$$

$$\mathcal{D}(z) = \begin{bmatrix} \mu_1 z[\alpha_1(z,w_0(z)) - \mu_2] & -2\mu_1^2 z & -[\lambda_2\alpha_1(z,w_0(z)) + \lambda_1\mu_1]zw_0(z) & -\mu_1 z[\lambda_2 w_0(z) - \frac{2\lambda_1\mu_1}{\alpha_2(z,w_0(z))}] \\ \mu_2 z & 2\mu_2 z & \lambda_1 z w_1(z) & z[\lambda_2 w_1(z) - \frac{2\lambda_1\mu_1}{\alpha_2(z,w_1(z))}] \end{bmatrix}.$$

and $\mathbf{b}_0(z) = 2\mu_2([z\alpha_1(z,w_0(z)) + \mu_1 w_0(z)]w_0^{R_2}(z),\ -w_1^{R_2+1}(z))^\tau$.

Let $\mathcal{P}_1 = (p_{1,0,0}, \cdots, p_{1,0,F_2})^\tau$, $\mathcal{P}_2 = (p_{2,0,0}, \cdots, p_{2,0,F_1})^\tau$, and

$\mathbf{\Pi}(z) = (\mathbf{\Pi}_1^\tau(z), \mathbf{\Pi}_2^\tau(z), \psi_{0R_2}(z))^\tau$, $\mathbf{P} = (\mathcal{P}_0^\tau, \mathcal{P}_1^\tau, \mathcal{P}_2^\tau)^\tau$, $\hat{\mathcal{P}} = (p_{0,0}, p_{0,1}, p_{1,0})^\tau$.

Then writing the equations (3.5)–(3.9) and (3.18) in matrix form, we have

$$\mathcal{M}(z)(\mathbf{\Pi}(z) - \mathbf{\Pi}(0)) = \mathcal{E}(z)\mathbf{P} + \mathcal{K}(z)\hat{\mathcal{P}}. \tag{3.19}$$

Here $\mathcal{M}(z)$ is the $(F_2+F_1+3)\times(F_2+F_1+3)$ matrix and $\mathcal{E}(z)$ is the $(F_2+F_1+3)\times(2F_2+F_1+3)$ matrix defined as follows

$$\mathcal{M}(z) = \begin{bmatrix} \mathcal{M}_{11}(z) & \mathcal{M}_{12}(z) & \mathbf{0}_{F_2} \\ \mathcal{M}_{21}(z) & \mathcal{M}_{22}(z) & \mathbf{0}_{F_1+1} \\ \mathcal{B}_1(z) & \mathcal{B}_2(z) & \mathbf{b}_0 \end{bmatrix}, \quad \mathcal{E}(z) = \begin{bmatrix} \mathcal{E}_{11}(z) & \mathcal{E}_{12}(z) & \mathcal{E}_{13}(z) \\ \mathcal{E}_{21}(z) & \mathcal{E}_{22}(z) & \mathcal{E}_{23}(z) \\ \mathcal{E}_{31}(z) & \mathcal{E}_{32}(z) & \mathcal{E}_{33}(z) \end{bmatrix},$$

where $\mathcal{M}_{11}(z)$: $F_2\times(F_2+1)$; $\mathcal{M}_{12}(z)$: $F_2\times(F_1+1)$; $\mathcal{M}_{21}(z)$: $(F_1+1)\times(F_2+1)$; $\mathcal{M}_{22}(z)$: $(F_1+1)\times(F_1+1)$ matrices, and $\mathcal{E}_{11}(z)$, $\mathcal{E}_{12}(z)$: $F_2\times(F_2+1)$; $\mathcal{E}_{13}(z)$: $F_2\times(F_1+1)$; $\mathcal{E}_{21}(z)$, $\mathcal{E}_{22}(z)$: $(F_1+1)\times(F_2+1)$; $\mathcal{E}_{23}(z)$: $(F_1+1)\times(F_2+1)$; $\mathcal{E}_{31}(z)$, $\mathcal{E}_{32}(z)$: $2\times(F_2+1)$; $\mathcal{E}_{33}(z)$:$2\times(F_1+1)$ matrices such that

$$\mathcal{M}_{11}(z)=\begin{bmatrix} \beta_1(z) & & & & & & \\ -\lambda_2 z & \beta_1(z) & & & & & \\ & \ddots & \ddots & & & & \\ & & -\lambda_2 z & \beta_1(z) & & & \\ & & & -\lambda_2 z & \beta_1(z) & -\mu_2 z & \\ & & & & \ddots & \ddots & \ddots \\ & & & & -\lambda_2 z & \beta_1(z) & -\mu_2 z \end{bmatrix},$$

$$\mathcal{M}_{12}(z)=\begin{bmatrix} 0 & & \\ \vdots & \ddots & \\ 0 & \cdots & -2\mu_1 z(\frac{\lambda_2}{\beta_3(z)}) \\ \vdots & & \vdots \\ 0 & \cdots & -2\mu_1 z(\frac{\lambda_2}{\beta_3(z)})^{F_2-F_1} \end{bmatrix},\quad \mathcal{M}_{21}(z)=\begin{bmatrix} -\mu_2 & & & & \\ & \ddots & & & \\ & & -\mu_2 & 0 & \cdots & 0 \end{bmatrix},$$

$$\mathcal{M}_{22}(z)=\begin{bmatrix} \beta_2(z) & & & \\ -\lambda_2 z & \beta_2(z) & & \\ & \ddots & \ddots & \\ & & -\lambda_2 z & \beta_2(z) \end{bmatrix},$$

$$\mathcal{E}_{11}(z)=\frac{2\mu_2\lambda_1 z}{\beta_0(z)}\begin{bmatrix} 1 & & & & \\ \frac{\lambda_2}{\beta_0(z)} & 1 & & & \\ \vdots & \vdots & \ddots & & \\ (\frac{\lambda_2}{\beta_0(z)})^{R_2} & (\frac{\lambda_2}{\beta_0(z)})^{R_2-1} & \cdots & 1 & \\ 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & \cdots & & \ddots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

$$\mathcal{E}_{12}(z)=\begin{bmatrix} -(\beta_1(z)-\mu_1) & \mu_2 z & & & \\ \lambda_2 z & -(\beta_1(z)-\mu_1) & \mu_2 z & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda_2 z & -(\beta_1(z)-\mu_1) & \mu_2 z \end{bmatrix},$$

$$\mathcal{E}_{13}(z)=\begin{bmatrix} 0 & 2\mu_1 z & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 2\mu_1 z(\frac{\lambda_1\lambda_2 z}{\beta_3(z)\beta_3(0)}+\frac{\lambda_2}{\beta_3(0)}) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 2\mu_1 z(\frac{\lambda_1\lambda_2^{F_2-F_1}z}{\beta_3^{F_2-F_1+1}(z)}\sum_{k=1}^{F_2-F_1}(\frac{\beta_3(z)}{\beta_3(0)})^k+(\frac{\lambda_2}{\beta_3(0)})^{F_2-F_1}) \end{bmatrix},$$

$$\mathcal{E}_{23}(z) = \begin{bmatrix} -(\beta_2(z)-2\mu_1) & & & \\ \lambda_2 z & -(\beta_2(z)-2\mu_1) & & \\ & \ddots & \ddots & \\ & & \lambda_2 z & -(\beta_2(z)-2\mu_1) \end{bmatrix}, \quad \mathcal{E}_{31}(z) = \mathcal{D}_0(z)$$

$$\mathcal{E}_{32}(z) = \begin{bmatrix} \mu_1 z[\alpha_1(z,w_0(z))-\mu_2] & 0 & \cdots & 0 \\ \mu_2 z & 0 & \cdots & 0 \end{bmatrix}, \quad \mathcal{E}_{33}(z) = \begin{bmatrix} -2\mu_1^2 z & 0 & \cdots & 0 \\ 2\mu_2 z & 0 & \cdots & 0 \end{bmatrix}.$$

and $\mathcal{E}_{21}(z) = \mathcal{E}_{22}(z) = \mathbf{0}_{(F_1+1)(F_2+1)}$, i.e., the $(F_1+1)\times(F_2+1)$ null matrix. $\mathbf{0}_k$ denotes k–dimensional null vector. Moreover, $\mathcal{K}(z)$ is the $(F_2+(F_2+1)+2)\times 3$ matrix such that

$$\mathcal{K}(z) = \begin{bmatrix} \mathcal{K}_1(z) \\ \mathcal{K}_2(z) \\ \mathcal{K}_3(z) \end{bmatrix}, \quad \text{where} \quad \mathcal{K}_1(z) = \begin{bmatrix} 0 & \lambda_1 z & \lambda_2 z \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{K}_2(z) = \begin{bmatrix} 0 & 0 & \lambda_1 z \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{K}_3(z) = \begin{bmatrix} 0 & -z[\lambda_2\alpha_1(z,w_0(z))+\lambda_1\mu_1]w_0(z) & -\mu_1 z[\lambda_2 w_0(z) - \frac{2\lambda_1\mu_1}{\alpha_2(z,w_0(z))}] \\ 0 & \lambda_1 z w_1(z) & z[\lambda_2 w_1(z) - \frac{2\lambda_1\mu_1}{\alpha_2(z,w_1(z))}] \end{bmatrix}.$$

Whenever $\mathcal{M}(z)$ is non-singular, the solution of (3.19) is given by

$$\mathbf{\Pi}(z) - \mathbf{\Pi}(0) = \mathcal{M}(z)^{-1}\{\mathcal{E}(z)\mathbf{P} + \mathcal{K}(z)\hat{\mathcal{P}}\} = \frac{[adj\mathcal{M}(z)]\{\mathcal{E}(z)\mathbf{P} + \mathcal{K}(z)\hat{\mathcal{P}}\}}{det\mathcal{M}(z)}. \tag{3.20}$$

Since we seek $\mathbf{\Pi}(z)-\mathbf{\Pi}(0)$ which is analytic in $|z|<1$, the numerator of the right-hand side of (3.20) must vanish at the zeros of $det\mathcal{M}(z)$ inside the unit circle. Therefore, in solving (3.19) we have to consider the characters of those zeros. Let $z_0, z_1, \cdots, z_{\kappa-1}$ be zeros of $det\mathcal{M}(z)$ in $|z|\le 1$, and let d_i be the multiplicity of z_i. The analyticity of $\mathbf{\Pi}(z)-\mathbf{\Pi}(0)$ implies that

$$\frac{d^k}{dz^k}[adj\mathcal{M}(z)]\{\mathcal{E}(z)\mathbf{P} + \mathcal{K}(z)\hat{\mathcal{P}}\}|_{z=z_i} = 0 \qquad 0\le k < d_j, \quad 0 \le i \le \kappa-1. \tag{3.21}$$

It can be verified that for every zero z_i, at most d_i of the $(F_2+F_1+3)d_i$ equations in (3.21) are independent. In general, it is particularly difficult to determine directly the value of κ and d_i for $i=0,1,\cdots,\kappa-1$ because of the complexity of $det\mathcal{M}$. Here we discuss the problem by the approach in Cohen and Down(1996), Boxma and Down(1997), where the fact that under the ergodicity condition the inherent Kolmogorov equations for the stationary state probabilities have a unique, absolutely convergent solution is used to show that the corresponding functional equation has the specified number of zeros in the required domain. Suppose, without loss of generality, that $d_i = 1$ for $i=0,1,\cdots,\kappa-1$, i.e., all zeros have multiplicity one. We shall argue that under the ergodicity condition: $\rho<1$, $\kappa = F_2+F_1+3$, i.e., F_2+F_1+3 independent equations can be obtained from (3.21). Indeed, the Kolmogrov equations for the equilibrium distribution of the continuous-time Markov chain $\{(I_1(t), I_2(t), Q_1(t), Q_2(t)\}_{t\ge 0}$, along with the normalizing condition (2.29), have a unique, absolutely convergent solution, and using the generating functions, we have transformed those Kolmogorov equation into the (F_2+F_1+3)-dimensional matrix equation (3.19). If $\kappa = F_2+F_1+3$, then (3.21) plus F_2+3 equations (2.2)–(2.4) and (2.7)–(2.8) for $0\le m < F_2$, there exists a unique solution for the $2F_2+F_1+6$ unknown constants $\mathbf{P}, \mathcal{P}$. Now suppose that $\kappa < F_2+F_1+3$. Then we would obtain not sufficient equations to determine all $2F_2+F_1+6$ unknown constants uniquely, and we would find multiple solutions for them–which is impossible. Finally, if $\kappa > F_2+F_1+3$, then we would find too many equations for the $2F_2 + F_1 + 6$ unknown constants. Once again, as it is known that there is a unique solution, there must be exactly $F_2 + F_1 + 3$ independent equations amongst those derived by using (3.21).

Note that $w_0(1) = 1$ is the unique zero of $\alpha_0(1, w)$ in $|w| \leq 1$. We have $\alpha_1(1, w_0(1)) = -\mu_1$ and $\alpha_2(1, w_0(1)) = 2\mu_1$. Then

$$[z\alpha_1(z, w_0(z)) + \mu_1 w_0(z)]_{z=1} = 0, \qquad [z - \tfrac{2\mu_1}{\alpha_2(z, w_0(z))}]_{z=1} = 0,$$
$$[z\alpha_1(z, w_0(z)) - (\mu_2 z - \tfrac{2\mu_1\mu_2}{\alpha_2(z, w_0(z))} - \mu_1 w_0(z))]_{z=1} = 0.$$

It follows that all entries of the $(F_2 + F_1 + 2)$th row of $\mathcal{M}(z)$ at $z = 1$ are zero. Thus $z = 1$ is a zero of $det\mathcal{M}(z)$. Without loss of generality, let $z_0 = 1$. From (3.21) we have that $[adj\mathcal{M}(z)]\{\mathcal{E}(z)\mathbf{P} + \mathcal{K}(z)\hat{\mathcal{P}}\}|_{z=1} = 0$. Then from (3.20) there exist two vectors $\mathbf{e}_1, \mathbf{k}_1$ such that

$$\mathbf{\Pi}(1) - \mathbf{\Pi}(0) = \mathbf{e}_1\mathbf{P} + \mathbf{k}_1\hat{\mathcal{P}}. \tag{3.22}$$

Furthermore, substituting (3.22) into (3.14) and noting that from (2.21)

$$\Psi_2(1,1) = \frac{\mu_2}{2\mu_1}\sum_{m=0}^{R_1}(\psi_{1m}(1) - \psi_{1m}(0)) + \sum_{m=0}^{F_1}(\psi_{2m}(1) - \psi_{2m}(0)) + \frac{\rho_1}{2}p_{1,0},$$

we get an expression of sum of $\Psi_i(1,1), i = 0, 1, 2$ as

$$\Psi_0(1,1) + \Psi_1(1,1) + \Psi_2(1,1) = \mathbf{eP} + \mathbf{k}\hat{\mathcal{P}}$$

where $\mathbf{e}$ and $\mathbf{k}$ are the known vectors. Now the normalizing condition (2.29) can be denoted as

$$\mathbf{eP} + \mathbf{k}\hat{\mathcal{P}} + p_{0,0} + p_{0,1} + p_{1,0} = 1. \tag{3.23}$$

Therefore, combining (3.23), (3.21) for $i = 1, \cdots, \kappa - 1$ with (2.2)–(2.4) and (2.7)–(2.8) for $0 \leq m < F_2$, we get a matrix system with $2F_2 + F_1 + 6$ independent equations for the unknown constants $\mathbf{P}, \hat{\mathcal{P}}$ as follows

$$\mathcal{Q}[\mathbf{P}^\tau, \hat{\mathcal{P}}^\tau]^\tau = \boldsymbol{\nu} \tag{3.24}$$

where $\boldsymbol{\nu} = (0, \cdots, 0, 1)^\tau$ is an $(2F_2 + F_1 + 6)$-dimensional vector. Solving the system and substituting the solution into (3.20), we can obtain the one-dimensional generating functions $\psi_{0m}(z)$ $(0 \leq m \leq R_2)$, $\psi_{1m}(z)$ $(0 \leq m \leq F_2)$ and $\psi_{2m}(z)$ $(0 \leq m \leq F_1)$. Then substituting these functions into (3.14) and (2.21), we can finally determine the two-dimensional generating functions $\Psi_i(z, w)$ $(i = 0, 1, 2)$.

4. The mean queue length and mean waiting time

In this section we derive the mean queue length and the mean waiting time for both queues by using the previous results concerning the generating functions. By $E[Q_i]$ and $E[W_i]$, we denote the mean queue length and the mean waiting time of the queue Q_i, respectively. Since the queue or waiting time may be built only when both two servers are busy, we merely need to consider the generating functions $\Psi_i(z_1, z_2)$ $(0 \leq i \leq 2)$. We have

$$E[Q_1] = \sum_{i=0}^{2}\frac{\partial}{\partial z}\Psi_i(z,1)|_{z=1}, \qquad E[Q_2] = \sum_{i=0}^{2}\frac{\partial}{\partial w}\Psi_i(1,w)|_{w=1}. \tag{4.1}$$

We first calculate the values of $\alpha_i(z, w)$ $(0 \leq i \leq 2)$ at $z = 1$ and $w = 1$ as

$$\alpha_0(1,1) = 0, \qquad \frac{\partial}{\partial z}\alpha_0(z,1)|_{z=1} = \lambda_1, \qquad \frac{\partial}{\partial w}\alpha_0(1,w)|_{w=1} = \lambda_2 - 2\mu_2,$$

$$\alpha_1(1,1)=-\mu_1,\quad \frac{\partial}{\partial z}\alpha_1(z,1)|_{z=1}=\lambda_1,\quad \frac{\partial}{\partial w}\alpha_1(1,w)|_{w=1}=\lambda_2-\mu_1-\mu_2,$$
$$\alpha_2(1,1)=2\mu_1,\quad \frac{\partial}{\partial z}\alpha_2(z,1)|_{z=1}=-\lambda_1,\quad \frac{\partial}{\partial w}\alpha_2(1,w)|_{w=1}=-\lambda_2.$$

Substituting $w=1$ into (2.21) and (3.14) yields

$$\Psi_2(z,1)=\frac{\mu_2\sum_{m=0}^{R_2}(\psi_{1m}(z)-\psi_{1m}(0))+2\mu_1\sum_{m=0}^{F_1}(\psi_{2m}(z)-\psi_{2m}(0))+\lambda_1 z p_{1,0}}{\alpha_2(z,1)},$$

$$\Psi_1(z,1)=\frac{(0\quad 1)\{\sum_{i=1}^{2}\mathcal{A}_i(z,1)(\mathbf{\Pi}_i(z)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(z,1)(\psi_{0R_2}(z)-\psi_{0R_2}(0))}{\alpha_1(z,1)}$$
$$+\frac{\mathcal{C}_0(z,1)\mathcal{P}_0+\mathcal{C}(z,1)\mathcal{P}\}}{\alpha_1(z,1)},$$

$$\Psi_0(z,1)=\frac{(\alpha_1(z,1)\quad-\mu_1)\{\sum_{i=1}^{2}\mathcal{A}_i(z,1)(\mathbf{\Pi}_i(z)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(z,1)(\psi_{0R_2}(z)-}{\alpha_0(z,1)\alpha_1(z,1)}$$
$$+\frac{\psi_{0R_2}(0))+\mathcal{C}_0(z,1)\mathcal{P}_0+\mathcal{C}(z,1)\mathcal{P}\}}{\alpha_0(z,1)\alpha_1(z,1)}.$$

Hence we have

$$\frac{\partial}{\partial z}\Psi_2(z,1)|_{z=1}=(\frac{1}{2}\rho_1-1)[\frac{\mu_2}{2\mu_1}\sum_{m=0}^{R_2}(\psi_{1m}(1)-\psi_{1m}(0))+\sum_{m=0}^{F_1}(\psi_{2m}(1)-\psi_{2m}(0))]$$
$$+\frac{\mu_2}{2\mu_1}\sum_{m=0}^{F_1}\frac{d}{dz}\psi_{1m}(z)|_{z=1}+\sum_{m=0}^{F_2}\frac{d}{dz}\psi_{2m}(z)|_{z=1}+\frac{1}{4}\rho_1^2p_{1,0}\tag{4.2}$$

$$\frac{\partial}{\partial z}\Psi_1(z,1)|_{z=1}=-\frac{2\mu_2}{\mu_1}(1-\rho_1)(\psi_{0R_2}(1)-\psi_{0R_2}(0))-(1-\rho_1+\frac{\mu_2}{\mu_1}(1-\frac{1}{2}\rho_1))$$
$$\times\sum_{m=0}^{R_1}(\psi_{1m}(1)-\psi_{1m}(0))-(1-\rho_1)\sum_{m=R_1+1}^{F_2}(\psi_{1m}(1)-\psi_{1m}(0))$$
$$-(2-\rho_1)\sum_{m=0}^{F_1}(\psi_{2m}(1)-\psi_{2m}(0))+\frac{2\mu_2}{\mu_1}\frac{d}{dz}\psi_{0R_2}(z)|_{z=1}+\sum_{m=0}^{F_2}\frac{d}{dz}\psi_{1m}(z)|_{z=1}$$
$$+\frac{2\rho_1}{2+\rho_2}\sum_{m=0}^{R_2-1}\sum_{k=m}^{R_2-1}(\rho_1+(k-m+1)\frac{\lambda_1}{\mu_2(2+\rho_2)})(\frac{\rho_2}{2+\rho_2})^{k-m}p_{0,0,m}$$
$$-\frac{\mu_2}{\mu_1}\rho_1p_{1,0,0}-2\rho_1p_{2,0,0}-\rho_1^2p_{0,1}+\rho_1(\frac{3}{2}\rho_1-\frac{\lambda_2}{\mu_1})p_{1,0}\tag{4.3}$$

$$\frac{\partial}{\partial z}\Psi_0(z,1)|_{z=1}=\frac{2\mu_2}{\lambda_1\mu_1}(\rho_1(1-\rho_1)-\mu_1)(\psi_{0R_2}(1)-\psi_{0R_2}(0))+(1-\rho_1-\frac{1}{\rho_1}-\frac{\mu_2}{2\mu_1}$$
$$+\frac{\mu_2}{\lambda_1}(2\rho_1-\frac{3}{4}\rho_1^2-1))\sum_{m=0}^{R_1}(\psi_{1m}(1)-\psi_{1m}(0))+\frac{1}{\lambda_1}(\rho_1(1-\rho_1)-\mu_1)$$
$$\times\sum_{m=R_1+1}^{F_2}(\psi_{1m}(1)-\psi_{1m}(0))+\frac{1}{\rho_1}(3\rho_1-\frac{3}{2}\rho_1^2-2)\sum_{m=0}^{F_1}(\psi_{2m}(1)-\psi_{2m}(0))$$

$$+\frac{2\mu_2}{\lambda_1}(1-\rho_1)\frac{d}{dz}\psi_{0R_2}(z)|_{z=1}+(\frac{1}{\rho_1}-1+\frac{\mu_2}{\mu_1}(1-\frac{1}{2}\rho_1))\sum_{m=0}^{R_1}\frac{d}{dz}\psi_{1m}(z)|_{z=1}$$

$$+\frac{1}{\rho_1}(1-\rho_1)\sum_{m=R_1+1}^{F_2}\frac{d}{dz}\psi_{1m}(z)|_{z=1}+\frac{2}{2+\rho_2}\sum_{m=0}^{R_2-1}\sum_{k=m}^{R_2-1}[\frac{\rho_1(1-\rho_1)-\mu_1}{\mu_1}$$

$$+(1-\rho_1)(1+(k-m+1)\frac{\lambda_2}{\mu_2(2+\rho_2)})](\frac{\rho_2}{2+\rho_2})^{k-m}p_{0,0,m}$$

$$-\frac{1}{\mu_1}\{2\mu_2\rho_1 p_{0,0,0}+\lambda_1 p_{1,0,0}-\lambda_1\rho_1 p_{0,1}+\frac{3}{4}\rho_1^2 p_{1,0}\}. \tag{4.4}$$

Similarly, substituting $z=1$ into (2.22) and (3.21) yields

$$\Psi_2(1,w)=\frac{\mu_2\sum_{m=0}^{R_2}(\psi_{1m}(1)-\psi_{1m}(0))w^m+2\mu_1\sum_{m=0}^{F_1}(\psi_{2m}(1)-\psi_{2m}(0))w^m+\lambda_1 p_{1,0}}{\alpha_2(1,w)},$$

$$\Psi_1(1,w)=\frac{(0\quad 1)\{\sum_{i=1}^{2}\mathcal{A}_i(1,w)(\mathbf{\Pi}_i(1)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(1,w)(\psi_{0R_2}(1)-\psi_{0R_2}(0))}{\alpha_1(1,w)}$$
$$+\frac{\mathcal{C}_0(1,w)\mathcal{P}_0+\mathcal{C}(1,w)\mathcal{P}\}}{\alpha_1(1,w)},$$

$$\Psi_0(1,w)=\frac{(\alpha_1(1,w)\quad-\mu_1)\{\sum_{i=1}^{2}\mathcal{A}_i(1,w)(\mathbf{\Pi}_i(1)-\mathbf{\Pi}_i(0))+\mathbf{a}_0(1,w)(\psi_{0R_2}(1)-}{\alpha_0(1,w)\alpha_1(1,w)}$$
$$+\frac{\psi_{0R_2}(0))+\mathcal{C}_0(1,w)\mathcal{P}_0+\mathcal{C}(1,w)\mathcal{P}\}}{\alpha_0(1,w)\alpha_1(1,w)}.$$

We have

$$\frac{\partial}{\partial z}\Psi_2(1,w)|_{w=1}=\frac{\mu_2}{4\mu_1}\sum_{m=0}^{R_2}(\lambda_2+2m\mu_1)(\psi_{1m}(1)-\psi_{1m}(0))+\frac{1}{2\mu_1}\sum_{m=0}^{F_1}(\lambda_2+2m\mu_1)$$
$$\times(\psi_{2m}(1)-\psi_{2m}(0))+\frac{\lambda_2}{4\mu_1}\rho_1 p_{1,0} \tag{4.5}$$

$$\frac{\partial}{\partial z}\Psi_1(1,w)|_{w=1}=\frac{2\mu_2}{\mu_1^2}(\lambda_2+\mu_1R_2-\mu_2)(\psi_{0R_2}(1)-\psi_{0R_2}(0))+\frac{1}{\mu_1}\sum_{m=0}^{R_1}(\frac{\lambda_2\mu_2}{2\mu_1}+\lambda_2+m\mu_1-\mu_2)$$
$$\times(\psi_{1m}(1)-\psi_{1m}(0))+\frac{1}{\mu_1}\sum_{m=R_1+1}^{F_2}(\lambda_2+m\mu_1-\mu_2)(\psi_{1m}(1)-\psi_{1m}(0))$$
$$+\frac{\lambda_2}{\mu_1}\sum_{m=0}^{F_1}(\psi_{2m}(1)-\psi_{2m}(0))+\frac{\rho_1}{\mu_1}\frac{2}{2+\rho_2}\sum_{m=0}^{R_2-1}\sum_{k=m}^{R_2-1}(\lambda_2+(k-1)\mu_1-\mu_2)$$
$$\times(\frac{\rho_2}{2+\rho_2})^{k-m}p_{0,0,m}-\frac{1}{\mu_1^2}\{(\lambda_2-\mu_1-\mu_2)(\mu_2 p_{1,0,0}+2\mu_1 p_{2,0,0})+\lambda_1(\lambda_2-\mu_2)p_{0,1}$$
$$+((\lambda_2-\lambda_1)(\lambda_2-\mu_1-\mu_2)+\lambda_2\mu_1(1-\frac{1}{2}\rho_1))p_{1,0}\} \tag{4.6}$$

$$\frac{\partial}{\partial z}\Psi_0(1,w)|_{w=1}=\frac{2\mu_2}{\mu_1^2}(\rho_2-1)(\lambda_2+(R_2-1)\mu_1-\mu_2)(\psi_{0R_2}(1)-\psi_{0R_2}(0))$$

$$+\frac{1}{2\mu_1}\sum_{m=0}^{R_1}\{2(\lambda_2-\mu_1-\mu_2)(\rho_2+m\frac{\mu_1}{\mu_2}-1+\frac{\lambda_2}{2\mu_1})+\frac{\lambda_2^2}{2\mu_1}+m((m-3)\mu_2-\rho_2)\}$$

$$\times(\psi_{1m}(1)-\psi_{1m}(0))+\frac{1}{\mu_1}(\rho_2-1)\sum_{m=R_1+1}^{F_2}(\lambda_2+(m-1)\mu_1-\mu_2)$$

$$\times(\psi_{1m}(1)-\psi_{1m}(0))+\frac{\rho_2}{\mu_1}\sum_{m=0}^{F_1}(\lambda_2+(m-1)\mu_1-\mu_2)(\psi_{2m}(1)-\psi_{2m}(0))$$

$$+\frac{\rho_1(1-\rho_1)}{\mu_1}\frac{2}{2+\rho_2}\sum_{m=0}^{R_2-1}\sum_{k=m}^{R_2-1}(\lambda_2+(k-1)\mu_1-\mu_2)(\frac{\rho_2}{2+\rho_2})^{k-m}p_{0,0,m}$$

$$+\frac{1}{\mu_1}(\rho_2-\frac{\mu_1}{\mu_2}-1)\{(\lambda_2-\mu_1-\mu_2)(2\mu_2p_{0,0,0}+\mu_1p_{1,0,0})+\lambda_1(\mu_1+\mu_2-\lambda_2)p_{0,1}\}$$

$$-\frac{\rho_2}{2\mu_1^2}(\frac{1}{2}\lambda_2\mu_1\rho_1-2(\lambda_2-\mu_1-\mu_2)(1-\frac{1}{2}\rho_1))p_{1,0}. \tag{4.7}$$

As shown in (4.2)–(4.7), the values of $\mathbf{\Pi}(1)-\mathbf{\Pi}(0)$ and $d/dz\mathbf{\Pi}(z)|_{z=1}$ are necessary to obtain $\partial/\partial z\Psi_i(z,1)$ and $\partial/\partial w\Psi_i(1,w)$ for $i=0,1,2$. From (3.22) we can calculate $\mathbf{\Pi}(1)-\mathbf{\Pi}(0)$. Furthermore, utilizing analyticities of $\mathbf{\Pi}(z)$ we can calculate $d/dz\mathbf{\Pi}(z)|_{z=1}$ by directly differentiating (3.20) in z, and then letting $z\to 1$. We have

$$\frac{d}{dz}\mathbf{\Pi}(z)|_{z=1}=\mathcal{M}(z)\left[(\frac{d}{dz}\mathcal{E}(z))\mathbf{P}+(\frac{d}{dz}\mathcal{K}(z))\hat{\mathcal{P}}-(\frac{d}{dz}\mathcal{M}(z))(\mathbf{\Pi}(z)-\mathbf{\Pi}(0))\right]_{z=1}. \tag{4.8}$$

The mean waiting times for the customers of both queues can be obtained by Little's result as

$$E[W_1]=\frac{E[Q_1]}{\lambda_1},\qquad E[W_2]=\frac{E[Q_2]}{\lambda_2}. \tag{4.9}$$

5. The summary

In this paper, we have considered the queueing model consisting of two-queue and two-server with the hysteretic control service policy. As it has been seen, the service schedule is more flexible because by choosing the values of the forward threshold levels (F_1,F_2) and the reverse threshold levels (R_1,R_2), one can easily assign a higher priority to some queue according to variety of the system state. For the model, we derived the generating functions of the stationary joint queue-length distribution, and obtained the mean queue length and the mean waiting time. Here we want to emphasize that it is still an opening problem to determine the number of zeros of the determinant $\mathcal{M}(z)$ by a direct method(for the special case $F_1=F_2=0$, i.e., the second queue has a non-preemptive priority over the first queue, we have proved directly that $det\mathcal{M}(z)$ has indeed $F_2+F_1+3=3$ zeros in $|z|\leq 1$ in Feng et al(1999)). For the further work, it should be worthwhile to consider the analysis of the general model with $m(>2)$ servers and the problem of determining the optimal threshold values of (F_1,F_2) and (R_1,R_2).

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